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BEST APPROXIMATION IN A HILBERT SPACE

E. Aghdassi[1 §], S.F. Rzaev[2]
[1,2]Faculty of Mathematical Sciences
University of Tabriz
Tabriz, IRAN
[1]e-mail: esaghdassi@tabrizu.ac.ir
[2]e-mail: rzseymur@hotmail.com

Abstract: In this paper we consider the shift operator on Hilbert spaces and by using this operator we define the modulus of continuity of fractional index, including relation to the K-functional and we prove the fractional analog we direct and inverse theorems of approximation theory.

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Let $A : D(A) \subseteq H \to H$ be a self-adjoint operator on a separable Hilbert space H, which possesses a complete orthogonal system $\{w_1, w_2, \dots\}$ of eigenvectors with

$$Aw_k = \lambda_k w_k \qquad \text{for all } n = 1, 2, \dots,$$

where

$$0 < c \leq |\lambda_1| \leq |\lambda_2| \leq \cdots \leq |\lambda_n| \leq \cdots \to \infty, \quad \text{as } n \to \infty.$$

It is known that (for example see [6])

$$A^\alpha f := \sum_{k=1}^{\infty} \lambda_k^\alpha \langle f, w_k \rangle w_k, \quad \alpha > 0\,,$$

where $f \in D(A^\alpha)$ iff $\sum_{k=1}^{\infty} |\lambda_k^\alpha \langle f, w_k \rangle|^2 < \infty$.

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§Correspondence author

Since $\{w_n\}$ is complete,

$$f=\sum_{k=1}^{\infty}\langle f,w_k\rangle w_k \quad \text{for all } f\in H\,. \tag{1}$$

Let H_n =span$\{w_1,w_2,\dots,w_n\}$ which is an n-dimensional subspace of a real Hilbert space H and let $S_n(f)=\sum_{k=1}^{n}\langle f,w_k\rangle w_k$. It is well-known that for $f\in H$ the element $S_n(f)$ is best approximation for f from H_n (Toepler's Best Approximation Theorem, see [5] or [3] p. 45, [1] p. 102) given by

$$E_n(f):=E_{H_n}(f)=\inf_{c_1,c_2,\dots,c_k}\|f-\sum_{k=1}^{n}c_kw_k\|$$
$$=\|f-S_n(f)\|=\left(\|f\|^2-\sum_{k=1}^{n}a_k^2\right)^{\frac{1}{2}}=\left(\sum_{k=n+1}^{\infty}a_k^2\right)^{\frac{1}{2}},$$

where $a_k=\langle f,w_k\rangle$.

Due to specific properties of Hilbert spaces we have the following theorem.

Theorem 1. *For the element $A^\alpha f\in H(\alpha>0)$ the element $A^\alpha S_n(f)$ is the best approximation for f from H_n such that*

$$E_n(A^\alpha f)=\|A^\alpha f-A^\alpha S_n(f)\|$$
$$=\left((\|A^\alpha f\|)^2-\sum_{k=1}^{n}a_k^2\lambda_k^{2\alpha}\right)^{\frac{1}{2}}=\left(\sum_{k=n+1}^{\infty}a_k^2\lambda_k^{2\alpha}\right)^{\frac{1}{2}}.$$

The proof of the case is similar to the proof of Toepler's Theorem (see [5]) and will not be repeated here.

Corollary 2. *Let $w_1,w_2,\dots,w_n\in H_n$ and $f\in D(A^\alpha)$, Then*

$$\sum_{k=1}^{n}a_k^2\lambda_k^{2\alpha}\le\|A^\alpha f\|^2\,.$$

Best approximations in Hilbert spaces for operator A^α can be obtained easily, i.e. the best approximation $A^\alpha S_n$ of $A^\alpha f$ from H_n is given by the formula

$$A^\alpha S_n=\sum_{k=1}^{n}\lambda_k^\alpha\langle f,w_k\rangle w_k.$$

Note. The best L_2-approximation of functions is a part of the general theory of best approximation in Hilbert spaces.

It is known that if $f \in L_2$ has Fourier series representation such as $f = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx + b_k \sin kx$ then s_n is the best approximation to f and s'_n is the best approximation to $f' \in L_2$. But this is not true for Fourier Legendre series in L_2.

Theorem 3. *Let $f \in D(A^\alpha)$, then the inequality $E_n(f) \leq \frac{E_n(A^\alpha f)}{|\lambda_{n+1}|^\alpha}$ holds.*

Indeed, by Theorem 1,

$$E_n(A^\alpha f) = \left(\sum_{k=n+1}^{\infty} a_k^2 \lambda_k^{2\alpha} \right)^{\frac{1}{2}} \geq \left(\lambda_{n+1}^{2\alpha} \sum_{k=n+1}^{\infty} a_k^2 \right)^{\frac{1}{2}} = |\lambda_{n+1}^\alpha| E_n(f).$$

From this the theorem is proved.

Corollary 4. *If $f \in D(A^\alpha)$, then $E_n(f) \leq \frac{\|A^\alpha f\|}{|\lambda_{n+1}|^\alpha}$.*

This inequality follows from properties $E_n(f) \leq \|f\|$.

Theorem 5. *If $0 < \alpha < \beta$, then $\|A^\alpha f\| \leq \|f\|^{1-\frac{\alpha}{\beta}} \|A^\beta f\|^{\frac{\alpha}{\beta}}$.*

The proof is the same as Theorem 1 in [2] (Landau inequality for derivative).

From this and by Theorem 1, we have the following corollary.

Corollary 6. *If $0 < \alpha < \beta$, then $E_n(A^\alpha f) \leq [E_n(f)]^{1-\frac{\alpha}{\beta}} \left[E_n(A^\beta f)\right]^{\frac{\alpha}{\beta}}$.*

Using Theorem 3, and the above inequality we get

Corollary 7. *If $0 < \alpha < \beta$, then $E_n(A^\alpha f) \leq \frac{E_n(A^\beta f)}{|\lambda_{n+1}|^{\beta-\alpha}}$.*

Corollary 8. *Let the polynomial $P_n(f)$ be the best approximation for $f \in H$, then $\forall \alpha > 0$ $E_n(f) \leq \sum_{\nu=0}^{\infty} \frac{\|A^\alpha P_{2^{\nu+1}n}\|}{|\lambda_{2^v \nu n+1}|^\alpha}$.*

Proof. By Corollary 4 we have

$$\|P_{2n}(f) - P_n(P_{2n}(f))\| \leq \frac{\|A^\alpha P_{2n}(f)\|}{|\lambda_{n+1}|^\alpha}.$$

On the other hand

$$\begin{aligned}\|P_{2n}(f) - P_n(P_{2n}(f))\| &= \|\,(f - P_n(P_{2n}(f))) - (f - P_{2n}(f))\,\| \\ &\geq \|f - P_n(P_{2n}(f))\| - \|f - P_{2n}(f)\|.\end{aligned}$$

Since the polynomial $P_n(P_{2n}(f))$ is the best approximation of order n, then $\|f - P_{n-1}(P_{2n-1}(f))\| = E_n(f)$ because

$$0 \leq E_n(f) - E_{2n}(f) \leq \frac{\|A^\alpha P_{2n}(f)\|}{|\lambda_{n+1}|^\alpha}.$$

In this relation instead of n we write $2^\nu n$ and we consider $\sum_{\nu=0}^{\infty}\{E_{2^\nu n} - E_{2^{\nu+1}n}\} = E_n(f)$ then we have Corollary 8. □

Theorem 9. *Let $f \in D(A^\alpha)$, $\alpha > 0$, then*

$$E_n(f) \leq |\lambda_{n+1}|^{-\alpha} K_\alpha\left(A^\alpha f; \frac{1}{|\lambda_{n+1}|}\right),$$

where $K_\alpha(f;t) = \inf_{g\in D(A^\alpha)}\{\|f-g\| + t^\alpha\|A^\alpha g\|\}$

Proof. $E_n(f) \leq E_n(f-g) + E_n(g) \leq \|f-g\| + E_n(g)$. Let $g \in D(A^\alpha)$, then by Corollary 4

$$\begin{aligned}
&E_n(f) \leq \|f-g\| + \frac{E_n(A^\alpha g)}{|\lambda_{n+1}|^\alpha} \\
&\Rightarrow E_n(f) \leq \inf_{g\in D(A^\alpha)}\left\{\|f-g\| + \frac{E_n(A^\alpha g)}{|\lambda_{n+1}|^\alpha}\right\} = K_\alpha\left(f; \frac{1}{|\lambda_{n+1}|}\right) \\
&\Rightarrow \frac{E_n(A^\alpha f)}{|\lambda_{n+1}|^\alpha} \leq \frac{K_\alpha\left(A^\alpha f; \frac{1}{|\lambda_{n+1}|}\right)}{|\lambda_{n+1}|^\alpha} \\
&\Rightarrow E_n(f) \leq |\lambda_{n+1}|^{-\alpha} K_\alpha\left(A^\alpha f; \frac{1}{|\lambda_{n+1}|}\right).
\end{aligned}$$ □

Lemma 10. (Inequality of Bernstein's Type) *Let $P_n(f) = \sum_{k=1}^{n} c_k w_k$, then $\|A^\alpha P_n\| \leq \lambda_n^\alpha \|P_n\|$.*

Proof. Indeed $\|P_n(f)\| = \left(\sum_{k=1}^{n} c_k^2\right)^{\frac{1}{2}}$ and $A^\alpha P_n(f) = \sum_{k=1}^{n} c_k \lambda_k^\alpha w_k$. Then

$$\|A^\alpha P_n(f)\|^2 = \sum_{k=1}^{n} c_k^2 \lambda_k^{2\alpha} \leq \lambda_n^{2\alpha}\|P_n\|^2 \Rightarrow \|A^\alpha P_n(f)\| \leq \lambda_n^\alpha \|P_n\|.$$ □

Let the function $\psi_k(h)$, $0 < h \leq h_0$ $(k = 0, 1, 2, \ldots)$ satisfies the following condition:

$\exists$ number $m_1 > 0$, $m_2 \leq 0$ and constant $c_1 > 0$, $c_2 > 0$, such that

$$|1 - \psi_k(h)| \leq c_1(kh)^{m_1}. \quad (2)$$

all the k, and h; and also

$$|1 - \psi_k(h)| \geq c_2(kh)^{m_2} \qquad \text{as } 0 \leq kh \leq \mu_0, \quad (3)$$

from (2) and (3) we have $\psi_0(h) \equiv 1$, also $\lim_{h\to 0}\psi_k(h) = 1$.

Now we introduce in H a family of bounded linear operator $\{T_h\}$ defined as follows

$$T_h f \equiv f_h = \sum_{k=1}^{\infty} \psi_k(h)(f; w_k) w_k \,. \tag{4}$$

From this definition we have:
(1) $\|T_h f\| \le \|f\|$, $0 < h < h_0$.
(2) $\|f - T_h f\| \to 0$, $h \to 0$.
(3) $A(T_h f) = T_h(Af)$.
Now we define the modulus of continuity of fractional index: Let

$$\Delta_h^r := (E - T_h)^r = \sum_{k=0}^{\infty} (-1)^k \begin{pmatrix} r \\ k \end{pmatrix} (T_h)^k \quad (r > 0)$$

then

$$\Delta_h^r f = \sum_{k=0}^{\infty} (1 - \psi_k(h))^r (f; w_k) w_k = \sum_{k=1}^{\infty} (1 - \psi_k(h))^r (f; w_k) w_k.$$

Definition. $\omega_r(f;\tau) := \sup\{\|\Delta_h^r f\| : 0 < h \le \tau\}$, then $\omega_r(f,\tau)$ is called the modulus of continuity of fractional index.

Lemma 11. *Let the function $\psi_k(h)$ satisfies the condition (2), then*

$$\left\{ \sum_{\nu=n}^{2n-1} a_\nu^2(f) \right\}^{\frac{1}{2}} \le C(\alpha)\omega_r\left(f, \frac{\mu_0}{2\alpha\pi}\right) ,$$

where $\alpha \ge \max\left(1, \frac{\mu_0}{2h_0}\right)$, $C(\alpha) = \left(\frac{2\alpha}{\mu_0}\right)^{rm_2} \cdot \frac{c_2}{r}$.

Proof. From (1) and (4) we have

$$\Delta_h^r f = \sum_{k=1}^{\infty} [\psi_k(h) - 1]^r (f, \omega_k)\omega_k \,,$$

and by virtue of Parseval's equality

$$\|\Delta_h^r f\|^2 = \sum_{k=1}^{\infty} |1 - \psi_k(h)|^{2r} a_k^2(f) \,. \tag{5}$$

From this and (3)

$$\sum_{k=n}^{2n-2} a_k^2(f) \le (2\alpha\mu_0^{-1})^{2rm_2} \sum_{k=n}^{2n-1} \left(k\frac{\mu_0}{2\alpha n}\right)^{2rm_2} a_k^2(f)$$
$$\le (2\alpha\mu_0^{-1})^{2rm_2} C_2^{-2r} \sum_{k=n}^{2n-1} |1-\psi_n\left(\frac{\mu_0}{2\alpha n}\right)|^{2r} a_k^2(f)$$
$$\le C^2(\alpha)\|\Delta^r_{\frac{\mu}{2\alpha h}} f\|^2 \le C^2(\alpha)\omega_r(f,\frac{\mu_0}{2\alpha n})$$

Thus the proof is complete. □

Theorem 12. *Under the condition in Lemma 11 the inequality*

$$E_n^2(f) \le 2C(\alpha) \sum_{m=n+1}^{\infty} \frac{1}{m}\omega_r^2(f,\frac{\mu_0}{\alpha m})$$

is true.

Proof. Indeed,

$$\sum_{m=n}^{\infty} a_m^2(f) = \sum_{k=0}^{\infty} \sum_{m=2^k n}^{2^{k+1}n-1} a_m^2 \le C^2(\alpha) \sum \omega_r^2(f,\frac{\mu_0}{2^{k+1}n\alpha}),$$

from this and by virtue of Toepler's Theorem we have

$$E_n^2(f) \le C^2(\alpha)\sum_{k=1}^{\infty} \omega_k^r(f,\frac{\mu_0}{2^k n\alpha}). \tag{6}$$

Since, by virtue of properties $\omega_r(f,\tau)$, for any $k \ge 1$

$$\omega_r^2(f,\frac{\mu_0}{2^k n\alpha} \le 2 \sum_{m=2^{k-1}n+1}^{2^k n} \frac{1}{m}\omega_r^2(f,\frac{\mu_0}{m\alpha}).$$

Thus

$$\sum_{k=1}^{\infty} \omega_r^2(f,\frac{\mu_0}{2^k n\alpha}) \le 2 \sum_{m=n+1}^{\infty} \frac{1}{m}\omega_r^2(f,\frac{\mu_0}{m\alpha}). \tag{7}$$

From (6) and (7) we have Theorem 12. □

The following theorem is analogous to lemma of S.B. Stechkin in ([4], Lemma 1).

Theorem 13. *Let the function $\{\psi_k(n)\}$ is satisfies the condition (2) then for $r \geq \frac{1}{2m_1}$ and $h < \tau^{-1}$* $(0 < \tau \leq 1)$

$$\omega_r^2(f,\tau) \leq c_3 n^{-2rm_1} \sum_{k=1}^{n} k^{2rm_1-1} E_k(f),$$

where $c_3 = 2rm_1 c_1^{2r} + 2^{rm_1+2r}$.

Proof. From (5) and the condition $|\psi_k(n)| \leq 1$ and (1) we have for all n and $h > 0$

$$\|\Delta_h^r f\|^2 \leq c_1^{2r} h^{2rm_1} \sum_{k=1}^{n} k^{2rm_1} a_k^2(f) + 2^{2r} \sum_{k=n+1}^{\infty} c_k^2(f).$$

If $n \leq \frac{1}{\tau}$, then $2^{2rm_1} \leq n^{-2rm_1}$. From this and by Toepler's Theorem for $n \leq \frac{1}{\tau}$. We have

$$\omega_r^2(f,\tau) \leq c_1^{2r} n^{-2rm_1} \sum_{k=1}^{n} k^{rm_1} a_k^2(f) + 2^{2r} E_{n+1}(f)\,. \tag{8}$$

Now by Toepler's Theorem we get

$$\begin{aligned} \sum_{k=1}^{n} k^{2rm_1} a_k^2(f) &= \sum_{k=1}^{n} k^{2rm_1} [E_k^2(f) - E_{k+1}^2(f)] \\ &\leq \sum_{k=1}^{n} [k^{2rm_1} - (k-1)^{2rm_1}] E_k^2(f) \leq 2rm_1 \sum_{k=1}^{n} k^{2rm_1-1} E_k^2(f)\,. \end{aligned} \tag{9}$$

Also

$$E_{n+1}^2(f) \leq 2^{2rm_1} n^{-2rm_1} \sum_{k=1}^{n} k^{2rm_1-1} E_k^2(f)\,. \tag{10}$$

From (8), (9) and (10) we have the theorem. □

Finally, we note that under some additional conditions all the theorems can be proved in a normed space.

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