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CURVATURE, DISCRETE GROUPS AND HAUSDORFF DISTANCE

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Abstract: In this short note we develop the notion of Hausdorff distance between the discrete subgroups of the isometry group on a simply connected Riemannian manifold with lower curvature bounds. We prove a rigidity theorem to show that close geometric structures will imply close group structures.

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1. Introduction

Let G be a Lie group. An interesting problem is to investigate the relationship between the group structure and the geometric structure of G. However, it is not easy, in the authors' opinion, to answer this problem for general Lie groups. It is well-known that the group of isometries of a Riemannian manifold with lower sectional curvature bounds is in fact a Lie group. In general, Fukaya and Yamaguchi proved in [4] that it is true for the ones of Alexandrove spaces with

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lower curvature bounds. Moreover, the structure of the isometry group of a given Riemannian manifold depends heavily on the geometric and topological properties of the manifold itself. For example, Wei showed in [9] that there are examples of complete manifolds of positive Ricci curvature with nilpotent isometry groups. Therefore, it will be helpful for one to study isometry groups if there are some new ideas about these groups.

In this paper, we consider the discrete subgroups of isometry group $\mathrm{Isom}\,(\tilde{M})$ of a simply connected Riemannian n-manifold $\tilde{M}$ with a lower sectional curvature bound $K_{\tilde{M}} \geq -1$. Our approach is to define a Hausdorff distance between these groups.

Let us first recall the classic Hausdorff distance d_H between subsets of metric spaces. We refer the readers to [5] or [7] for details. If (X, d) is a metric space and $A,\ B \subseteq X$, we define $\mathrm{dist}\,(A, B) \equiv \inf\{d(a, b)|a \in A,\ \ b \in B\}$, $B(A, \epsilon) = \{x \in X|d(x, A) < \epsilon\}$ and then

$$d_H(A, B) \ = \ \inf\{\epsilon \mid A \subset B(B, \epsilon),\ B \subset B(A, \epsilon)\}.$$

Let $(\tilde{M}, d)$ denote a topological n-manifold with a metric d. Then the distance d induces a natural *pseudometric* $\bar{d}$ on $\mathrm{Isom}\,(\tilde{M})$ as follows. Given any two isometries g and γ in $\mathrm{Isom}\,(\tilde{M})$ we define this distance $\bar{d}$ by

$$\bar{d}(g, \gamma) \ = \ \sup_{\tilde{x} \in \tilde{M}} d(g\tilde{x}, \gamma\tilde{x}).$$

Note that this distance $\bar{d}$ may not be finite (this is why we call it a pseudometric). However, the triangle inequality for $\bar{d}$ still holds. Thus one can also consider the Hausdorff distance $d_H(G, \Gamma)$ between two subgroups G and Γ of $\mathrm{Isom}\,(\tilde{M})$.

We are to investigate the relationship between $d_H(G, \Gamma)$ and the group structure of G and Γ. Note that we say a group G acting *properly* on a topological space X if $\{g \in G|gK \cap K \neq \phi\}$ is finite for each compact subset $K \subset X$; and *freely* if, for every $x \in X$ and every $g \in G - \{e\}$, one has $gx \neq x$. G is a *cocompact group* if there exists a compact set $K \subseteq X$ such that $X = GK = \{gx|g \in G \text{ and } x \in K\}$. First let us review some examples.

Example 1.1. Let $\mathbf{H}^n$ denote the hyperbolic n-space and G and Γ be subgroups of $\mathrm{Isom}\,(\mathbf{H}^n)$. The famous Mostow's Rigidity Theorem (c.f. [8], Chapter 11) implies that if $n \geq 3$ and $\mathbf{H}^n/G$ and $\mathbf{H}^n/\Gamma$ are both compact oriented manifolds with constant sectional curvature $K \equiv -1$, then G and Γ are conjugate in the group $\mathrm{Isom}\,(\mathbf{H}^n)$ provided G and Γ are isomorphic. In our terminology, it means that if $G, \Gamma \subseteq \mathrm{Isom}\,(\mathbf{H}^n)$ are both discrete and cocompact groups acting freely on $\mathbf{H}^n$, then G and Γ are isomorphic implies that there exists $\tilde{f} \in \mathrm{Isom}\,(\mathbf{H}^n)$ such that $d_H(G, \tilde{f}^{-1}\Gamma\tilde{f}) = 0$.

The next example is a result shown in [2] by the author. It can be viewed in some sense as a converse of Example 1 for negative curved manifolds.

Example 1.2. Let $\tilde{M}$ be a simply connected Riemannian n-manifold with $-1 \leq K_{\tilde{M}} < 0$ and $\Gamma \subseteq \mathrm{Isom}\,(\tilde{M})$ is a discrete and cocompact subgroup acting freely on $\tilde{M}$. Set $N_\Gamma \equiv \{a \in \mathrm{Diff}(\tilde{M}) | a\Gamma a^{-1} = \Gamma\}$ to be the normalizer of Γ in the diffeomorphism group of $\tilde{M}$. The author showed in [1] that for $G \subset N_\Gamma$ and G is a finite extension of Γ (i.e. G/Γ is a finite group) and

$$\sup\{d(g\tilde{x}, \Gamma\tilde{x}) | g \in G, \tilde{x} \in \tilde{M}\} \;\leq\; 4^{-(n+4)},$$

then $G = \Gamma$. This implies that if $G, \Gamma \subseteq \mathrm{Isom}\,(\tilde{M}^n)$ are both discrete cocompact groups acting freely on $\tilde{M}^n$, then $G = \Gamma$ provided $G \subset N_\Gamma$, G/Γ is finite and $d_H(G, \Gamma) < 4^{-(n+4)}$.

Our main result is the following theorem.

Main Theorem 1.3. *Let $(\tilde{M}, d)$ be a simply connected Riemannian n-manifold with an induced metric d and sectional curvature $K_{\tilde{M}} \geq -1$. Denote d_H be the Hausdorff distance in the isometry group* $\mathrm{Isom}\,(\tilde{M})$ *of $\tilde{M}$ induced by d. Given a discrete and cocompact subgroup G of* $\mathrm{Isom}\,(\tilde{M})$*. Then there exists $\epsilon = \epsilon(G) > 0$ such that if Γ is another discrete and cocompact subgroup of* $\mathrm{Isom}\,(\tilde{M})$ *with $d_H(G, \Gamma) < \epsilon$, then G and Γ are conjugate in the group of homeomorphisms of $\tilde{M}$.*

Remark 1.4. If G is a discrete subgroup of $\mathrm{Isom}\,(\tilde{M})$, then G acts properly on $\tilde{M}$ by the theory of group action. Moreover, it can be shown that the quotient space $\tilde{M}/G$ is a Riemannian manifold if in addition G acts freely on $\tilde{M}$. In Main Theorem, however, $\tilde{M}/G$ can only be a compact *Alexandrov space* (see [1]) with curvature bounded from below.

2. Proof of Main Theorem

Now we are in the position to prove Main Theorem 1.3. We first recall some properties about a discrete and cocompact group $G \subset \mathrm{Isom}\,(\tilde{M})$ acting on a simply connected Riemannian manifold $\tilde{M}$. The readers are referred to [8] for details. Let $G_{\tilde{x}} = \{g \in G | g\tilde{x} = \tilde{x}\}$ denote the stabilizer of G at $\tilde{x} \in \tilde{M}$ and $G\tilde{x} = \{g\tilde{x} | g \in G\}$ denote the G-orbit through $\tilde{x}$. Let $P : \tilde{M} \to \tilde{M}/G$ be the quotient map and $B_{\tilde{x}}(r)$ denote the open ball centered at $\tilde{x}$ with radius r. Then for each $\tilde{x} \in \tilde{M}$ with $P(\tilde{x}) = x$, the map P induces a homeomorphism from $B_{\tilde{x}}(r)/G_{\tilde{x}}$ onto $B_x(r)$ for all r such that $0 < r \leq \frac{1}{2}\mathrm{dist}\,(\tilde{x}, G\tilde{x} - \{\tilde{x}\})$.

Moreover, P also induces an isometry from $B_{\tilde{x}}(r)/G_{\tilde{x}}$ onto $B_x(r)$ for all r such that $0 < r \leq \frac{1}{4}\text{dist}\,(\tilde{x}, G\tilde{x} - \{\tilde{x}\})$. A point $\tilde{x} \in \tilde{M}$ is called a *regular* point if $G_{\tilde{x}} = \{e\}$ is the trivial group, otherwise $\tilde{x}$ is called a *singular* point of $\tilde{M}$. The set of all regular points of $\tilde{M}$ is a connect, open, dense subset and then the set of all singular points of $\tilde{M}$ is a close nowhere dense subset.

It is clear that $\tilde{x}$ is regular if and only if $g\tilde{x}$ is regular for each $g \in G$ and so are singular points. So, for the sake of brief, we also call a point $x = P(\tilde{x}) \in \tilde{M}/G$ a regular (or singular) point if $\tilde{x}$ is a regular (or singular) point in $\tilde{M}$.

Let $\tilde{x}_0 \in \tilde{M}$ be a regular point. The *closed Dirichlet domain* for $\tilde{x}_0$ is the set

$$\bar{D}_G(\tilde{x}_0) \equiv \{u \in \tilde{M} | d(u, \tilde{x}_0) \leq d(u, g\tilde{x}_0) \text{ for all } g \in G\}.$$

The open interior $D_G(\tilde{x}_0)$ is called the *open Dirichlet domain* for the regular orbit $G\tilde{x}_0$. Since $\tilde{M} = \cup_{g\in G} g\bar{D}_G(\tilde{x}_0)$, we can find a *fundamental set* F_G in $\tilde{M}$ containing $\tilde{x}_0$, which means that F_G meets each orbit in exactly one point, for the action of G satisfying $D_G(\tilde{x}_0) \subseteq F_G \subset \bar{D}_G(\tilde{x}_0)$. Note that each point in $D_G(\tilde{x}_0)$ is a regular point and then $gD_G(\tilde{x}_0)$ consists of regular points for all $g \in G$. So the set of all singular points of $\tilde{M}$ is contained in the set $\{g(\bar{D}_G(\tilde{x}_0) - D_G(\tilde{x}_0)) | g \in G\}$.

Consider two discrete and cocompact subgroups G and Γ of $\text{Isom}\,(\tilde{M})$. It can be shown that the quotient spaces $\tilde{M}/G$ and $\tilde{M}/\Gamma$ are both compact Alexandrov spaces with lower curvature bound. Let $P_1 : \tilde{M} \to \tilde{M}/G$ and $P_2 : \tilde{M} \to \tilde{M}/\Gamma$ be the quotient maps. Then we have the following lemma.

Lemma 2.1. *If $d_H(G, \Gamma) < \epsilon$, then $d_H(\tilde{M}/G, \tilde{M}/\Gamma) < \epsilon$.*

Proof. Since the groups G and Γ are both discrete and cocompact subgroups in $\text{Isom}\,(\tilde{M})$, there exist compact subsets $\bar{D}_G$ and $\bar{D}_\Gamma$ of $\tilde{M}$ containing the same point $\tilde{x}_0$ such that $G\bar{D}_G = \tilde{M}$ and $\Gamma\bar{D}_\Gamma = \tilde{M}$. For each $\tilde{x} \in \tilde{M}$ the two sets $G\tilde{x} \cap \bar{D}_G$ and $\Gamma\tilde{x} \cap \bar{D}_\Gamma$ are both finite. To prove this lemma, it suffices to show that there exists $g \in G$ such that $d_H(g\bar{D}_\Gamma, \bar{D}_G) < \epsilon$. We prove it by contradiction.

Suppose there is $\tilde{x} \in \bar{D}_G \setminus \bar{D}_\Gamma$ such that $d(\tilde{x}, g\tilde{y}) \geq \epsilon$ for all $g \in G$ and $\tilde{y} \in \bar{D}_\Gamma$. Since $\tilde{x} \notin \bar{D}_\Gamma$, there exist $\tilde{y} \in \bar{D}_\Gamma$ and $\gamma \in \Gamma$ such that $\gamma\tilde{y} = \tilde{x}$. This implies that $d(\gamma\tilde{y}, g\tilde{y}) = d(\tilde{x}, g\tilde{y}) \geq \epsilon$ for all $g \in G$. It contradicts to the assumption that $d_H(G, \Gamma) < \epsilon$. □

Next we apply Perelman's work. Since $\tilde{M}/G$ and $\tilde{M}/\Gamma$ are both compact Alexandrov spaces with lower sectional curvature bound, Perelman showed in [6] that there exists $\epsilon_1 = \epsilon_1(G) > 0$ such that if $d_H(\tilde{M}/G, \tilde{M}/\Gamma) < \epsilon_1$ then there exists a homeomorphism $f : \tilde{M}/G \to \tilde{M}/\Gamma$ and f is also an ϵ_1-*Hausdorff approximation*, which means that

$$|d(f(x_1), f(x_2)) - d(x_1, x_2)| \ < \ \epsilon_1$$

for all $x_1,\ x_2 \in \tilde{M}/G$. Similar argument also applies to the inverse $f^{-1} : \tilde{M}/\Gamma \to \tilde{M}/G$ of f.

Lemma 2.2. *Let G and Γ be as in Main Theorem 1.3. Then there exists $\epsilon = \epsilon(G) > 0$ such that if $d_H(G, \Gamma) < \epsilon$ then for each $g \in G$ there is a unique $\gamma \in \Gamma$ such that $\bar{d}(g, \gamma) < \epsilon$.*

Moreover, if $g_1,\ g_2 \in G$ and $\gamma_1,\ \gamma_2 \in \Gamma$ satisfy $\bar{d}(g_1, \gamma_1) < \epsilon$ and $\bar{d}(g_2, \gamma_2) < \epsilon$ then $\bar{d}(g_1 g_2, \gamma_1\gamma_2) < \epsilon$.

Proof. Let $\tilde{x}_0$ be a regular point in $\tilde{M}$, $x_0 = P_1(\tilde{x}_0)$, $y_0 = f(x_0)$, $P_2(\tilde{y}_0) = y_0$ and $D_G(\tilde{x}_0)$ be the open Dirichlet domain for $\tilde{x}_0$. We can choose the point $\tilde{x}_0$ such that the open ball $B_{\tilde{x}_0}(r_0)$ with $r_0 = \frac{1}{2}\mathrm{dist}\,(\tilde{x}_0, G\tilde{x}_0 - \{\tilde{x}_0\})$ is the largest ball contained in $D_G(\tilde{x}_0)$. Here the radius r_0 is dependent on the group G. Let $\epsilon = \epsilon(G) = \min\{\epsilon_1, \frac{1}{10}r_0\}$ depending on the group G and $d(G, \Gamma) < \epsilon$. Then $P_1(B_{\tilde{x}_0}(5\epsilon))$ is isometric to the ball $B_{x_0}(5\epsilon)$. Since by Lemma 2.1 f is an ϵ-Hausdorff approximation and a homeomorphism, we have $B_{y_0}(3\epsilon) \subset f(B_{x_0}(5\epsilon))$ and consists of regular points. This means that $d(\tilde{y}_0, \gamma\tilde{y}_0) > 2\epsilon$ for each nontrivial $\gamma \in \Gamma$.

Now suppose there are $\gamma_1,\ \gamma_2 \in \Gamma$ such that $\bar{d}(g, \gamma_1) < \epsilon$ and $\bar{d}(g, \gamma_2) < \epsilon$. Then, by the triangle inequality, $\bar{d}(\gamma_1, \gamma_2) \le 2\epsilon$. Denote $\gamma = \gamma_1^{-1}\gamma_2$ and then $\bar{d}(e, \gamma) \le 2\epsilon$. However, $d(\tilde{y}_0, \gamma\tilde{y}_0) > 2\epsilon$ for each nontrivial $\gamma \in \Gamma$. Therefore γ is the identity and then $\gamma_1 = \gamma_2$. This proves the first part of Lemma 2.2.

Since $G,\ \Gamma \subset \mathrm{Isom}\,(\tilde{M})$, one has by the triangle inequality that

$$\bar{d}(g_1g_2, \gamma_1\gamma_2) \le \bar{d}(g_1g_2, g_1\gamma_2) + \bar{d}(g_1\gamma_2, \gamma_1\gamma_2) = \bar{d}(g_2, \gamma_2) + \bar{d}(g_1, \gamma_1) \le 2\epsilon.$$

Hence $\gamma_1\gamma_2$ is the unique element in Γ with $\bar{d}(g_1g_2, \gamma_1\gamma_2) < \epsilon$. □

Next we will lift the map $f : \tilde{M}/G \to \tilde{M}/\Gamma$ to a homeomorphism $\tilde{f} : \tilde{M} \to \tilde{M}$.

Lemma 2.3. *Let ϵ be as in Lemma 2.2. Then there exists a homeomorphism $\tilde{f} : \tilde{M} \to \tilde{M}$ with $\tilde{f}(\tilde{x}_0) = \tilde{y}_0$ such that the following diagrams commute.*

$$\begin{array}{ccc} (\tilde{M}, \tilde{x}_0) & \xrightarrow{\tilde{f}} & (\tilde{M}, \tilde{y}_0) \\ P_1 \downarrow & & \downarrow P_2 \\ (\tilde{M}/G, x_0) & \xrightarrow{f} & (\tilde{M}/\Gamma, y_0) \end{array} \qquad \begin{array}{ccc} (\tilde{M}, \tilde{x}_0) & \xleftarrow{\tilde{f}^{-1}} & (\tilde{M}, \tilde{y}_0) \\ P_1 \downarrow & & \downarrow P_2 \\ (\tilde{M}/G, x_0) & \xleftarrow{f^{-1}} & (\tilde{M}/\Gamma, y_0) \end{array}$$

Proof. Since $f : \tilde{M}/G \to \tilde{M}/\Gamma$ is a homeomorphism, f lifts a homeomorphism $\tilde{f}_1$ from the open Dirichlet fundamental domain $D_G(\tilde{x}_0)$ to an open set D_Γ containing $\tilde{y}_0$ in $\tilde{M}$. Choose a fixed fundamental set F_G with $D_G(\tilde{x}_0) \subseteq F_G \subset \bar{D}_G(\tilde{x}_0)$. For each $\tilde{x}' \in F_G - D_G(\tilde{x}_0)$, there is a sequence $\{\tilde{x}_i\}$ in $D_G(\tilde{x}_0)$ such that $\tilde{x}_i \to \tilde{x}'$ as $i \to \infty$. Define a new map f_2, which extend $\tilde{f}_1$, by $\tilde{f}_2(\tilde{x}) \equiv \tilde{f}_1(\tilde{x})$ if $\tilde{x} \in D_G(\tilde{x}_0)$; and $\tilde{f}_2(\tilde{x}') \equiv \lim_{i\to\infty} \tilde{f}_1(\tilde{x}_i)$. Set $\tilde{y}' \equiv \tilde{f}_2(\tilde{x}')$ and $F_\Gamma \equiv \tilde{f}_2(F_G)$. Then we claim that the map $\tilde{f}_2 : F_G \to F_\Gamma$ is a homeomorphism and F_Γ is a fundamental set for Γ.

Suppose $\{\tilde{u}_i\}$ is another sequence in $D_G(\tilde{x}_0)$ such that $\tilde{f}_1(\tilde{u}_i) \to \gamma\tilde{y}'$ in $\tilde{M}$ as $i \to \infty$ for some nontrivial $\gamma \in \Gamma$ with $\gamma\tilde{y}' \neq \tilde{y}'$. Without lose of generality, we can assume that both of the sequences $\{\tilde{f}_1(\tilde{x}_i)\}$ and $\{\tilde{f}_1(\tilde{u}_i)\}$ consist of regular points. Let α_i denote a minimal geodesic from $\tilde{x}_i$ to $\tilde{u}_i$, $\tilde{\beta}_i = \tilde{f}_1(\alpha_i)$, $\alpha_i = P_1(\tilde{\alpha}_i)$ and $\beta_i = P_2(\tilde{\beta}_i)$. Then f maps α_i homeomorphically to β_i for each i. However, α_i tends to a point and β_i tends to a loop. It is impossible. Therefore the map $\tilde{f}_2$ is well-defined. Since F_G is a fundamental set for G, F_Γ is a fundamental set for Γ and hence $\tilde{f}_2 : F_G \to F_\Gamma$ is a homeomorphism.

Now we extend $\tilde{f}_2$ to be a map $\tilde{f}$ defined on the whole $\tilde{M}$. For each $\tilde{x} \in \tilde{M}$ there is a unique $g \in G$ such that $\tilde{x} \in gF_G$. By Lemma 2.2, there is a unique $\gamma \in \Gamma$ such that $\bar{d}(g, \gamma) < \epsilon$. So we define the map $\tilde{f} : \tilde{M} \to \tilde{M}$ by

$$\tilde{f}(\tilde{x}) \equiv \gamma \circ \tilde{f}_2 \circ g^{-1}(\tilde{x}).$$

It is clear that $\tilde{f}$ is a bijection. Moreover, $\tilde{f}$ maps homeomorphically the open dense set $\{gD_G(\tilde{x}_0) | g \in G\}$ in $\tilde{M}$ to the open set $\{\gamma D_\Gamma | \gamma \in \Gamma\}$ in $\tilde{M}$. Here we check that $\tilde{f} : \tilde{M} \to \tilde{M}$ is a homeomorphism. Let $\tilde{x}_i \in gF_G$ for all i and $\tilde{x}_i \to \tilde{x}''$ as $i \to \infty$. If $\tilde{x}'' \in gF_G$ then, by the above argument, we have

$$\tilde{f}(\tilde{x}_i) = \gamma \circ \tilde{f}_2 \circ g^{-1}(\tilde{x}_i) \to \gamma \circ \tilde{f}_2 \circ g^{-1}(\tilde{x}'') = \tilde{f}(\tilde{x}'')$$

as $i \to \infty$. On the other hand, it suffices to show that if $\tilde{x}_i \in F_G$ for all i, $\tilde{x}_i \to \tilde{x}'' \in \bar{F}_G - F_G$ and $\tilde{x}'' \in g'F_G$ for some nontrivial $g' \in G$, then $\tilde{f}(\tilde{x}_i) \to \tilde{f}(\tilde{x}'') \in \bar{F}_\Gamma - F_\Gamma$ and $\tilde{f}(\tilde{x}'') \in \gamma' F_G$ with $\bar{d}(g', \gamma') < \epsilon$.

Indeed, we have $\tilde{f}(\tilde{x}'') \in \bar{F}_\Gamma - F_\Gamma$ since if $\bar{d}(g', \gamma') < \epsilon$ then $g'\bar{F}_G \cap \bar{F}_G \neq \emptyset$ if and only if $\gamma'\bar{F}_\Gamma \cap \bar{F}_\Gamma \neq \emptyset$. Since $\tilde{f}(\tilde{x}_i) \in F_\Gamma$ for all i, there exists $\tilde{y}'' \in \bar{F}_\Gamma - F_\Gamma$ such that $\tilde{f}(\tilde{x}_i) \to \tilde{y}''$ as $i \to \infty$. Moreover, we have

$$P_2(\tilde{y}'') = P_2(\tilde{f}(\tilde{x}'')) = \tilde{f}(P_1(\tilde{x}'')).$$

Hence $\tilde{y}'' = \tilde{f}(\tilde{x}'')$. This completes the proof of Lemma 2.3. □

Finally we give a proof of our main theorem.

Proof of Main Theorem 1.3. Let $\tilde{f}^{-1}\Gamma\tilde{f} \equiv \{\tilde{f}^{-1}\circ\gamma\circ\tilde{f} | \gamma \in \Gamma\}$. We claim that

$$\tilde{f}^{-1}\Gamma\tilde{f} = G.$$

Indeed, for given $\gamma \in \Gamma$ and $\tilde{x} \in \tilde{M}$, we have by Lemma 2.3 that

$$\begin{aligned} P_1 \circ \tilde{f}^{-1} \circ \gamma \circ \tilde{f}(\tilde{x}) = f^{-1} \circ P_2 \circ \gamma \circ \tilde{f}(\tilde{x}) = f^{-1} \circ P_2 \circ \tilde{f}(\tilde{x}) \\ = f^{-1} \circ f \circ P_1(\tilde{x}) = P_1(\tilde{x}). \end{aligned}$$

This shows that $\tilde{f}^{-1}\circ\gamma\circ\tilde{f}(\tilde{x}) = g_{\tilde{x}}\tilde{x}$ for some $g_{\tilde{x}} \in G$ depending on the point $\tilde{x}$. Note that for all $\tilde{x} \in F_G$ we have $g_{\tilde{x}} = g$, where $\bar{d}(g,\gamma) < \epsilon$. Let $\tilde{x}_k = g_k\tilde{x}$ for $g_k \in G$, $\bar{d}(g_k,\gamma_k) < \epsilon$ and $\tilde{f}(\tilde{x}) = \tilde{y}$. Then by Lemma 2.2 we have

$$\tilde{f}^{-1}\circ\gamma\circ\tilde{f}(\tilde{x}_k) = \tilde{f}\circ\gamma\gamma_k(\tilde{y}) = gg_k(\tilde{x}) = g(\tilde{x}_k).$$

This shows that for each $\gamma \in \Gamma$, $\tilde{f}^{-1}\circ\gamma\circ\tilde{f} \in G$ and then $\tilde{f}^{-1}\Gamma\tilde{f}$ is a subgroup of G. Also, $\tilde{f}G\tilde{f}^{-1} \subseteq \Gamma$ is a subgroup of Γ. Therefore,

$$G \;=\; \tilde{f}^{-1}\tilde{f}G\tilde{f}^{-1}\tilde{f} \;\subseteq\; \tilde{f}^{-1}\Gamma\tilde{f} \;\subseteq\; G.$$

Hence $\tilde{f}^{-1}\Gamma\tilde{f} = G$ and the proof of our main theorem is complete. □

The following is a Ricci version of our main theorem.

Remark 2.4. In [3], Theorem A.1.12, Cheeger and Colding prove that compact Riemannian n-manifolds with lower Ricci curvature bound $-(n-1)$ and close Hausdorff distance will be diffeomorphic. So our approach can be applied to manifolds with lower Ricci curvature bound and gives the following result.

Theorem 2.5. *Let $(\tilde{M}, d)$ be a simply connect Riemannian n-manifold with Ricci curvature $Ric_{\tilde{M}} \geq -(n-1)$. Denote d_H be the Hausdorff distance in* Isom$(\tilde{M})$ *induced by d. Consider a discrete and cocompact subgroup G of* Isom$(\tilde{M})$ *acting freely on $\tilde{M}$. Then there exists $\epsilon = \epsilon(G) > 0$ such that if Γ is another discrete and cocompact subgroup of* Isom$(\tilde{M})$ *acting freely on $\tilde{M}$ with $d_H(G,\Gamma) < \epsilon$, then G and Γ are conjugate in the group of homeomorphisms of $\tilde{M}$.*

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