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APPROXIMATION OF A SOLUTION FOR A K-POSITIVE DEFINITE OPERATOR EQUATION IN REAL UNIFORMLY SMOOTH BANACH SPACES

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Abstract: Suppose that X be a real uniformly smooth Banach space and $A : D(A) \subseteq X \to X$ is a K-positive definite operator with $D(A) = D(K)$. It is proved that the iteration scheme introduced by Bai (see *J. Math. Anal. Appl.*, **236** (1999), 236-242), for arbitrary initial vector in $D(A)$ and for any $f \in X$, converges strongly to the unique solution of the equation $Ax = f$. Moreover, our iteration parameters are completely independent of the geometry of the underlying Banach space and of any property of the operator. The results in this note extend, improve and unify the results due to Bai, Chidume and Aneke, Chidume and Osilike.

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1. Introduction

Let X be a normed linear space and let J denote the normalized duality mapping from X into 2^{X^*} given by

$$Jx = \{f \in X^* : \mathrm{Re}\langle x, f\rangle = \|x\|^2 = \|f\|^2\}, \quad x \in X,$$

where X^* denotes the dual space of X and $\langle\cdot,\cdot\rangle$ denotes the generalized duality pairing. It is known that X is a uniformly smooth Banach space if and only if X^* is a uniformly convex Banach space. On the other hand, if X^* is uniformly convex, then J is the single-valued and uniformly continuous on any bounded subset of X. In the sequel, we shall denote the single-valued normalized duality mapping by j. The symbols $D(A)$ and $R(A)$ stand for the domain and the range of an operator A, respectively.

In 1962, Petryshyn [5] introduced first the concept of a K-positive definite operator in complex Hilbert spaces and established the following excellent result:

Theorem P. (see [5]) *Let H be a complex separable Hilbert space and $A : D(A) \subseteq H \to H$ be a K-positive definite operator with $D(A) = D(K)$. Then there exists a constant $\alpha > 0$ such that for any $u \in D(K)$,*

$$\|Au\| \le \alpha\|Ku\|.$$

Furthermore, the operator A is closed, $R(A) = H$, and the equation $Au = f$, $f \in H$ has a unique solution.

In 1993, Chidume and Aneke [3] extended the concept of a K-positive definite operator to real separable Banach space X with strictly convex dual X^* and obtained the following result.

Theorem CA. (see [3]) *Let X be a real separable Banach space with a strictly convex dual X^* and let A be a K-positive definite operator with $D(A) = D(K)$. Suppose that for all $x, y \in D(A)$, $\langle Ax, j(Ky)\rangle = \langle Kx, j(Ay)\rangle$. Then there exists a constant $\alpha > 0$ such that for any $x \in D(A)$,*

$$\|Ax\| \le \alpha\|Kx\|.$$

Furthermore, the operator A is closed, $R(A) = X$, and the equation $Ax = f$, for each $f \in X$, has a unique solution.

Meanwhile, they gave an application of Theorem CA in L_p(or l_p) spaces with $p \ge 2$. That is, they constructed an iteration scheme which converges strongly to the unique solution of the equation $Ax = f$, $f \in X$.

In 1997, Chidume and Osilike [4] generalized the convergence result of Chidume and Aneke from L_p(or l_p), $p \geq 2$, to separable q-uniformly smooth real Banach spaces.

In 1999, Bai [1] constructed an iteration procedure and extended the convergence result of Chidueme and Osilike to separable uniformly smooth real Banach spaces. He proved the following result.

Theorem B. (see [1]) *Let X be a real uniformly smooth separable Banach space and let $A : D(A) \subseteq X \to X$ be a K-positive definite operator with $D(A) = D(K)$. Suppose that*

$$\langle Ax, j(Ky)\rangle = \langle Kx, j(Ky)\rangle, \quad x, y \in D(A).$$

For arbitrary $f \in X$, define the sequence $\{x_n\}_{n=0}^{\infty}$ iteratively from an arbitrary $x_0 \in D(A)$ by

$$y_n = x_n + r_n\mu_n, \quad x_{n+1} = y_n + t_n\gamma_n, \quad n \geq 0; \tag{1.1}$$

$$\mu_n = K^{-1}f - K^{-1}Ax_n, \quad \gamma_n = K^{-1}f - K^{-1}Ay_n, \quad n \geq 0; \tag{1.2}$$

$$\lim_{n\to\infty} r_n = 0; \tag{1.3}$$

$$\sum_{n=0}^{\infty} t_n = \infty, \quad \lim_{n\to\infty} t_n = 0; \tag{1.4}$$

$$0 \leq t_n, \quad r_n \leq \frac{1}{2c}, \quad n \geq 0; \tag{1.5}$$

$$b(\alpha t_n) \leq \frac{2c}{B\alpha}, \quad b(\alpha r_n) \leq \frac{2c}{B\alpha}, \quad n \geq 0, \tag{1.6}$$

where α and c are the constants appearing in (1.7) *and* (2.3), *respectively, b is the Reich's function appearing in Lemma 2.1, and $B = \max\{1, \|K\mu_0\|\}$. Then $\{x_n\}_{n=0}^{\infty}$ converges strongly to the unique solution of $Ax = f$.*

Recently, Zhou, Kang and Cho [7] obtained the following result.

Theorem ZKC. (see [7]) *Let X be a real uniformly smooth Banach space and let $A : D(A) \subseteq X \to X$ be a K-positive define operator with $D(A) = D(K)$. Then there exists a constant $\alpha > 0$ such that for all $x \in D(A)$,*

$$\|Ax\| \leq \alpha\|Kx\|. \tag{1.7}$$

Furthermore, the operator A is closed, $R(A) = X$ and the equation $Ax = f$, for each $f \in X$ has a unique solution.

Note that the iteration parameters $\{t_n\}_{n=0}^{\infty}$ and $\{r_n\}_{n=0}^{\infty}$ in Theorem B depend heavily both on the Reich's function b, on the K-positive definite operator

A, on the initial value x_0 and on constants α and c. Thus it is very difficult for one to choose suitable iteration parameters $\{t_n\}_{n=0}^{\infty}$ and $\{r_n\}_{n=0}^{\infty}$, which fulfill (1.5) and (1.6).

The purpose of this note is to study the iterative approximation of solutions for K-positive definite operator equations in real uniformly smooth Banach spaces. By using a new proof technique, we prove that the iteration method introduced by Bai can be used to approximate the solution of the equation $Ax = f$ and that our iteration parameters are completely independent of the K-positive definite operator A, the initial value x_0, constants α and c and the geometry of the underlying Banach spaces. The results in this note extend, improve and unify the results due to Bai [1], Chidume and Aneke [3], Chidume and Osilike [4].

2. Preliminaries

Lemma 2.1. (see [6]) *Let X be a real uniformly smooth Banach space. Then there exists a continuous nondecreasing function $b : [0,\infty) \to [0,\infty)$ such that*

$$b(0) = 0, \quad b(rt) \le rb(t), \quad r \ge 1, \quad t \ge 0; \tag{2.1}$$

$$\|x+y\|^2 \le \|x\|^2 + 2\langle y, j(x)\rangle + \max\{\|x\|, 1\}\|y\|b(\|y\|), \quad x, y \in X. \tag{2.2}$$

The function b in Lemma 2.1 and the inequality (2.2) are called Reich's function and Reich's inequality, respectively.

Lemma 2.2. (see [2]) *Suppose that $\{\alpha_n\}_{n=0}^{\infty}, \{\beta_n\}_{n=0}^{\infty}$ and $\{\omega_n\}_{n=0}^{\infty}$ are nonnegative sequences such that*

$$\alpha_{n+1} \le (1-\omega_n)\alpha_n + \beta_n\omega_n, \quad n \ge 0$$

with $\{\omega_n\}_{n=0}^{\infty} \subset [0,1]$, $\sum_{n=0}^{\infty} \omega_n = \infty$ and $\lim_{n\to\infty} \beta_n = 0$. Then

$$\lim_{n\to\infty} \alpha_n = 0\,.$$

Definition 2.1. (see [3], [7]) Let X be a real Banach space and let X_1 be a subspace of X. A operator A with domain $D(A) \supseteq X_1$, is called *continuously X_1-invertible* if A, as an operator restricted to X_1, has a bounded inverse on $R(A)$. A linear unbounded operator A with domain $D(A)$ in X and range $R(A)$ in X is called *K-positive definite* if there exist a continuously $D(A)$-invertible closed linear operator K with $D(A) \subseteq D(K)$ and a constant $c > 0$ such that, for every $x \in D(A)$, there exists $j(Kx) \in J(Kx)$ satisfying

$$\langle Ax, j(Kx)\rangle \geq c\|Kx\|^2. \tag{2.3}$$

3. Convergence Results

Theorem 3.1. *Let X be a real uniformly smooth Banach space and let $A : D(A) \subseteq X \to X$ be a K-positive definite operator with $D(A) = D(K)$. For any $x_0 \in D(A)$ and $f \in X$, assume that the sequences $\{x_n\}_{n=0}^{\infty}$, $\{y_n\}_{n=0}^{\infty}$, $\{\mu_n\}_{n=0}^{\infty}$ and $\{\gamma_n\}_{n=0}^{\infty}$ are as in (1.1) and (1.2). Suppose that $\{r_n\}_{n=0}^{\infty}$ and $\{t_n\}_{n=0}^{\infty}$ are arbitrary nonnegative sequences satisfying*

$$\lim_{n\to\infty} r_n = \lim_{n\to\infty} t_n = 0; \tag{3.1}$$

$$\sum_{n=0}^{\infty}(r_n + t_n) = \infty. \tag{3.2}$$

Then the sequence $\{x_n\}_{n=0}^{\infty}$ converges strongly to the unique solution of $Ax = f$.

Proof. The existence of a unique solution to $Ax = f$ follows from Theorem ZKC. The linearity of A and K and (1.2) ensure that

$$K\mu_{n+1} = K\gamma_n - t_n A\gamma_n, \quad K\gamma_n = K\mu_n - r_n A\mu_n, \quad n \geq 0. \tag{3.3}$$

In view of Lemma 2.1, (2.3), (1.7) and (3.3), we infer that for all $n \geq 0$,

$$\begin{aligned}
\Big\|\frac{K\mu_{n+1}}{1+\|K\gamma_n\|}\Big\|^2 &= \Big\|\frac{K\gamma_n}{1+\|K\gamma_n\|} - t_n\frac{A\gamma_n}{1+\|K\gamma_n\|}\Big\|^2 \\
&= \Big(\frac{\|K\gamma_n\|}{1+\|K\gamma_n\|}\Big)^2 - 2t_n\Big\langle\frac{A\gamma_n}{1+\|K\gamma_n\|}, j\Big(\frac{K\gamma_n}{1+\|K\gamma_n\|}\Big)\Big\rangle \\
&\quad + \max\Big\{\frac{\|K\gamma_n\|}{1+\|K\gamma_n\|}, 1\Big\} t_n\frac{\|A\gamma_n\|}{1+\|K\gamma_n\|} b\Big(t_n\frac{\|A\gamma_n\|}{1+\|K\gamma_n\|}\Big) \\
&\leq \Big(\frac{\|K\gamma_n\|}{1+\|K\gamma_n\|}\Big)^2 - 2ct_n\Big(\frac{\|K\gamma_n\|}{1+\|K\gamma_n\|}\Big)^2 \\
&\quad + t_n\alpha\frac{\|K\gamma_n\|}{1+\|K\gamma_n\|} b\Big(t_n\alpha\frac{\|K\gamma_n\|}{1+\|K\gamma_n\|}\Big) \\
&\leq (1-2ct_n)\Big(\frac{\|K\gamma_n\|}{1+\|K\gamma_n\|}\Big)^2 + t_n\alpha\frac{\|K\gamma_n\|}{1+\|K\gamma_n\|} b(\alpha t_n),
\end{aligned} \tag{3.4}$$

and

$$\begin{aligned}\Big\|\frac{K\gamma_n}{1+\|K\mu_n\|}\Big\|^2 &= \Big\|\frac{K\mu_n}{1+\|K\mu_n\|} - r_n\frac{A\mu_n}{1+\|K\mu_n\|}\Big\|^2 \\ &= \Big(\frac{\|K\mu_n\|}{1+\|K\mu_n\|}\Big)^2 - 2r_n\Big\langle\frac{A\mu_n}{1+\|K\mu_n\|}, j\Big(\frac{K\mu_n}{1+\|K\mu_n\|}\Big)\Big\rangle \\ &\quad + \max\Big\{\frac{\|K\mu_n\|}{1+\|K\mu_n\|}, 1\Big\}r_n\frac{\|A\mu_n\|}{1+\|K\mu_n\|}b\Big(r_n\frac{\|A\mu_n\|}{1+\|K\mu_n\|}\Big) \\ &\le \Big(\frac{\|K\mu_n\|}{1+\|K\mu_n\|}\Big)^2 - 2r_nc\Big(\frac{\|K\mu_n\|}{1+\|K\mu_n\|}\Big)^2 \\ &\quad + \alpha r_n\frac{\|K\mu_n\|}{1+\|K\mu_n\|}b\Big(\alpha r_n\frac{\|K\mu_n\|}{1+\|K\mu_n\|}\Big) \\ &\le (1-2cr_n)\Big(\frac{\|K\mu_n\|}{1+\|K\mu_n\|}\Big)^2 + \alpha r_n\frac{\|K\mu_n\|}{1+\|K\mu_n\|}b(\alpha r_n). \qquad (3.5)\end{aligned}$$

It follows from (3.1), (2.1) and the continuity of b that for a fixed $s \in (0, 2c)$, there exists a positive integer N such that

$$\max\{t_n, r_n\} < \frac{1}{s}, \quad n \ge N, \qquad (3.6)$$

and

$$\frac{3}{2}\alpha\max\{b(\alpha t_n), b(\alpha r_n)\} < 2c - s, \quad n \ge N. \qquad (3.7)$$

Using (3.5)-(3.7), we get that

$$\begin{aligned}\|K\gamma_n\|^2 &\le (1-2cr_n)\|K\mu_n\|^2 + \alpha r_n\|K\mu_n\|(1+\|K\mu_n\|)b(\alpha r_n) \\ &\le (1-2cr_n)\|K\mu_n\|^2 + \frac{1}{2}\alpha r_n(1+3\|K\mu_n\|^2)b(\alpha r_n) \\ &= [1-2cr_n+\frac{3}{2}\alpha r_n b(\alpha r_n)]\|K\mu_n\|^2 + \frac{1}{2}\alpha r_n b(\alpha r_n) \\ &\le (1-sr_n)\|K\mu_n\|^2 + \frac{1}{2}\alpha r_n b(\alpha r_n), \qquad (3.8)\end{aligned}$$

for all $n \ge N$. By virtue of (3.4), (3.6)-(3.8), we conclude that

$$\begin{aligned}\|K\mu_{n+1}\|^2 &\le (1-2ct_n)\|K\gamma_n\|^2 + \alpha t_n\|K\gamma_n\|(1+\|K\gamma_n\|)b(\alpha t_n) \\ &\le (1-2ct_n)\|K\gamma_n\|^2 + \frac{1}{2}\alpha t_n(1+3\|K\gamma_n\|^2)b(\alpha t_n) \\ &= [1-2ct_n+\frac{3}{2}\alpha t_n b(\alpha t_n)]\|K\gamma_n\|^2 + \frac{1}{2}\alpha t_n b(\alpha t_n)\end{aligned}$$

$$
\begin{aligned}
&\leq (1-st_n)\|K\gamma_n\|^2 + \frac{1}{2}\alpha t_n b(\alpha t_n) \\
&\leq (1-st_n)(1-sr_n)\|K\mu_n\|^2 + \frac{1}{2}(1-st_n)\alpha r_n b(\alpha r_n) \\
&+ \frac{1}{2}\alpha t_n b(\alpha t_n) \leq [1 - s(t_n+r_n) + s^2 t_n r_n]\|K\mu_n\|^2 \\
&+ \frac{1}{2}\alpha[r_n b(\alpha r_n) + t_n b(\alpha t_n)] \leq [1 - s(t_n+r_n) + \min\{st_n, sr_n\}]\|K\mu_n\|^2 \\
&+ \frac{1}{2}\alpha[r_n b(\alpha r_n) + t_n b(\alpha t_n)] \leq [1 - \frac{1}{2}s(t_n+r_n)]\|K\mu_n\|^2 + s_n(t_n+r_n)\,, \quad (3.9)
\end{aligned}
$$

for all $n \geq N$, where

$$
s_n = \begin{cases} \frac{1}{2}\alpha\left[\frac{r_n b(\alpha r_n)}{t_n+r_n} + \frac{t_n b(\alpha t_n)}{t_n+r_n}\right], & t_n + r_n \neq 0 \\ 0, & t_n + r_n = 0. \end{cases}
$$

Observe that

$$
0 \leq s_n \leq \frac{1}{2}\alpha[b(\alpha r_n) + b(\alpha t_n)], \quad n \geq N. \tag{3.10}
$$

It follows from (3.1), (2.1) and the continuity of b that

$$
\lim_{n\to\infty} s_n = 0. \tag{3.11}
$$

Let

$$
\alpha_n = \|K\mu_n\|^2, \quad \omega_n = \frac{1}{2}s(t_n+r_n), \quad \beta_n = \frac{2}{s}s_n, \quad n \geq N.
$$

Using Lemma 2.2, (3.1), (3.2), (3.9) and (3.11), we obtain that $\alpha_n \to 0$ as $n \to \infty$. That is,

$$
\|K\mu_n\| = \|Ax_n - f\| \to 0 \quad \text{as} \quad n \to \infty. \tag{3.12}
$$

Note that A has a bounded inverse. Thus (3.12) yields that $x_n \to A^{-1}f$, the unique solution of $Ax = f$. This completes the proof. □

Remark 3.1. Theorem 3.1 extends [1, Theorem 3; 3, Theorem 2; 4, Theorem] in the following sense:

(i) L_p(or l_p) spaces with $p \geq 2$ in [3], real q-uniformly smooth separable Banach spaces with $q > 1$ in [4], real uniformly smooth separable Banach spaces in [1] are replaced by the more general class of real uniformly smooth Banach spaces.

(ii) The commutativity of A and K in [1] and [4] are omitted.

(iii) The assumption conditions of $\max\{t_n, r_n\} \leq \frac{1}{2c}, n \geq 0$ and

$$
\max\{b(\alpha t_n), b(\alpha r_n)\} \leq \frac{2c}{B\alpha}, n \geq 0
$$

in [1] are superfluous.

(iv) Condition (3.2) is much weaker than $\sum_{n=0}^{\infty} t_n = \infty$ of [1].

Remark 3.2. In our theorem, the iteration parameters $\{t_n\}_{n=0}^{\infty}$ and $\{r_n\}_{n=0}^{\infty}$ both are independent of any properties of the operator and the geometry of the underlying Banach space and can be chosen at the start of the iteration process. A proto type for the parameters is

$$t_n = r_n = \frac{1}{n+1}, \quad n \geq 0.$$

As a consequent of Theorem 3.1, we have the following result.

Corollary 3.1. *Let X, A, K, f, x_0 be as in Theorem 3.1 and let $\{t_n\}_{n=0}^{\infty}$ be an arbitrary nonnegative sequence satisfying*

$$\lim_{n\to\infty} t_n = 0, \quad \sum_{n=0}^{\infty} t_n = \infty.$$

Define the sequence $\{x_n\}_{n=0}^{\infty}$ iteratively from x_0 by

$$x_{n+1} = x_n + t_n\gamma_n, \quad \gamma_n = K^{-1}f - K^{-1}Ax_n, \quad n \geq 0.$$

Then the sequence $\{x_n\}_{n=0}^{\infty}$ converges strongly to the unique solution of $Ax = f$.

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References

[1] C.Z. Bai, Approximation of a solution for K-positive define operator equation in uniformly smooth separable Banach spaces, *J. Math. Anal. Appl.*, **236** (1999), 236-242.

[2] S.S. Chang, Y.J. Cho, B.S. Lee, J.S. Jung, S.M. Kang, Iterative approximations of fixed points and solutions for strongly accretive and strongly pseudo-contractive mappings in Banach spaces, *J. Math. Anal. Appl.*, **224** (1998), 146-165.

[3] C.E. Chidume, S.J. Aneke, Existence, uniqueness and approximation of a solution for a K-positive definite operator equation, *Appl. Anal.*, **50** (1993), 285-294.

[4] C.E. Chidume, M.O. Osilike, Approximation of a solution for a K-positive definite operator equation, *J. Math. Anal. Appl.*, **210** (1997), 1-7.

[5] W.V. Petryshyn, Direct and iterative methods for the solution of linear operator equations in Hilbert spaces, *Trans. Amer. Math. Soc.*, **105** (1962), 136-175.

[6] S. Reich, An iterative procedure for constructing zeros of accretive sets in Banach spaces, *Nonlinear Anal.*, **2** (1978), 85-92.

[7] H.Y. Zhou, S.M. Kang, Y.J. Cho, Constructive solvability of K-positive definite operator equations in uniformly smooth Banach spaces, *Submitted*.