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A GENERALIZED HARDY INEQUALITY TO THE BAOUENDI-GRUSHIN TYPE OPERATOR

Haifeng Liu[1 §], Pengcheng Niu[2]
[1,2]Department of Applied Mathematics
Northwestern Polytechnical University
No. 127, Youyi Road, Xi'an, 710072, P.R. CHINA
[1]e-mail: hfeng_l@yahoo.com
[2]e-mail: Pengchengniu@yahoo.com

Abstract: By choosing the appropriate test functions and using an inequality on N-dimensional vector, a generalized Hardy type inequality with singular weight to the degenerate elliptic operators $\mathcal{L} = \sum_{j=1}^{N} X_j^2$ generated by the Baouendi-Grushin type vector fields $X_1 = \frac{\partial}{\partial x_1}, \cdots, X_k = \frac{\partial}{\partial x_d}, Y_1 = |x|^\alpha \frac{\partial}{\partial y_1}, \cdots, Y_k = |x|^\alpha \frac{\partial}{\partial y_k}$ ($\alpha > 0$), is proved, which generalizes the related results in previous literature.

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1. Introduction

The pre-eminent rule of the Hardy inequality in the study of linear and nonlinear partial differential equations is well known (see L. D'Ambrosio [1], L. D'Ambrosio and S. Lucente [2], B. Franchi, C. Gutierrez and R. Wheedem [4], J.D. Fernandes, J. Groisman and S.T. Melo [5], N. Garofalo [6], J.A. Goldstein and Q.S. Zhang [7] and the reference therein). I. Kombe [8] has recently established a sharp weighted Hardy type inequality on the Heisenberg group and extended the result to the Heisenberg type group and the Carnot group with an arbitrary step. In this paper, we will prove a generalized sharp Hardy in-

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§Correspondence author

equality which extends the results in previous literatures. It is known that the operators considered on the Carnot group including the Heisenberg group and the Heisenberg type group satisfy Hörmander's condition and are left-invariant, but in general, the Baouendi-Grushin type operators are not.

In order to state our theorem clearly, we need to recall some known facts about the Baouendi-Grushin type operators.

Let $\xi = (x_1, \cdots, x_d, y_1, \cdots, y_k) = (x, y) \in \mathbf{R}^d \times \mathbf{R}^k = \mathbf{R}^N$ with $d, k \geq 1$ and $N = d + k$. We denote by $|x|$ (resp. $|y|$) the Euclidean norm in $\mathbf{R}^d$ (resp. $\mathbf{R}^k$): $|x| = (\sum_{i=1}^d x_i^2)^{\frac{1}{2}}$ (resp. $|y| = (\sum_{j=1}^k y_j^2)^{\frac{1}{2}}$).

Let $\alpha \geq 0$. For $i = 1, \cdots, d$, and $j = 1, \cdots, k$ consider the Baouendi-Grushin type vector fields

$$Z_i = X_i = \frac{\partial}{\partial x_i},\ Z_{d+j} = Y_j = |x|^\alpha \frac{\partial}{\partial y_j}, \tag{1}$$

and the associate gradient as follows:

$$Z = (X_1, \cdots, X_d, Y_1, \cdots, Y_k) = (\nabla_x, |x|^\alpha \nabla_y).$$

Here ∇_x and ∇_y are the usual gradients on $x \in \mathbf{R}^d$ and $y \in \mathbf{R}^k$ respectively. When α is even, the vector fields satisfy the Hörmander's condition. But in general, it does not satisfy the Hörmander's condition and does not have left invariant property. The Baouendi-Grushin type operator $\mathcal{L}$ is the degenerate operator defined as

$$\mathcal{L}u = \sum_{i=1}^N Z_i^2 u = \sum_{i=1}^d X_i^2 u + \sum_{j=1}^k Y_j^2 u. \tag{2}$$

The Baouendi-Grushin type subelliptic p-Laplacian is the quasilinear degenerate operator defined as

$$\mathcal{L}_p u = \sum_{i=1}^N Z_i(|Zu|^{p-2} Z_i u). \tag{3}$$

When $\alpha = 0$, one gets the classical Laplacian and p-Laplacian on the Euclidean space $\mathbf{R}^N$.

Defining on $\mathbf{R}^N$ the dilation δ_λ as

$$\delta_\lambda(x, y) = (\lambda x, \lambda^{1+\alpha} y), \tag{4}$$

it is easy to check that Z_i are homogeneous of degree one with respect to the dilation: $Z_i(\delta_\lambda) = \lambda \delta_\lambda(Z_i)$ for $\forall 1 \leq i \leq N$.

Let $\|\xi\|_\alpha = \|(x,y)\|_\alpha$ be the distance from the origin defined on $\mathbf{R}^N$ as

$$\|\xi\|_\alpha = \|(x,y)\|_\alpha = \left[|x|^{2(1+\alpha)} + (1+\alpha)^2|y|^2\right]^{\frac{1}{2(1+\alpha)}}. \tag{5}$$

Clearly, $\|\xi\|_\alpha$ is homogeneous of degree one with respect to δ_λ. The main result of this paper is the following generalized Hardy inequality.

Theorem 1. *Let $\varphi \in C_0^\infty(\mathbf{R}^N \setminus \{0\})$ and $1 < p < +\infty, p \neq Q$. Then for any $\sigma, \tau, \gamma \in \mathbf{R}$ we have*

$$\int_{\mathbf{R}^N} |x|^\sigma |y|^\tau \|\xi\|_\alpha^\gamma |Z\varphi|^p \mathrm{d}\xi \geq C_{\sigma,\tau,\gamma} \int_{\mathbf{R}^N} |x|^\sigma |y|^\tau \|\xi\|_\alpha^{\gamma-p} |Z\|\xi\|_\alpha|^p \varphi^p \mathrm{d}\xi\,. \tag{6}$$

Moreover, the constant $C_{\sigma,\tau,\gamma} = \left|\frac{p-Q}{p}\right|^p - \frac{p-Q}{p}\left|\frac{p-Q}{p}\right|^{p-2}[\sigma + (1+\alpha)\tau + \gamma]$ is sharp. Here $Q = d + k(\alpha+1)$ is the homogeneous dimension associated with the Baouendi-Grushin type operators.

2. Preliminaries

For $|x| \neq 0$ and $|y| \neq 0$, a simple calculation shows that

$$\begin{aligned} &\frac{\partial}{\partial x_i}|x| = \frac{x_i}{|x|}, \quad \frac{\partial}{\partial x_i}\|\xi\|_\alpha = \|\xi\|_\alpha^{-2\alpha-1}|x|^{2\alpha}x_i, \\ &\frac{\partial}{\partial y_j}|y| = \frac{y_j}{|y|}, \quad |x|^\alpha \frac{\partial}{\partial y_j}\|\xi\|_\alpha = (1+\alpha)\|\xi\|_\alpha^{-2\alpha-1}|x|^\alpha y_j. \end{aligned}$$

Thus, one can see that

$$\begin{aligned} &Z|x| = |x|^{-1}(x,0), \quad Z|y| = |x|^\alpha |y|^{-1}(0,y), \\ &Z\|\xi\|_\alpha = \|\xi\|_\alpha^{-2\alpha-1}|x|^\alpha(|x|^\alpha x, (1+\alpha)y). \end{aligned}$$

By an easy computation one gets the following result.

Lemma 2. *Let $Z_i, 1 \leq N = d+k$ be the Baouendi-Grushin type vector fields defined as (1), $\|\xi\|_\alpha$ is the associated distance from origin defined as (5), then*

$$|Z\|\xi\|_\alpha|^2 = \frac{|x|^{2\alpha}}{\|\xi\|_\alpha^{2\alpha}}. \tag{7}$$

Moreover, $|Z|x|| = 1$, $|Z|y|| = |x|^\alpha$ and $Z|x| \cdot Z|y| = 0$, $Z|x| \cdot Z\|\xi\|_\alpha = |x|^{2\alpha+1}\|\xi\|_\alpha^{-2\alpha-1}$, $Z|y| \cdot Z\|\xi\|_\alpha = (1+\alpha)|x|^{2\alpha}|y|\|\xi\|_\alpha^{-2\alpha-1}$.

A slight complicated computation gives

$$\mathcal{L}(|x|) = \Delta_x|x| = \frac{d-1}{|x|},$$
$$\mathcal{L}(|y|) = |x|^{2\alpha}\Delta_y|y| = |x|^{2\alpha}\frac{k-1}{|y|}, \tag{8}$$

where $\mathcal{L}$ is defined by (2). Moreover,

$$\begin{aligned}\sum_{i=1}^{d} X_i^2\|\xi\|_\alpha &= -(2\alpha+1)|x|^{4\alpha+2}\|\xi\|_\alpha^{-4\alpha-3} \\ &\quad +2\alpha|x|^{2\alpha}\|\xi\|_\alpha^{-2\alpha-1} + d|x|^{2\alpha}\|\xi\|_\alpha^{-2\alpha-1}, \\ \sum_{j=1}^{k} Y_j^2\|\xi\|_\alpha &= -(2\alpha+1)(1+\alpha)^2|x|^{2\alpha}|y|^2\|\xi\|_\alpha^{-4\alpha-3} \\ &\quad +k(\alpha+1)|x|^{2\alpha}\|\xi\|_\alpha^{-2\alpha-1}.\end{aligned}$$

Thus, we have the following lemma.

Lemma 3. *Let $\mathcal{L}$ be the degenerate elliptic operator defined as (2), $\|\xi\|_\alpha$ is the associated distance from origin defined as (5), then*

$$\mathcal{L}\|\xi\|_\alpha = \frac{Q-1}{\|\xi\|_\alpha}\frac{|x|^{2\alpha}}{\|\xi\|_\alpha^{2\alpha}} = \frac{Q-1}{\|\xi\|_\alpha}|Z\|\xi\|_\alpha|^2, \tag{9}$$

where $Q = d + k(\alpha+1)$ is the homogeneous dimension.

To prove our main result we need a result concerning fundamental solution of Baouendi-Grushin type subelliptic p-Laplacian obtained by H. Zhang [12] and the following elementary inequality.

Lemma 4. *For any $a, b \in \mathbf{R}^N$ and $p > 1$, the following inequality holds*

$$|a+b|^p - |a|^p \geq p|a|^{p-2}a\cdot b.$$

Proof. If $a = 0$ or $b = 0$, the inequality is trivial. We assume $a \not\equiv 0$, $b \not\equiv 0$ and prove the inequality in three cases.

1. When $1 < p < 2$, G. Barbatis, S. Filippas and A. Terikas [3] (see also P. Lindqvist [9]) proved that

$$|a+b|^p - |a|^p \geq c(p)\frac{|b|^2}{(|a|^2+|b|^2)^{2-p}} + p|a|^{p-2}a\cdot b\,,$$

where $c(p) > 0$.

2. When $p = 2$, it is easy to see that

$$|a+b|^2 =< a+b, a+b >= |a|^2 + 2a \cdot b + |b|^2 \geq |a|^2 + 2a \cdot b.$$

3. When $p > 2$, it is elementary to show that $(1+t)^{\frac{p}{2}} \geq 1 + \frac{p}{2}t$ for any $-1 \leq t \in \mathbf{R}$. Noting that $|2a \cdot b| \leq |a|^2 + |b|^2$ we have

$$\begin{aligned} |a+b|^p &= (|a+b|^2)^{\frac{p}{2}} = (|a|^2 + |b|^2 + 2a \cdot b)^{\frac{p}{2}} \\ &= (|a|^2 + |b|^2)^{\frac{p}{2}} \left(1 + \frac{2a \cdot b}{|a|^2 + |b|^2}\right)^{\frac{p}{2}} \geq (|a|^2 + |b|^2)^{\frac{p}{2}} \left(1 + \frac{p}{2}\frac{2a \cdot b}{|a|^2 + |b|^2}\right) \\ &= (|a|^2 + |b|^2)^{\frac{p}{2}} + p(|a|^2 + |b|^2)^{\frac{p}{2}-1} a \cdot b \geq |a|^p + p|a|^{p-2} a \cdot b. \quad \square \end{aligned}$$

The following lemma is one of the main results obtained by H. Zhang [12].

Lemma 5. *When $p \neq Q$, there exists a positive constant C such that $\Gamma = C^{-1}\|\xi\|_\alpha^{\frac{p-Q}{p-1}}$ is the fundamental solution of $\mathcal{L}_p$ at the origin, i.e., the identity*

$$\mathcal{L}_p(\|\xi\|_\alpha^{\frac{p-Q}{p-1}}) = C\delta$$

holds in the weak sense.

Now we are in the place to prove Theorem 1. We assume all the integrations appeared in this paper making sense.

3. Proof of Theorem 1

Proof. Let $\varphi = \|\xi\|_\alpha^\beta \psi$ with $\beta \in \mathbf{R} \setminus \{0\}$ and $\psi \in C_0^\infty(\mathbf{R}^N \setminus \{0\})$. A direct calculation shows that

$$Z\varphi = \beta\|\xi\|_\alpha^{\beta-1}\psi Z\|\xi\|_\alpha + \|\xi\|_\alpha^\beta Z\psi. \tag{10}$$

In view of Lemma 4 one gets for $1 < p < +\infty$,

$$\begin{aligned} |Z\varphi|^p &\geq |\beta|^p |\|\xi\|_\alpha^{\beta-1}\psi|^p |Z\|\xi\|_\alpha|^p \\ &\quad + p\big|\beta\|\xi\|_\alpha^{\beta-1}\psi\big|^{p-2} |Z\|\xi\|_\alpha|^{p-2} \beta\|\xi\|_\alpha^{\beta-1}\psi Z\|\xi\|_\alpha \cdot \|\xi\|_\alpha^\beta Z\psi \\ &= |\beta|^p |\|\xi\|_\alpha^{\beta-1}\psi|^p |Z\|\xi\|_\alpha|^p + p|\beta|^{p-2}\beta\|\xi\|_\alpha^\beta |\psi|^{p-1} |Z\|\xi\|_\alpha|^{p-2} Z\|\xi\|_\alpha \cdot Z|\psi|. \end{aligned}$$

Multiplying both sides of the equation by $|x|^\sigma|y|^\tau\|\xi\|_\alpha^\gamma$ and integrating over $\mathbf{R}^N$ gives

$$\begin{aligned}&\int_{\mathbf{R}^N}|x|^\sigma|y|^\tau\|\xi\|_\alpha^\gamma|Z\varphi|^p\mathrm{d}\xi\\&\geq|\beta|^p\int_{\mathbf{R}^N}|x|^\sigma|y|^\tau\|\xi\|_\alpha^\gamma\|\xi\|_\alpha^{p(\beta-1)}|\psi|^p|Z\|\xi\|_\alpha|^p\mathrm{d}\xi+I,\end{aligned}\tag{11}$$

where $I=\beta|\beta|^{p-2}\int_{\mathbf{R}^N}|x|^\sigma|y|^\tau\|\xi\|_\alpha^\gamma\|\xi\|_\alpha^{\beta p-p+1}|Z\|\xi\|_\alpha|^{p-2}Z\|\xi\|_\alpha\cdot Z|\psi|^p\mathrm{d}\xi$. Applying integration by parts to equation I, we get

$$\begin{aligned}I&=-\beta|\beta|^{p-2}\int_{\mathbf{R}^N}|\psi|^pZ(|x|^\sigma|y|^\tau\|\xi\|_\alpha^\gamma\|\xi\|_\alpha^{\beta p-p+1}|Z\|\xi\|_\alpha|^{p-2}Z\|\xi\|_\alpha)\mathrm{d}\xi\\&=-\beta|\beta|^{p-2}\\&\quad\times\Big\{\sigma\int_{\mathbf{R}^N}|x|^{\sigma-1}|y|^\tau\|\xi\|_\alpha^\gamma\|\xi\|_\alpha^{\beta p-p+1}|Z\|\xi\|_\alpha|^{p-2}(Z\|\xi\|_\alpha\cdot Z|x|)|\psi|^p\,\mathrm{d}\xi\\&\quad+\tau\int_{\mathbf{R}^N}|x|^\sigma|y|^{\tau-1}\|\xi\|_\alpha^\gamma\|\xi\|_\alpha^{\beta p-p+1}|Z\|\xi\|_\alpha|^{p-2}(Z\|\xi\|_\alpha\cdot Z|y|)|\psi|^p\mathrm{d}\xi\\&\quad+\gamma\int_{\mathbf{R}^N}|x|^\sigma|y|^\tau\|\xi\|_\alpha^{\gamma-1}\|\xi\|_\alpha^{\beta p-p+1}|Z\|\xi\|_\alpha|^{p-2}\big|Z\|\xi\|_\alpha\big|^2|\psi|^p\mathrm{d}\xi\\&\quad+\int_{\mathbf{R}^N}|x|^\sigma|y|^\tau\|\xi\|_\alpha^\gamma Z(\|\xi\|_\alpha^{\beta p-p+1}|Z\|\xi\|_\alpha|^{p-2}Z\|\xi\|_\alpha)|\psi|^p\mathrm{d}\xi\Big\}.\end{aligned}$$

Keep in mind $\varphi=\|\xi\|_\alpha^\beta\psi$ and Lemma 2. Equation I becomes

$$\begin{aligned}I&=-\beta|\beta|^{p-2}\Big\{[\sigma+(1+\alpha)\tau+\gamma]\int_{\mathbf{R}^N}|x|^\sigma|y|^\tau\|\xi\|_\alpha^\gamma\|\xi\|_\alpha^{\beta p-p}|Z\|\xi\|_\alpha|^p|\psi|^p\mathrm{d}\xi\\&\quad+\int_{\mathbf{R}^N}|x|^\sigma|y|^\tau\|\xi\|_\alpha^\gamma Z(\|\xi\|_\alpha^{\beta p-p+1}|Z\|\xi\|_\alpha|^{p-2}Z\|\xi\|_\alpha)|\psi|^p\mathrm{d}\xi\Big\}\\&=-\beta|\beta|^{p-2}\Big\{[\sigma+(1+\alpha)\tau+\gamma]\int_{\mathbf{R}^N}|x|^\sigma|y|^\tau\|\xi\|_\alpha^{\gamma-p}|Z\|\xi\|_\alpha|^p|\varphi|^p\mathrm{d}\xi\\&\quad+\int_{\mathbf{R}^N}|x|^\sigma|y|^\tau\|\xi\|_\alpha^\gamma Z(\|\xi\|_\alpha^{\beta p-p+1}|Z\|\xi\|_\alpha|^{p-2}Z\|\xi\|_\alpha)|\psi|^p\mathrm{d}\xi\Big\}.\end{aligned}$$

Now we choose $\beta=\frac{p-Q}{p}$. The second term in the right hand sides of the above equation becomes

$$\int_{\mathbf{R}^N}|x|^\sigma|y|^\tau\|\xi\|_\alpha^\gamma Z(\|\xi\|_\alpha^{p-Q-p+1}|Z\|\xi\|_\alpha|^{p-2}Z\|\xi\|_\alpha)|\psi|^p\mathrm{d}\xi$$

$$
\begin{aligned}
&= \int_{\mathbf{R}^N} |x|^\sigma |y|^\tau \|\xi\|_\alpha^\gamma Z((\|\xi\|_\alpha^{\frac{p-Q}{p-1}-1})^{p-1}|Z\|\xi\|_\alpha|^{p-2} Z\|\xi\|_\alpha)|\psi|^p \mathrm{d}\xi \\
&= \left|\frac{p-Q}{p-1}\right|^{-(p-1)} \\
&\quad \times \int_{\mathbf{R}^N} |x|^\sigma |y|^\tau \|\xi\|_\alpha^\gamma Z(|\|\xi\|_\alpha^{\frac{p-Q}{p-1}-1} Z\|\xi\|_\alpha|^{p-2}(\|\xi\|_\alpha^{\frac{p-Q}{p-1}-1} Z\|\xi\|_\alpha))|\psi|^p \mathrm{d}\xi \\
&= \left|\frac{p-Q}{p-1}\right|^{-(p-1)} \int_{\mathbf{R}^N} |x|^\sigma |y|^\tau \|\xi\|_\alpha^\gamma \mathcal{L}_p(\|\xi\|_\alpha^{\frac{p-Q}{p-1}})|\psi|^p \mathrm{d}\xi \equiv 0.
\end{aligned}
$$

Here we have used $\psi \in C_0^\infty(\mathbf{R}^N \setminus \{0\})$ and Lemma 5. Thus, we get

$$
I = -\frac{p-Q}{p}\left|\frac{p-Q}{p}\right|^{p-2}[\sigma + (1+\alpha)\tau + \gamma] \times \int_{\mathbf{R}^N} |x|^\sigma |y|^\tau \|\xi\|_\alpha^{\gamma-p} |Z\|\xi\|_\alpha|^p |\varphi|^p \mathrm{d}\xi\,.
$$

Noting $\beta = \frac{p-Q}{p}$, Equation (11) gives us

$$
\int_{\mathbf{R}^N} |x|^\sigma |y|^\tau \|\xi\|_\alpha^\gamma |Z\varphi|^p \mathrm{d}\xi \geq \left\{ \left|\frac{p-Q}{p}\right|^p - \frac{p-Q}{p}\left|\frac{p-Q}{p}\right|^{p-2} \times [\sigma + (1+\alpha)\tau + \gamma] \right\} \int_{\mathbf{R}^N} |x|^\sigma |y|^\tau \|\xi\|_\alpha^{\gamma-p} |Z\|\xi\|_\alpha|^p |\psi|^p \mathrm{d}\xi. \quad \square
$$

4. Corollaries

By specializing partial parameters in Theorem 1 we can get some interesting corollaries.

When $\sigma = \tau = 0$, one gets the following result.

Theorem 6. *Let $\varphi \in C_0^\infty(\mathbf{R}^N \setminus \{0\})$ and $1 < p < +\infty, p \neq Q$. Then for any $\gamma \in \mathbf{R}$, we have*

$$
\int_{\mathbf{R}^N} \|\xi\|_\alpha^\gamma |Z\varphi|^p \mathrm{d}\xi \geq C \int_{\mathbf{R}^N} \|\xi\|_\alpha^\gamma |Z\|\xi\|_\alpha|^p \frac{\varphi^p}{\|\xi\|_\alpha^p} \mathrm{d}\xi. \tag{12}
$$

Here the constant $C = \left|\frac{p-Q}{p}\right|^p - \gamma\frac{p-Q}{p}\left|\frac{p-Q}{p}\right|^{p-2}$.

Furthermore, if $\gamma = 0$, we have the next result.

Corollary 7. *Let $\varphi \in C_0^\infty(\mathbf{R}^N \setminus \{0\})$ and $1 < p < +\infty, p \neq Q$. Then we have*

$$\int_{\mathbf{R}^N} |Z\varphi|^p \mathrm{d}\xi \geq C \int_{\mathbf{R}^N} |Z\|\xi\|_\alpha|^p \frac{\varphi^p}{\|\xi\|_\alpha^p} \mathrm{d}\xi. \tag{13}$$

Moreover, the constant $C = \left|\frac{p-Q}{p}\right|^p$ is sharp (see L. D'Ambrosio [1], Theorem 3.1).

When $p = 2$, by reiterating the program in proof of Theorem 1 with a small modification, we get the following theorem.

Theorem 8. *Let $\varphi \in C_0^\infty(\mathbf{R}^N \setminus \{0\})$. For any $\sigma, \tau, \gamma \in \mathbf{R}$, we have a generalized Hardy inequality of the form*

$$\int_{\mathbf{R}^N} |x|^\sigma |y|^\tau \|\xi\|_\alpha^\gamma |Z\varphi|^2 \mathrm{d}\xi \geq C(\sigma, \tau, \gamma) \int_{\mathbf{R}^N} |x|^\sigma |y|^\tau \|\xi\|_\alpha^{\gamma-2} \frac{|x|^{2\alpha}}{\|\xi\|_\alpha^{2\alpha}} \varphi^2 \mathrm{d}\xi. \tag{14}$$

Here the constant $C(\sigma, \tau, \gamma) = \frac{1}{4}[(\gamma + Q - 2) + (\sigma + (1 + \alpha)\tau)]^2$.

We are particularly interested in the case $\sigma = \tau = 0$. By taking directly $\sigma = \tau = 0$ in Theorem 8 we obtain next corollary.

Corollary 9. *Let $\varphi \in C_0^\infty(\mathbf{R}^N \setminus \{0\})$, $\gamma \in \mathbf{R}$. Then we have*

$$\int_{\mathbf{R}^N} \|\xi\|_\alpha^\gamma |Z\varphi|^2 \mathrm{d}\xi \geq C_{0,0} \int_{\mathbf{R}^N} \|\xi\|_\alpha^{\gamma-2} \frac{|x|^{2\alpha}}{\|\xi\|_\alpha^{2\alpha}} \varphi^2 \mathrm{d}\xi. \tag{15}$$

Moreover, the constant $C_{0,0} = \frac{1}{4}(Q + \gamma - 2)^2$ is sharp.

Furthermore, if $\gamma = 0$, we get the following corollary.

Corollary 10. *Suppose that $\varphi \in C_0^\infty(\mathbf{R}^N \setminus \{0\})$. Then*

$$\int_{\mathbf{R}^N} |Z\varphi|^2 \mathrm{d}\xi \geq \left(\frac{Q-2}{2}\right)^2 \int_{\mathbf{R}^N} \frac{|Z\|\xi\|_\alpha|^2}{\|\xi\|_\alpha^2} \varphi^2 \mathrm{d}\xi. \tag{16}$$

Moreover, the constant $\frac{1}{4}(Q-2)^2$ is sharp.

Remark 11. Equation (16) including the best constant is proved by P. Niu, Y. Chen and Y. Han [11] (Theorem 4.1 there, also, see equation (3.5) in L. D'Ambrosio [1]), which is a generalization of Theorem 8.3.4 in H. Zhang, P. Niu [13].

Remark 12. In the case $\alpha = 0$, the above inequality gives the classical Hardy inequality in $(d + k)$-dimensional Euclidean space.

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