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APPROXIMATION OF ABSOLUTELY CONTINUOUS INVARIANT MEASURES FOR MARKOV SWITCHING POSITION DEPENDENT RANDOM MAPS

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Abstract: A Markov switching position dependent random map is a random map of a finite number of measurable transformations where the probability of switching from one transformation to another is controlled by a position dependent irreducible stochastic matrix W. Existence of absolutely continuous invariant measures (acim) for a Markov switching position dependent random map was proved in [1] using spectral properties of Frobenius-Perron operator and geometric conditions respectively. In this note, we present a bounded variation proof for the existence of absolutely continuous invariant measures and we describe a method of approximating the invariant measures for Markov switching position dependent random maps. The method is known as Ulam's method.

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1. Introduction

Invariant measure of a dynamical system $(X, \mathcal{B}, \lambda, \tau)$ describes asymptotic behavior of orbits $\{x, \tau(x), \tau^2(x), \ldots\}$ of the dynamical system $(X, \mathcal{B}, \lambda, \tau)$ for almost every $x \in X$. The Birkhoff's Ergodic Theorem [4] establishes the dynamical importance of invariant measure. In [10] it is proved that computer simulations of orbits of the system reveal only invariant measures which are absolutely continuous with respect to Lebesgue measure. It is well known that fixed point of Frobenius-Perron operator of a dynamical system is the invariant density of the system. The Frobenius-Perron equation is a functional equation and it is very difficult to solve this equation except some simple cases. So it is important to approximate invariant measures of a dynamical system. This can be done by approximating the fixed point of Frobenius-Perron operator by fixed point of some matrix operators. Approximation of invariant measures was suggested by Ulam [24]. For a single transformation [15], Li [14] first proved convergence of Ulam's approximation.

Ulam and von Neuman [25] suggested the study of more general dynamical systems, namely random dynamical systems. Random dynamical systems provide a useful framework for modeling and analyzing various physical, social, and economic phenomena [5, 22]. A random dynamical system of special interest is a random map $T = \{\tau_1, \tau_2, \ldots, \tau_K;\ p_1, p_2, \ldots, p_K\}$ where the process switches from one map to another according to fixed probabilities [19] or, more generally, position dependent probabilities [1, 2, 8, 9, 12, 13]. The existence and properties of invariant measures for random maps reflect their long time behavior and play an important role in understanding their chaotic nature. Such dynamical systems have application in the study of fractals [3], in modeling interference effects in quantum mechanics [5], in computing metric entropy [23], and in forecasting the financial markets [1]. In [7] Froyland extended Ulam's method for a single transformation to a method for random map with constant probabilities [19]. Góra and Boyarsky in [8] proved the convergence of Ulam's approximation for position dependent random maps.

A Markov switching position dependent random map T associated with transformations $\tau_1, \tau_2, \ldots, \tau_K$, probabilities $p_1, p_2, \ldots, p_K$ and stochastic matrix W is a more general discrete-time dynamical system in which at each step one of K transformations τ_k is randomly selected and applied to the previous iteration of the process. The selection is controlled by a K by K position dependent stochastic irreducible matrix W such that the probability of switching from transformation τ_k to transformation τ_l is the position dependent probability $W_{k,l}$, the (k, l)-th entry of W. Froyland [7] considered the constant stochastic matrix W and he proved the convergence of Ulam's approximation of invariant measure. The existence of absolutely continuous invariant measure for Markov switching position dependent random maps was proved by Bahsoun, Góra and Boyarsky [1] using spectral properties of Frobenious-Perron operator. In this note, we present a bounded variation proof for the existence of absolutely continuous invariant measures and we describe a method of approximating the invariant measures for Markov switching position dependent random maps. We generalize Froyland's result for Markov switching random maps with constant stochastic irreducible matrix W to Markov switching random maps with position dependent stochastic irreducible matrix W.

In Section 2 we present the notation and summarize results we shall need in the sequel. In Section 3 we present a bounded variation proof for the existence of absolutely continuous invariant measures. In Section 4 we prove the convergence of Ulam's approximation for Markov switching position dependent random maps. In Section 4 we present error bounds. A numerical example is presented in Section 5.

2. Preliminaries

In [1] Bahsoun, Góra and Boyarsky defined Markov switching position dependent random maps and corresponding Frobenius-Perron operator. We are interested in approximating acim for Markov switching position dependent random maps. Frobenius-Perron Operator plays an important role for existence and approximation of acim for Markov switching position dependent random maps. For the convenience of the reader, we present a brief discussion of Markov switching position dependent random maps. In this section we closely follow [1].

2.1. Markov Switching Position Dependent Random Maps

Let $X = ([a,b], \mathcal{B}, \lambda)$ be a measure space where λ is Lebesgue measure on $[a,b]$. Let $\tau_k : X \to X, k = 1, 2, \ldots, K$, be piecewise monotonic non-singular transformations on a common partition $\mathcal{P}$ of $[a,b] : P = \{J_1, J_2, \ldots, J_q\}$ and $\tau_{k,i} = \tau_k | J_i,\ i = 1, 2, \ldots, q,\ k = 1, 2, \ldots, K$. A Markov switching position dependent random map T is a Markov process which is defined as follows: at time $n = 1$, we select a transformation τ_k randomly according to initial probabilities $p_k, k = 1, 2, \ldots, K$. Then we define a $K \times K$ position dependent stochastic irreducible matrix W such that $W_{k,l}$, the (k,l)-th entry of W, is the probability of switching from τ_k to τ_l. That is after choosing $\tau_{k_{N-1}}$, choosing the transformation τ_{k_N} at time $N, N = 2, 3, \dots$, depends only on the transformation applied at the previous time step and position at the previous time step. Therefore, if we choose τ_{k_1} at time $n = 1$, where we are at position x, the Markov process at time N is given by

$$T^N(x) = \tau_{k_N} \circ \tau_{k_{N-1}} \circ \dots \circ \tau_{k_1}(x) \tag{2.1}$$

with probability

$$W_{k_{N-1},k_N}(\tau_{k_{N-1}} \circ \dots \circ \tau_{k_1}(x)) \cdot W_{k_{N-2},k_{N-1}}(\tau_{k_{N-2}} \circ \dots \circ \tau_{k_1}(x)) \cdots W_{k_1,k_2}(x). \tag{2.2}$$

We assume that the probabilities $W_{k,l}(x)$ are piecewise continuous on the same partition $\mathcal{P}$. Let $\Omega = \{1, 2, \dots, K\}$. The transition function of the Markov process on $\Omega \times X$ is defined as follows:

$$\mathbb{P}((k, x), \{l\} \times A) = W_{k,l}(x)\chi_A(\tau_k(x)),$$

where A is any measurable set and χ_A denotes the characteristic function of the set A. The random map T is the projection of the process on the space X. The transition function $\mathbb{P}$ induces an operator $\mathbb{P}_*$ on measures $\bar{\mu}$ on $\Omega \times X$ as follows:

$$\begin{aligned} \mathbb{P}_*\bar{\mu}(\{l\} \times A) &= \int_{\Omega \times X} \mathbb{P}((k, x), \{l\} \times A) d\bar{\mu}(k, x) \\ &= \int_{\Omega \times X} W_{k,l}(x)\chi_A(\tau_k(x)) d\bar{\mu}(k, x). \end{aligned}$$

Let ν be a measure on $\Omega \times X$ such that $\nu(\{s\} \times A) = \lambda(A)$. If $\bar{\mu}$ has density f with respect to ν, $f(s, x) = \sum_{k=1}^K f_k(x)\chi_{\{k\}\times X}(s, x)$, where $\sum_{k=1}^K \int_X f_k(x) = 1$, then $\mathbb{P}_*\bar{\mu}$ also has a density which we denote by $P_T f$. By a change of variables, we obtain

$$\begin{aligned} \int_{l \times A} P_T f(s, x) d\nu(s, x) &= \sum_{k=1}^K \int_X W_{k,l}(x)\chi_A(\tau_k(x)) f_k(x) d\lambda(x) \\ &= \sum_{k=1}^K \int_{\tau_k^{-1}(A)} W_{k,l}(x) f_k(x) d\lambda(x). \end{aligned} \tag{2.3}$$

Using the definition of P_{τ_k}, the Frobenius-Perron operator associated with transformation τ_k [4] and (2.3), we obtain

$$\int_A \hat{f}_l(x) d\lambda(x) = \sum_{k=1}^K \int_A P_{\tau_k}(W_{k,l} f_k)(x) d\lambda(x), \tag{2.4}$$

where $P_T f(s,x) = \sum_{l=1}^{K} \hat{f}_l \chi_{\{l\}\times X}(s,x)$. Since (2.4) is true for any $A \in \mathcal{B}$, we obtain an a.e. equality

$$\hat{f}_l(x) = \sum_{k=1}^{K} P_{\tau_k}(W_{k,l} f_k)(x). \tag{2.5}$$

Thus, the density $f^*(s,x) = \sum_{l=1}^{K} f_l^*(x)\chi_{\{l\}\times X}(s,x)$ is $T-$invariant if

$$f_l^*(x) = \sum_{k=1}^{K} P_{\tau_k}(W_{k,l} f_k^*)(x) \tag{2.6}$$

for $l = 1, 2, \ldots, K$. If we denote

$$w_l = \int_X f_l^*(x) d\lambda(x), \quad l = 1, 2, \ldots, K,$$

then integrating (2.6) with respect to λ, we obtain

$$w_l = \sum_{k=1}^{K} w_k \int_X W_{k,l}(x) \frac{f_l^*(x)}{\int_X f_k^*(x) d\lambda(x)} d\lambda(x). \tag{2.7}$$

Note that, in the special case when $W_{k,l}$'s are constant, (2.7) reduces to $w_l = \sum_{k=1}^{K} w_k W_{k,l}$, i.e., to the case when $(w_1, w_2, \ldots, w_K)$ is a left invariant eigenvector of the matrix W.

3. Existence of Acim Using Bounded Variation

In this section we present a bounded variation proof for the existence of acim for Markov switching position dependent random maps.

Denote by $V(\cdot)$ the standard one dimensional variation of a function, and by $BV([a,b])$ the space of functions of bounded variations on $[a,b]$ equipped with the norm $\| \, . \, \|_{BV} = V(\cdot) + \| \cdot \|_1$, where $\| \cdot \|_1$ denotes the norm on $L^1([a,b], \mathcal{B}, \lambda)$. Let $\widehat{BV} = \prod_{k=1}^{K} BV$ denote the $K-$fold product of the space BV of functions of bounded variation and we define a norm on $\widehat{BV}$ as $\| (f_1, f_2, \ldots, f_K) \|_{\widehat{BV}} = \sum_{k=1}^{K} \| f_k \|_{BV}$. We also define L^1 norm on $\widehat{BV}$: $\| f_1, f_2, \ldots, f_K \|_1 = \sum_{k=1}^{K} \| f_k \|_1$. We define an operator $\widehat{P_T} : \widehat{BV} \to \widehat{BV}$ by

$$\widehat{P_T}(f_1, f_2, \ldots, f_K)$$

$$= \left(\sum_{k=1}^{K} P_{\tau_k}(W_{k,1}f_k), \sum_{k=1}^{K} P_{\tau_k}(W_{k,2}f_k), \ldots, \sum_{k=1}^{K} P_{\tau_k}(W_{k,K}f_k) \right). \quad (3.1)$$

If $(f_1^*, f_2^*, \ldots, f_K^*)$ is fixed point of $\widehat{P_T}$, we call

$$f^* = \sum_{k=1}^{K} f_k^*$$

an invariant density of the Markov switching position dependent random map T. For more details about $\widehat{P_T}$ see [1].

Lemma 3.1. *Let τ_k be piecewise C^2 on $I = [0,1]$ and $W_{k,l}$ be piecewise of class C^1, for $k = 1, 2, \ldots, K$ and $l = 1, 2, \ldots, K$,. Let*

$$\alpha_l = \max_k (\sup_x \frac{2 \cdot W_{k,l}(x)}{|\tau_k'(x)|}), \ l = 1, 2, \ldots, K.$$

Then,

$$V_I(\widehat{P}_T f)_l \le \alpha_l \sum_{k=1}^{K} V_I f_k + B_l \sum_{k=1}^{K} \|f_k\|_1, \quad (3.2)$$

where

$$h_k(x) = \frac{W_{k,l}(x)}{|\tau_k'(x)|}, \delta = \min_i \lambda(J_i)$$

and

$$B_l = \frac{2}{\delta}(\max_k \sup_x h_k(x)) + (\max_k \sup_x |h_k'(x)|).$$

Proof. Since f_k is Riemann integrable, for arbitrary $\epsilon > 0$, we can find a number θ such that for any $J_i \in \mathcal{P}$ and any partition finer than : $J_i = \cup_{p=1}^{L_i}[s_{p-1}, s_p]$ with $|s_p - s_{p-1}| < \theta$, we have

$$\sum_{p=1}^{L_i} |f_k(s_{p-1})||s_p - s_{p-1}| \le \int_{J_i} |f_k| d\lambda + \epsilon. \quad (3.3)$$

Let $0 = x_0 < x_1 < \ldots \le x_r = 1$ be such a fine partition of $I = [0,1]$. Define $\phi_{k,i} = \tau_{k,i}^{-1}$. Let $h_k(x) = \frac{W_{k,l}(x)}{|\tau_k'(x)|}$. We have,

$$V_I(\widehat{P}_T f)_l \le \sum_{k=1}^{K} V_I P_{\tau_k}(W_{k,l} f_k). \quad (3.4)$$

We estimate $V_I P_{\tau_k}(W_{k,l}f_k)$:

$$\sum_{j=1}^{r}|P_{\tau_k}(W_{k,l}f_k)(x_j) - P_{\tau_k}(W_{k,l}f_k)(x_{j-1})|$$

$$= \sum_{j=1}^{r}|(\sum_{i=1}^{q} h_k(\phi_{k,i}(x_j))f_k(\phi_{k,i}(x_j))\chi_{\tau_k(J_i)}(x_j)$$

$$- \sum_{i=1}^{q} h_k(\phi_{k,i}(x_{j-1}))f_k(\phi_{k,i}(x_{j-1}))\chi_{\tau_k(J_i)}(x_{j-1}))|$$

$$\leq \sum_{j=1}^{r}\sum_{i=1}^{q}|h_k(\phi_{k,i}(x_j))f_k(\phi_{k,i}(x_j))\chi_{\tau_k(J_i)}(x_j) -$$

$$h_k(\phi_{k,i}(x_{j-1}))f_k(\phi_{k,i}(x_{j-1}))\chi_{\tau_k(J_i)}(x_{j-1}))|.$$

We divide the sum on the right hand side into three parts:

(I) the summands for which $\chi_{\tau_k(J_i)}(x_j) = \chi_{\tau_k(J_i)}(x_{j-1}) = 1$;

(II) the summands for which $\chi_{\tau_k(J_i)}(x_j) = 1$ and $\chi_{\tau_k(J_i)}(x_{j-1}) = 0$;

(III) the summands for which $\chi_{\tau_k(J_i)}(x_j) = 0$ and $\chi_{\tau_k(J_i)}(x_{j-1}) = 1$.

First, we will estimate (I):

$$\sum_{j=1}^{r}\sum_{i=1}^{q}|h_k(\phi_{k,i}(x_j))f_k(\phi_{k,i}(x_j)) - h_k(\phi_{k,i}(x_{j-1}))f_k(\phi_{k,i}(x_{j-1}))|$$

$$\leq \sum_{i=1}^{q}\sum_{j=1}^{r}|f_k(\phi_{k,i}(x_j))[h_k(\phi_{k,i}(x_j)) - h_k(\phi_{k,i}(x_{j-1}))]|$$

$$+ \sum_{i=1}^{q}\sum_{j=1}^{r}|h_k(\phi_{k,i}(x_{j-1}))[f_k(\phi_{k,i}(x_j)) - f_k(\phi_{k,i}(x_{j-1}))]|$$

$$\leq \sup_x|h_k'(x)|\sum_{i=1}^{q}\sum_{j=1}^{r}|f_k(\phi_{k,i}(x_j))[\phi_{k,i}(x_j) - \phi_{k,i}(x_{j-1})]|$$

$$+ (\sup_x h_k(x))\sum_{i=1}^{q} V_{J_i}f_k$$

$$\leq \sup_x|h_k'(x)|\sum_{i=1}^{q}\left(\int_{J_i}|f_k|d\lambda(x) + \epsilon\right) + (\sup_x h_k(x))\sum_{i=1}^{q} V_{J_i}f_k, \quad \text{using } (3.3)$$

$$\leq \sup_x|h_k'(x)|\int_I|f_k|d\lambda(x) + (\sup_x h_k(x))V_I f_k + q(\sup_x|h_k(x)|)\epsilon.$$

We now consider (II) and (III) together. Notice that $\chi_{\tau_k(J_i)}(x_j) = 1$ and $\chi_{\tau_k(J_i)}(x_{j-1}) = 0$ occurs only if $x_j \in \tau_l(J_i)$ and $x_{j-1} \notin \tau_l(J_i)$, i.e., if x_j and x_{j-1} are on the opposite sides of an end point of $\tau_l(J_i)$. We can have at most one pair x_j, x_{j-1} like this and another pair $x_{j'} \notin \tau_l(J_i)$ and $x_{j'-1} \in \tau_l(J_i)$. Thus,

$$\begin{aligned} &\sum_{i=1}^{q}(|h_k(\phi_{k,i}(x_j))f_k(\phi_{k,i}(x_j))| + |h_k(\phi_{k,i}(x_{j'-1}))f_k(\phi_{k,i}(x_{j'-1}))|) \\ &\le \sup_x h_k(x)\sum_{i=1}^{q}(|f_k(\phi_{k,i}(x_j))| + |f_k(\phi_{k,i}(x_{j'-1}))|). \end{aligned} \tag{3.5}$$

Since $s_i = \phi_{k,i}(x_j)$ and $r_i = \phi_{k,i}(x_{j'-1})$ are both points in J_i, we can write

$$\sum_{i=1}(|f_k(s_i)| + |f_k(r_i)|) \le \sum_{i=1}^{q}(2|f_k(v_i)| + |f_k(v_i) - f_k(r_i)| + |f_k(v_i) - f_k(s_i)|),$$

where $v_i \in J_i$ is such that $|f_k(v_i)| \le \frac{1}{\lambda(J_i)}\int_{I_i}|f_k|d\lambda(x)$. Thus,

$$\begin{aligned} &\sup_x h_k(x)\sum_{i=1}(|f_k(\phi_{k,i}(x_j))| + |f_k(\phi_{k,i}(x'_{j-1}))|) \\ \le\ &\sup_x h_k(x)\sum_{i=1}\left(V_{I_i}f_k + \frac{2}{\lambda(J_i)}\int_{I_i}|f_k|d\lambda(x)\right) \\ \le\ &\sup_x|h_k(x)|V_I f_k + \frac{2\sup_x h_k(x)}{\delta}\int_I|f_k|d\lambda(x). \end{aligned}$$

Therefore,

$$\begin{aligned} V_I P_{\tau_k}(W_{k,l}f_k) \le 2\sup_x|h_k(x)|V_I f_k + (\frac{2}{\delta}(\sup_x h_k) \\ +(\sup|h'_k(x)|))\|f_k\|_1 + q(\sup_x h_k(x))\epsilon. \end{aligned}$$

Thus,

$$\begin{aligned} V_I(\widehat{P}_T f)_l \le \sum_{k=1}^{K}(2\max_k\sup_x|h_k(x)|V_I f_k + (\frac{2}{\delta}(\max_k\sup_x h_k) \\ +(\max_k\sup|h'_k(x)|))\|f_k\|_1 + q(\sup_x h_k(x))\epsilon). \end{aligned}$$

Since ϵ is arbitrarily small this proves the lemma. □

Theorem 3.2. *Let τ_k be piecewise C^2 on $I=[0,1]$ and $W_{k,l}$ be piecewise of class C^1, for $k=1,2,\dots,K$ and $l=1,2,\dots,K$,and*

$$\alpha_l = \max_k \left(\sup_x \frac{2W_{k,l}(x)}{|\tau_k'(x)|} \right), \ l=1,2,\dots,K,$$

and $\sum_{l=1}^K \alpha_l < 1$. Then the operator $\widehat{P}_T$ is quasi-compact and admits a fixed point in $\widehat{BV}$, i.e., the Markov switching random map T admits an absolutely continuous invariant measure.

Proof. The space $\widehat{BV}$ is a Banach space with norm $\|\cdot\|_{\widehat{BV}} = \sum_{k=1}^K \|\cdot\|_{BV}$. First, if $f=(f_1,f_2,\dots,f_K)$ with $f_k \geq 0$, then we have

$$\begin{aligned} \|\widehat{P}_T f\|_1 &= \sum_{l=1}^K \|(\widehat{P}_T f)_l\|_1 = \sum_{l=1}^K \int_I \sum_{k=1}^K P_{\tau_k}(W_{k,l} f_k) d\lambda \\ &= \sum_{k=1}^K \int_I \sum_{l=1}^K P_{\tau_k}(W_{k,l} f_k) d\lambda = \sum_{k=1}^K \int_I \sum_{l=1}^K P_{\tau_k}(f_k) d\lambda = \|f\|_1. \end{aligned}$$

For a general f it is easy to show that $\|\widehat{P}_T f\|_1 \leq \|f\|_1$. For $f \in \widehat{BV}$, by the above lemma, we obtain

$$\begin{aligned} \| \widehat{P_T} f \|_{\widehat{BV}} &= \sum_{l=1}^K \| (\widehat{P}_T f)_l \|_{BV} = \sum_{l=1}^K V_I(\widehat{P}_T f)_l + \| \widehat{P}_T f \|_1 \\ &\leq \sum_{l=1}^K V_I(\widehat{P}_T f)_l + \| \widehat{P}_T f \|_1 \\ &\leq \sum_{l=1}^K \alpha_l \left(\sum_{k=1}^K V_I f_k \right) + \sum_{l=1}^K \left(B_l \sum_{k=1}^K \| f_k \|_1 \right) + \| f \|_1 \\ &\leq \sum_{l=1}^K \alpha_l \| f \|_{\widehat{BV}} + \left(B_l + 1 - \sum_{l=1}^K \alpha_l \right) \cdot \| f \|_1 . \end{aligned}$$

Thus, by Ionescu-Tulcea and Marinescu Theorem [4], $\widehat{P}_T$ is quasi-compact on $\widehat{BV}$ and admits a fixed point f in $\widehat{BV}$. □

Example 3.3. Consider the Markov switching position dependent random map

$$T = \{\tau_1, \tau_2; p_1, p_2; W\},$$

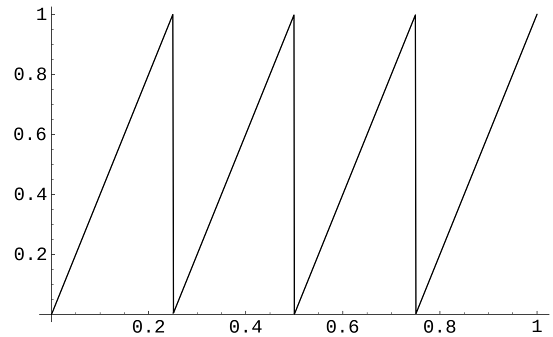

Figure 1: The graph of τ_1

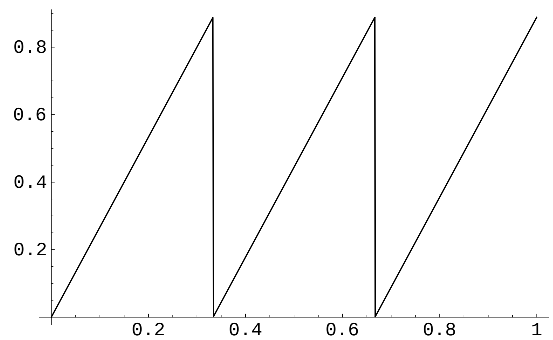

Figure 2: The graph of τ_2

where τ_1, τ_2 are maps on $I = [0,1]$ defined by (see Figure 1 and Figure 2)

$$\tau_1(x) = \begin{cases} 4x, & 0 \le x \le \frac{1}{4}, \\ 4x-1, & \frac{1}{4} < x \le \frac{1}{2}, \\ 4x-2, & \frac{1}{2} < x \le \frac{3}{4}, \\ 4x-3, & \frac{3}{4} < x \le 1 \end{cases} \tag{3.6}$$

and

$$\tau_2(x) = \begin{cases} \frac{8}{3}x\,, & 0 \le x \le \frac{1}{3}, \\ \frac{8}{3}x - \frac{8}{9}\,, & \frac{1}{3} < x \le \frac{2}{3}, \\ \frac{8}{3}x - \frac{16}{9}\,, & \frac{2}{3} < x \le 1. \end{cases} \tag{3.7}$$

and W is a stochastic switching matrix defined by

$$W = \begin{bmatrix} \frac{1}{2}x + \frac{1}{10} & \frac{9}{10} - \frac{1}{2}x \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix},$$

and p_1, p_2 are initial probabilities.

It is easy to show that $\alpha_1 = \frac{1}{2}$ and $\alpha_2 = \frac{9}{20}$. Hence the Markov switching random map $T = \{\tau_1, \tau_2\}$ with switching matrix W satisfies the condition of Theorem 3.1 and T has an acim. Although this is a very simple Markov switching random map, there is no known method to calculate the acim.

4. Approximation of Invariant Measure for Markov Switching Position Dependent Random Maps

In this section we consider Markov switching position dependent random maps T with position dependent switching matrix W and describe a method of approximating the fixed point of the operator $\widehat{P_T}$ by the fixed points of a matrix operators. Let $\tau_k : [0,1] \to [0,1], k = 1, 2, \ldots, K$, be piecewise C^2 maps on a common partition $\mathcal{P}$ of $[0,1] : \mathcal{P} = \{J_1, J_2, \ldots, J_q\}$ and

$$W = \begin{bmatrix} W_{1,1}(x) & W_{1,2}(x) & \ldots & W_{1,K}(x) \\ W_{2,1}(x) & W_{2,2}(x) & \ldots & W_{2,K}(x) \\ \vdots & \vdots & \ldots & \vdots \\ W_{K,1}(x) & W_{K,2}(x) & \ldots & W_{K,K}(x) \end{bmatrix}$$

be a switching matrix piecewise continuous on the same partition $\{J_1, J_2, \ldots, J_K\}$ of $I = [0,1]$, satisfying the condition of Theorem 3.1. Hence, the Markov switching random map $T = \{\tau_1, \tau_2, \ldots, \tau_K\}$ has an acim μ. We want to approximate the invariant density f of μ for the random map T. Let $\mathcal{P}^{(n)} = \{I_1, I_2, \ldots, I_n\}$ be a partition of $[0,1]$ into subintervals of equal length and let $M_n(k)$ be the matrix of transition probabilities between the elements of $\mathcal{P}^{(n)}$ for the map $\tau_k, k = 1, 2, \ldots, K$:

$$M_n(k) = \left(\frac{\lambda(I_i \cap \tau_k^{-1}(I_j))}{\lambda(I_i)} \right)_{1 \le i,j \le n}.$$

Let $L^{(n)} = \{f \in BV(I) : f = \sum_{i=1}^{n} f_n^i \chi_{I_i} = (f_n^1, f_n^2, \ldots, f_n^n)\}$. Define an operator $Q^{(n)} : BV(I) \to L^{(n)}$ by

$$Q^{(n)}(f) = \sum_{i=1}^{n} n \left(\int_{I_i} f d\lambda \right) \chi_{I_i} = \left(n \int_{I_1} f d\lambda, n \int_{I_2} f d\lambda, \ldots, n \int_{I_n} f d\lambda \right).$$

Let P_{τ_k} be the Frobenius-Perron operator of τ_k and $P_\tau^{(n)} : L^{(n)} \to L^{(n)}$ be a finite approximation of P_τ defined by

$$P_{\tau_k}^{(n)} f = (M_n(k))^t f.$$

Li [14] proved the following results:

1. For $f \in L_1$, $Q^{(n)} f \longrightarrow f$ in L_1 as $n \to \infty$;
2. For $f \in L^{(n)}, P_{\tau_k}^{(n)} f = Q^{(n)} P_{\tau_k} f$;
3. For $f \in BV(I), V_I Q^{(n)} f \le V_I f$;
4. For $f \in L_1, P_{\tau_k}^{(n)} f \longrightarrow P_{\tau_k} f$ in L_1 as $n \to \infty$;

Let $\widehat{L^{(n)}} = \prod_{k=1}^K L^{(n)}$ and define $\widehat{Q^{(n)}} : \widehat{BV} \to \widehat{L^{(n)}}$ by

$$\widehat{Q^{(n)}}(f_1, f_2, \ldots, f_K) = (Q^{(n)}(f_1), Q^{(n)}(f_2), \ldots, Q^{(n)}(f_n)).$$

We define an operator $\widehat{P_T^{(n)}} : \widehat{L^{(n)}} \to \widehat{L^{(n)}}$ by

$$\widehat{P_T^{(n)}}(f_1, f_2, \ldots, f_K) = (\sum_{k=1}^K M_n(k) Q^{(n)}(W_{k,1}) f_k, \sum_{k=1}^K M_n(k) Q^{(n)}(W_{k,2}) f_k, \ldots, \sum_{k=1}^K M_n(k) Q^{(n)}(W_{k,K}) f_k). \quad (4.1)$$

For a fixed n, let

$$f_k = (f_{k,n}^1, f_{k,n}^2, \ldots, f_{k,n}^n), \; k = 1, 2, \ldots, K,$$

and

$$Q^{(n)}(W_{k,l}) = (W_{k,l,n}^1, W_{k,l,n}^2, \ldots, W_{k,l,n}^n), \; k, l = 1, 2, \ldots, K.$$

Then, for any $1 \le i \le n$, $\sum_{l=1}^K W_{k,l,n}^i = \sum_{l=1}^K n \int_{I_i} W_{k,l} d\lambda = 1$ and hence

$$\widehat{P_T^{(n)}}(f_1, f_2, \ldots, f_K) = \left(\sum_{k=1}^K M_n(k) Q^{(n)}(W_{k,1}) f_k, \sum_{k=1}^K M_n(k) Q^{(n)}(W_{k,2}) f_k, \ldots, \sum_{k=1}^K M_n(k) Q^{(n)}(W_{k,K}) f_k \right)$$

$$= (\sum_{k=1}^{K} M_n(k) \left[W^1_{k,1,n} f^1_{k,n}, W^2_{k,1,n} f^2_{k,n}, \dots, W^n_{k,1,n} f^n_{k,n}\right],$$
$$\sum_{k=1}^{K} M_n(k) \left[W^1_{k,2,n} f^1_{k,n}, W^2_{k,2,n} f^2_{k,n}, \dots, W^n_{k,2,n} f^n_{k,n}\right],$$
$$\dots, \sum_{k=1}^{K} M_n(k) \left[W^1_{k,K,n} f^1_{k,n}, W^2_{k,2,n} f^2_{k,n}, \dots, W^n_{k,K,n} f^n_{k,n}\right]). \quad (4.2)$$

Lemma 4.1. *For $f \in \widehat{L^{(n)}}$, we have $\widehat{P^{(n)}_T} f = \widehat{Q^{(n)}} \widehat{P_T} f$.*

Proof. Let $f = (f_1, f_2, \dots, f_K) \in \widehat{L}^{(n)}$, where $f_k = (f^1_{k,n}, f^2_{k,n}, \dots, f^n_{k,n})$, $k = 1, 2, \dots, K$. Then,

$$\widehat{Q^{(n)}} \widehat{P_T} f = \widehat{Q^{(n)}} \left(\sum_{k=1}^{K} \mathcal{P}_{\tau_k} (Q^{(n)}(W_{k,1}) f_k), \dots, \sum_{k=1}^{K} \mathcal{P}_{\tau_k} Q^{(n)}(W_{k,K}) f_k) f_k \right)$$
$$= \left(Q^{(n)} \sum_{k=1}^{K} \mathcal{P}_{\tau_k} \left(Q^{(n)}(W_{k,1}) f_k\right), \dots, Q^{(n)} \sum_{k=1}^{K} \mathcal{P}_{\tau_k} \left(Q^{(n)}(W_{k,K}) f_k\right) \right)$$
$$= \left(\sum_{k=1}^{K} Q^{(n)} \left(\mathcal{P}_{\tau_k} Q^{(n)}(W_{k,1}) f_k\right), \dots, \sum_{k=1}^{K} Q^{(n)} \left(\mathcal{P}_{\tau_k} Q^{(n)}(W_{k,K}) f_k\right) \right)$$
$$= \left(\sum_{k=1}^{2} (M^n_k)^t Q^{(n)}(W_{k,1}) f_k, \sum_{k=1}^{2} (M^n_k)^t Q^{(n)}(W_{k,2}) f_k), \dots, \right.$$
$$\left. \sum_{k=1}^{K} (M^n_k)^t Q^{(n)}(W_{k,2}) f_k) \right) = \widehat{P^{(n)}_T} f. \quad \square$$

Using the properties of $Q^{(n)}$and $\widehat{Q}^{(n)}$, we can prove the following lemmas.

Lemma 4.2. *For $f \in \widehat{BV}$, $\widehat{Q}^{(n)} f$ converges to f in $\widehat{BV}$.*

Lemma 4.3. *For $f \in \widehat{L^{(n)}}$, $\widehat{P}^{(n)}_T f$ converges to $\widehat{P}_T f$ in $\widehat{BV}$.*

Proof. By Lemma 4.2, for $f \in \widehat{L^{(n)}}$ we have $\widehat{P^{(n)}_T} f = \widehat{Q^{(n)}} \widehat{P_T} f$. By Lemma 4.3 we have $\widehat{Q^{(n)}} \widehat{P_T} f$ converges to $\widehat{P_T} f$. $\square$

Now we prove a theorem which will be useful later.

Theorem 4.4. *Let τ_k be of class C^2 and $W_{k,l}$ be of class C^1, for $k = 1, 2$ satisfying the conditions of Theorem 3.1, i.e.,*

$$\alpha_l = \max_k (\sup_x \frac{2W_{k,l}(x)}{|\tau_k'(x)|}), \quad l = 1, 2, \ldots, K$$

and $\sum_{l=1}^{K} \alpha_l < 1$. Then for any positive integer n, $\widehat{P_T^{(n)}}$ has a fixed point f_n in $\widehat{L^{(n)}}$.

Proof.

$$V_I(\widehat{P_T^{(n)}} f)_l = V_I(\widehat{Q^n}\widehat{P_T} f)_l \le V_I(\widehat{P_T} f)_l \le \alpha_l \sum_{k=1}^{K} V_I f_k + B_l \sum_{k=1}^{K} \| f_k \|_1 ,$$

where $\alpha_l = \max_k (\sup_x \frac{2 \cdot W_{k,l}(x)}{|\tau_k'(x)|}), h_k(x) = \frac{W_{k,l}(x)}{|\tau_k'(x)|}, \delta = \min_i \lambda(I_i)$ and $B_l = \frac{2}{\delta}(\max_k \sup_x h_k(x)) + (\max_k \sup_x |h_k'(x)|)$.

Now,

$$\begin{aligned} \| \widehat{P_T^{(n)}} f \|_{\widehat{BV}} = \sum_{l=1}^{K} \| (\widehat{P_T^{(n)}} f)_l \|_{BV} = \sum_{l=1}^{K} V_I(\widehat{P_T^{(n)}} f)_l + \| \widehat{P_T^{(n)}} f \|_1 \\ \le \sum_{l=1}^{K} V_I(\widehat{P_T} f)_l + \| \widehat{P_T^{(n)}} f \|_1 \\ \le \sum_{l=1}^{K} \alpha_l \left(\sum_{k=1}^{K} V_I f_k \right) + \sum_{l=1}^{K} \left(B_l \sum_{k=1}^{K} \| f_k \|_1 \right) + \| f \|_1 \\ \le \sum_{l=1}^{K} \alpha_l \| f \|_{\widehat{BV}} + \left(B_l + 1 - \sum_{l=1}^{K} \alpha_l \right) \cdot \| f \|_1 . \end{aligned}$$

Thus, by the Ionescu-Tulcea and Marinescu Theorem [8], $\widehat{P_T^{(n)}}$ is quasi-compact on $\widehat{L^{(n)}}$ and admits a fixed point f_n in $\widehat{L^{(n)}}$. □

This theorem can be also proved using simple matrix theorem. It can be shown that the operator $\widehat{P_T^{(n)}}$ can be represented by the following $(K \times n)$ by $(K \times n)$ stochastic matrix S_n,

$$\begin{bmatrix} M_n(1)\mathrm{diag}\,[Q^{(n)}(W_{1,1})] & M_n(1)\mathrm{diag}\,[Q^{(n)}(W_{1,2})] & \dots \\ M_n(2)\mathrm{diag}\,[Q^{(n)}(W_{2,1})] & M_n(2)\mathrm{diag}\,[Q^{(n)}(W_{2,2})] & \dots \\ \vdots & \vdots & \ddots \\ M_n(K)\mathrm{diag}\,[Q^{(n)}(W_{K,1})] & M_n(K)\mathrm{diag}\,[Q^{(n)}(W_{K,2})] & \dots \end{matrix}$$
$$\begin{matrix} M_n(1)\mathrm{diag}\,[Q^{(n)}(W_{1,K})] \\ M_n(2)\mathrm{diag}\,[Q^{(n)}(W_{2,K})] \\ \vdots \\ M_n(K)\mathrm{diag}\,[Q^{(n)}(W_{K,K})] \end{bmatrix}.$$

Since 1 is a left eigenvalue of any stochastic matix [6, 18], the matrix S_n has 1 as a left eigenvalue. Let

$$s_n = (s_{n,1}, s_{n,2}, \dots, s_{n,K})$$

be a left eigenvector of S_n associated with the eigenvalue 1 and

$$s_{n,k} = \left(s^1_{n,k}, s^2_{n,k}, \dots, s^n_{n,k}\right), \quad \sum_{i=1}^{n} s^i_{n,k} = 1,\ k = 1, 2, \dots, K.$$

Define the approximating invariant density

$$d_n = \sum_{i=1}^{n} \left(\frac{\sum_{k=1}^{K} s^i_{n,k}}{\lambda(I_i)} \right) \chi_{I_i}. \tag{4.3}$$

Theorem 4.5. *Suppose that the Markov switching random map T satisfies the hypothesis of Theorem 3.2 and the operator $\widehat{P}_T$ has a unique invariant density d. Then $\| d - d_n \|_1 \to 0$ as $n \to 0$, for the approximate density d_n in (4.3).*

Proof. Let $f = (f_{n,1}, f_{n,2}, \dots, f_{n,K}) \in \widehat{L}^{(n)}$ be a fixed point of $\widehat{P}^{(n)}_T$. Then

$$\begin{aligned} \max_{1\le l\le K} V_I(\widehat{P}^{(n)}_T f)_l &= \max_{1\le l\le K} V_I(\widehat{Q}^{(n)} \widehat{P}_T\,(f_{n,1}, f_{n,2}, \dots, f_{n,K}))_l \\ &= \max_{1\le l\le K} V_I(Q^{(n)}(\widehat{P}_T\,(f_{n,1}, f_{n,2}, \dots, f_{n,K})))_l \\ &\le \max_{1\le l\le K} V_I(\widehat{P}_T\,(f_{n,1}, f_{n,2}, \dots, f_{n,K}))_l \\ &\le \max_{1\le l\le K} \alpha_l \sum_{k=1}^{K} V_I f_{n,k} + \max_{1\le l\le 2} B_l \sum_{k=1}^{K} \| f_{n,k} \|_1 . \end{aligned} \tag{4.4}$$

Thus the sequence $\{V_I(P_T^n f)_l)\}_{n\geq 1}$ is uniformly bounded. So by Helly's Theorem, the set $C = \{(f_{n,1}, f_{n,2}, \cdots, f_{n,K}); n = 1, 2, \cdots\}$ is sequentially compact in $\prod_{k=1}^{K} L^1$. Let $\{(f_{n_j,1}, f_{n_j,2}, \cdots, f_{n_j,K})\}_{j\geq 1}$ be any subsequence of C and assume that $\{(f_{n_j,1}, f_{n_j,2}, \cdots, f_{n_j,K})\}_{j\geq 1}$ converges to $(f_1, f_2, \ldots, f_K)$ as $j \to \infty$. Then

$$\begin{aligned}
& \| (f_1, f_2, \ldots, f_K) - \widehat{P}_T (f_1, f_2, \ldots, f_K) \|_1 \\
\leq \;& \| (f_1, f_2, \ldots, f_K) - (f_{n_j,1}, f_{n_j,2}, \cdots, f_{n_j,K}) \|_1 \\
& + \|(f_{n_j,1}, f_{n_j,2}, \cdots, f_{n_j,K}) - \widehat{Q}^{(n_j)} \widehat{P}_T (f_{n_j,1}, f_{n_j,2}, \cdots, f_{n_j,K})\|_1 \\
+ \;& \|\widehat{Q}^{(n_j)} \widehat{P}_T (f_{n_j,1}, f_{n_j,2}, \cdots, f_{n_j,K}) - \widehat{Q}^{(n_j)} \widehat{P}_T (f_1, f_2, \cdots, f_K)\|_1 \\
& + \|\widehat{Q}^{(n_j)} \widehat{P}_T (f_1, f_2, \cdots, f_K) - \widehat{P_T} (f_1, f_2, \ldots, f_K) \|_1.
\end{aligned}$$

Note that

$$\widehat{Q}^{(n_j)} \widehat{P}_T (f_{n_j,1}, f_{n_j,2}, \cdots, f_{n_j,K}) = \widehat{P}_T^{n_j} (f_{n_j,1}, f_{n_j,2}, \cdots, f_{n_j,K})$$

and $(f_{n_j,1}, f_{n_j,2}, \cdots, f_{n_j,K})$ is a fixed point of $\widehat{P}_T^{n_j}$. Thus

$$\|(f_{n_j,1}, f_{n_j,2}, \cdots, f_{n_j,K}) - \widehat{Q}^{(n_j)} \widehat{P}_T (f_{n_j,1}, f_{n_j,2}, \cdots, f_{n_j,K})\|_1 = 0.$$

Moreover,

$$\begin{aligned}
& \|\widehat{Q}^{(n_j)} \widehat{P}_T (f_{n_j,1}, f_{n_j,2}, \cdots, f_{n_j,K}) - \widehat{Q}^{(n_j)} \widehat{P_T} (f_{n,1}, f_{n,2}, \cdots, f_{n,K})\|_1 \\
& \quad \leq \|\widehat{Q}^{(n_j)} \widehat{P}_T\|_1 \|(f_{n_j,1}, f_{n_j,2}, \cdots, f_{n_j,K}) - (f_{n,1}, f_{n,2}, \cdots, f_{n,K})\|_1
\end{aligned}$$

and

$$\widehat{Q}^{n_j} (\widehat{P}_T h) \to \widehat{P}_T h.$$

Hence $\widehat{P_T} (f_1, f_2, \ldots, f_K) = (f_1, f_2, \ldots, f_K)$.

Therefore, any convergent subsequence of C converges to a fixed point of $\widehat{P_T}$. By assumption, $\widehat{P_T}$ has a unique fixed point d, that is, $\|d - d_n\|_1 \to 0$ as $n \to \infty$. ☐

Now, we restrict our attention to the space $\widehat{BV}_0 = \{\hat{f} \in \widehat{BV} : \int f_k = 0$ for all $k = 1, 2, \ldots, K\} \subset \widehat{BV}$. In this situation we have $\| f_k \|_1 \leq \frac{1}{2} V_I f_k, k = 1, 2, \ldots, K$ [11]. Moreover, the operator $\widehat{P}_T$ is a contraction on $\widehat{BV}_0$. We have the following lemma.

Lemma 4.6. *Let $f = (f_1, f_2, \ldots, f_K) \in \widehat{BV}_0$. Then*

$$\| (\widehat{P}_T f) \|_{\widehat{BV}} \leq \frac{1}{2} \left(\sum_{l=1}^{K} \alpha_l + \sum_{l=1}^{K} B_l + 1 \right) \| f \|_{\widehat{BV}} . \tag{4.5}$$

Proof.

$$\begin{aligned}
\| \widehat{P_T} f \|_{\widehat{BV}} &= \sum_{l=1}^{K} \| (\widehat{P_T} f)_l \|_{BV} = \sum_{l=1}^{K} V_I(\widehat{P_T} f)_l + \| \widehat{P_T} f \|_1 \\
&\leq \sum_{l=1}^{K} \alpha_l \left(\sum_{k=1}^{K} V_I f_k \right) + \sum_{l=1}^{K} B_l \left(\sum_{k=1}^{K} \| f_k \|_1 \right) + \| f \|_1 \\
&= \sum_{l=1}^{K} \alpha_l \| f \|_{\widehat{BV}} + \left(\sum_{l=1}^{K} B_l + 1 - \sum_{l=1}^{K} \alpha_l \right) \cdot \| f \|_1 \\
&\leq \sum_{l=1}^{K} \alpha_l \| f \|_{\widehat{BV}} + \left(\sum_{l=1}^{K} B_l + 1 - \sum_{l=1}^{K} \alpha_l \right) \cdot \sum_{l=1}^{K} \frac{1}{2} V_I(f_l) \\
&= \sum_{l=1}^{K} \alpha_l \| f \|_{\widehat{BV}} + \frac{1}{2} \left(\sum_{l=1}^{K} B_l + 1 - \sum_{l=1}^{K} \alpha_l \right) \| f \|_{\widehat{BV}} \\
&= \frac{1}{2} \left(\sum_{l=1}^{K} \alpha_l + \sum_{l=1}^{K} B_l + 1 \right) \| f \|_{\widehat{BV}} . \quad \square
\end{aligned}$$

5. Error Bounds for Markov Switching Position Dependent Random Map

In this section we restrict our attention to the space $\widehat{BV}_0 = \{\hat{f} \in \widehat{BV} : \int f_k = 0$ for all $k = 1, 2, \ldots, K\} \subset \widehat{BV}$. Let $d = \sum_{k=1}^{K} d_k$, where $\widehat{P}_T(d_1, d_2, \ldots, d_K) = (d_1, d_2, \ldots, d_K)$ is the unique invariant density of the invariant measure μ for a Markov switching position dependent random map T, and for a fixed n let d_n be the approximate invariant density of T. In this section we want to find a bound for $\|d - d_n\|_1$. The measure μ can be decomposed as $\mu = \sum_{k=1}^{K} \mu_k$, where $(\mu_1, \mu_2, \ldots, \mu_K)$ is fixed by the operator

$$(\mu_1, \mu_2, \ldots, \mu_K) \mapsto \left(\sum_{k=1}^{K} \int_{\tau_k^{-1}} W_{k,1} d\mu_k, \sum_{k=1}^{K} \int_{\tau_k^{-1}} W_{k,2} d\mu_k, \ldots, \sum_{k=1}^{K} \int_{\tau_k^{-1}} W_{k,K} d\mu_k \right).$$

Let $\widehat{M}_n(k)$ be the matrix of transition probabilities between the elements of $\mathcal{P}^{(n)}$ for the map τ_k with respect to measure $\mu_k, k = 1, 2, \ldots, K$:

$$\widehat{M}_n(k) = \left(\frac{\mu_k(I_i \cap \tau_k^{-1}(I_j))}{\mu_k(I_i)} \right)_{1\le i,j\le n} .$$

Now, consider the following $(K \times n)$ by $(K \times n)$ matrix $\hat{S}_n$

$$\begin{bmatrix} \widehat{M}_n(1)\mathrm{diag}\,[Q^{(n)}(W_{1,1})] & \widehat{M}_n(1)\mathrm{diag}\,[Q^{(n)}(W_{1,2})] & \ldots & \widehat{M}_n(1)\mathrm{diag}\,[Q^{(n)}(W_{1,K})] \\ \widehat{M}_n(2)\mathrm{diag}\,[Q^{(n)}(W_{2,1})] & \widehat{M}_n(2)\mathrm{diag}\,[Q^{(n)}(W_{2,2})] & \ldots & \widehat{M}_n(2)\mathrm{diag}\,[Q^{(n)}(W_{2,K})] \\ \vdots & \vdots & \ldots & \vdots \\ \widehat{M}_n(K)\mathrm{diag}\,[Q^{(n)}(W_{K,1})] & \widehat{M}_n(K)\mathrm{diag}\,[Q^{(n)}(W_{K,2})] & \ldots & \widehat{M}_n(K)\mathrm{diag}\,[Q^{(n)}(W_{K,K})] \end{bmatrix}.$$

Let $\hat{s}_n = (\hat{s}_{n,1}, \hat{s}_{n,2}, \ldots, \hat{s}_{n,K})$ be a left eigenvector of $\hat{S}_n$ associated with eigenvalue 1 and $\hat{s}_{n,k} = \left(\hat{s}^1_{n,k}, \hat{s}^2_{n,k}, \ldots, \hat{s}^n_{n,k}\right)$, $\sum_{i=1}^n \hat{s}^i_{n,k} = 1, k = 1, 2, \ldots, K$. It can be shown that

$$\hat{s}_{n,k} = \left(\hat{s}^1_{n,k}, \hat{s}^2_{n,k}, \ldots, \hat{s}^n_{n,k}\right) = (\mu_k(I_1), \mu_k(I_2), \ldots, \mu_k(I_n)) .$$

Lemma 5.1. *For $f_n \in L^{(n)}, \|f_n\|_m \le n\|f_n\|_{BV}$ and $\|f_n\|_{BV} \le 3\|f_n\|_m$, where $\|f_n\|_{BV} = V(f_n) + \|f_n\|_1$ and $\|f_n\|_m = \sum_{i=1}^n |f^i_n|$.*

Proof. $\|f_n\|_m = \sum_{i=1}^n |f^i_n| = n\left(\sum_{i=1}^n |f^i_n| \frac{1}{n}\right) = n\|f_n\|_1 \le nV(f_n) + n\|f_n\|_1 = n\|f_n\|_{BV}$. Notice that $V(f_n) \le 2\|f_n\|_m$. Thus,

$$\|f_n\|_{BV} = V(f_n) + \|f_n\|_1 \le 2\|f_n\|_m + \frac{1}{n}\|f_n\|_m \le 2\|f_n\|_m + \|f_n\|_m = 3\|f_n\|_m. \quad \square$$

Define $\|s_n\|_m = \sum_{k=1}^K \sum_{i=1}^n |s^i_{n,k}|$.

Lemma 5.2. *For $\hat{f}_n = (f_{n,1}, f_{n,2}, \ldots, f_{n,K}) \in \widehat{L}^{(n)}, \|\hat{f}_n\|_m \le n\|\hat{f}_n\|_{\widehat{BV}}$ and $\|\hat{f}_n\|_{\widehat{BV}} \le 3\|\hat{f}_n\|_m$.*

Proof. $\|\hat{f}_n\|_m = \sum_{k=1}^{K} \|f_{n,k}\|_m \leq n \sum_{k=1}^{K} \|f_{n,k}\|_{BV} \leq n\|\hat{f}_n\|_{\widehat{BV}}$. On the other hand, $\|\hat{f}_n\|_{\widehat{BV}} = \sum_{k=1}^{K} \|f_{n,k}\|_{BV} \leq 3 \sum_{k=1}^{K} \|f_{n,k}\|_m = 3\|\hat{f}_n\|_m$. □

Theorem 5.3.

$$\|d - d_n\|_1 \leq \frac{1}{n} \sum_{k=1}^{K} \mathrm{Lip}\,(d_k) + \inf_{0<\delta<1} \left(2 + \frac{\delta}{1-\delta}\right) \left(\left[\frac{\log(6K^2 n/\delta)}{-\log\gamma}\right] + 1\right) - 1\}.$$

Proof. $\|d - d_n\|_1 \leq \|d - Q^{(n)}(d)\|_1 + \|Q^{(n)}(d) - d_n\|_1$. First we find a bound for $\|d - Q^{(n)}(d)\|_1$.

$$\begin{aligned}
\|d - Q^{(n)}(d)\|_1 &= \|\sum_{k=1}^{K} d_k - Q^{(n)}(\sum_{k=1}^{K} d_k)\|_1 \leq \sum_{k=1}^{K} \|d_k - Q^{(n)}(d_k)\|_1 \\
&\leq \sum_{k=1}^{K} \left(\int_I |d_k - Q^{(n)}(d_k)| d\lambda\right) \leq \sum_{k=1}^{K} \left(\sum_{i=1}^{n} \int_{I_i} |d_k - n\int_{I_i} d_k d\lambda| d\lambda\right) \\
&\leq \sum_{k=1}^{K} \left(\sum_{i=1}^{n} \frac{1}{2n} \left(\sup_{I_i} d_k - \inf_{I_i} d_k\right)\right) \leq \sum_{k=1}^{K} \left(\sum_{i=1}^{n} \frac{1}{n} \cdot \mathrm{Lip}\,(d_k) \cdot \frac{1}{2n}\right) \\
&= \sum_{k=1}^{K} \mathrm{Lip}\,(d_k) \cdot \frac{1}{2n},
\end{aligned}$$

where $\mathrm{Lip}\,(d_k)$ is the maximum Lipschitz constant calculated over each of the Lipschitz pieces of d_k separately.

Now, we want to bound $\|Q^{(n)}(d) - d_n\|_1$:

$$\begin{aligned}
\|Q^{(n)}(d) - d_n\|_1 &= \|\sum_{i=1}^{n} n\int_{I_i} d - \sum_{i=1}^{n} n\left(\sum_{k=1}^{K} s_{n,k}^i\right)\chi_{I_i}\|_1 \\
&= \|\sum_{i=1}^{n} n\left(\sum_{k=1}^{K} \hat{s}_{n,k}^i\right)\chi_{I_i} - \sum_{i=1}^{n} n\left(\sum_{k=1}^{K} s_{n,k}^i\right)\chi_{I_i}\|_1 \\
&= \sum_{i=1}^{n} |\sum_{k=1}^{K} (\hat{s}_{n,k}^i - s_{n,k}^i)| \leq \sum_{i=1}^{n}\sum_{k=1}^{K} |\hat{s}_{n,k}^i - s_{n,k}^i| \\
&= \sum_{k=1}^{K}\sum_{i=1}^{n} |\hat{s}_{n,k}^i - s_{n,k}^i| = \|\hat{s_n} - s_n\|_m,
\end{aligned}$$

where $\|.\|_m$ denotes the L^1 vector norm. Now the stochastic matrix S_n has a unique left eigenvector and we have by [21],

$$\|\hat{s_n} - s_n\|_m \leq \|\hat{S_n} - S_n\|_m \| \left(I_{Kn} - S_n + {S_n}^{\infty}\right)^{-1} \|_m,$$

where each row of ${S_n}^\infty$ is the unique left eigenvector of S_n.

Now, we want to bound $\|\hat{S_n} - S_n\|_m$:

$$\begin{aligned}
|M_{n,ij}(k) - \hat{M}_{n,ij}(k)| &= M_{n,ij}(k)|1 - \frac{\mu_k(I_i \cap \tau_k^{-1}(I_j))}{\mu_k(I_i)} \frac{\lambda(I_i)}{\lambda(I_i \cap \tau_k^{-1}(I_j))}| \\
&= M_{n,ij}(k)|1 - \frac{\int_{I_i \cap \tau_k^{-1}(I_j)} d_k}{\lambda(I_i \cap \tau_k^{-1}(I_j))} \frac{\lambda(I_i)}{\int_{I_i} d_k}| \\
&\le M_{n,ij}(k)|1 - \frac{\sup_{I_i \cap \tau_k^{-1}} d_k}{\inf_{I_i} d_k}| \le \frac{M_{n,ij}(k)}{\inf_{I_i} d_k} |\sup_{I_i \cap \tau_k^{-1}} d_k - \inf_{I_i} d_k| \\
&\le \frac{M_{n,ij}(k)}{\inf_{I_i} d_k} |\sup_{I_i} d_k - \inf_{I_i} d_k| \le \frac{M_{n,ij}(k)}{\inf_I d_k} \mathrm{Lip}\,(d_k) . \frac{1}{2n} \,,
\end{aligned}$$

where Lip (d_k) is as before.

Now,

$$\begin{aligned}
\|\hat{M}_n(k) - M_n(k)\|_m &= \max_{1\le i\le n} \sum_{j=1}^{n} |M_{n,ij}(k) - \hat{M}_{n,ij}(k)| \\
&\le \max_{1\le i\le n} \sum_{j=1}^{n} \frac{M_{n,ij}(k)}{\inf_I d_k} \mathrm{Lip}\,(d_k) . \frac{1}{2n} = \frac{\mathrm{Lip}\,(d_k)}{\inf_I d_k} . \frac{1}{2n} .
\end{aligned}$$

$$\begin{aligned}
\|\hat{S}_n - S_n\|_m &= \max_{1\le k\le K} \sum_{l=1}^{K} \|\hat{M}_n(k)\mathrm{diag}[Q^n(W_{k,l}] - M_n(k)\mathrm{diag}[Q^n(W_{k,l}]\|_m \\
&= \max_{1\le k\le K} \sum_{l=1}^{K} \left(\max_{1\le i\le n} \sum_{j=1}^{n} |\Big(\hat{M}_{n,ij}(k)\mathrm{diag}[Q^n(W_{k,l}]\Big)_{ij} \right. \\
&\qquad \left. - \left(M_{n,ij}(k)\mathrm{diag}[Q^n(W_{k,l}]\right)_{ij}| \right) \\
&\le \max_{1\le k\le K} \sum_{l=1}^{K} \left(\left(\max_{I_i\in\mathcal{P}^n} n \int_{I_i} W_{k,l} \right) \max_{1\le i\le n} \sum_{j=1}^{n} |\hat{M}_{n,ij}(k) - M_{n,ij}(k)| \right) \\
&\le \max_{1\le k\le K} \sum_{l=1}^{K} \left(\left(\max_{I_i\in\mathcal{P}^n} n \int_{I_i} W_{k,l} \right) \frac{\mathrm{Lip}\,(d_k)}{\inf_I d_k} . \frac{1}{2n} \right) .
\end{aligned}$$

Now,

$$\| \left(I_{Kn} - S_n + S_n^\infty\right)^{-1} \|_m \le 1 + \sum_{N=1}^{\infty} \|S_n^N - S_n^\infty\|_m .$$

Define $\widehat{L^{(n)}}_0 = \{(f_{n,1}, f_{n,2}, \ldots, f_{n,K}) \in \widehat{L^{(n)}} : \sum_{i=1}^{n} f_{n,k}^i = 0$ for all $k = 1, 2, \ldots, K\}$.

$$\begin{aligned}
&\|S_n^N - S_n^\infty\|_m \\
&\quad = \sup_{\hat{f}_{n,0}\in\widehat{L^{(n)}}_0} \frac{\|(S_n^N - S_n^\infty)\hat{f}_{n,0}\|_m}{\|\hat{f}_{n,0}\|_m} = \sup_{\hat{f}_{n,0}\in\widehat{L^{(n)}}_0} \frac{\|S_n^N(\hat{f}_{n,0} - S_n^\infty \hat{f}_{n,0})\|_m}{\|\hat{f}_{n,0}\|_m} \\
&\qquad\qquad \le \|S_n^N|_{\widehat{L^{(n)}}_0}\|_m \sup_{\hat{f}_{n,0}\in\widehat{L^{(n)}}_0} \frac{\|(\hat{f}_{n,0} - S_n^\infty \hat{f}_{n,0})\|_m}{\|\hat{f}_{n,0}\|_m} \\
&\qquad\qquad \le 2\|S_n^N|_{\widehat{L^{(n)}}_0}\|_m = 2 \sup_{\hat{f}_{n,0}\in\widehat{L^{(n)}}_0} \frac{\|[\hat{Q}^n \widehat{P}_T]^N \hat{f}_{n,0}\|_m}{\|\hat{f}_{n,0}\|_m} \\
&\qquad\qquad \le 2 \sup_{\hat{f}_{n,0}\in\widehat{L^{(n)}}_0} \frac{n\|[\hat{Q}^n \widehat{P}_T]^N \hat{f}_{n,0}\|_{\widehat{BV}}}{\frac{\|\hat{f}_{n,0}\|_{\widehat{BV}}}{3}} \le 6n\|\widehat{P}_T|_{\widehat{BV}_0}\|_{\widehat{BV}}^N \le 6n\gamma^N,
\end{aligned}$$

where $\gamma = \frac{1}{2}\left(\sum_{l=1}^K \alpha_l + \sum_{l=1}^K B_l + 1\right)$. Using Lemma 6.11 and Lemma 6.15 of [7], we get,

$$\begin{aligned}
\|(I_{Kn} - S_n + S_n^\infty)^{-1}\|_m &\le 1 + \sum_{N=1}^{\infty} \|S_n^N - S_n^\infty\|_m \\
&= 1 + \sum_{N=1}^{m_n} \|S_n^N - S_n^\infty\|_m + \sum_{N=m_n}^{\infty} \|S_n^N - S_n^\infty\|_m \\
\le 1 + \sum_{N=1}^{m_n} 2 + \sum_{N=m_n}^{\infty} \beta^{[n/m_n]} &= \left(2 + \frac{\beta}{1-\beta}\right) m_n - 1 \\
&\le \inf_{0<\beta<1} \left(2 + \frac{\beta}{1-\beta}\right)\left(\left[\frac{\log(6n/\beta)}{-\log\gamma}\right] + 1\right) - 1\}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\|d - d_n\|_1 \le \|d - Q^{(n)}(d)\|_1 + \|Q^{(n)}(d) - d_n\|_1 &\le \sum_{k=1}^{K} \mathrm{Lip}\,(d_k).\frac{1}{2n} \\
&+ \inf_{0<\beta<1} \left(2 + \frac{\beta}{1-\beta}\right)\left(\left[\frac{\log(6n/\beta)}{-\log\gamma}\right] + 1\right) - 1\}. \quad \square
\end{aligned}$$

6. Numerical Example

In this section we present a numerical example. We use *Maple, Version 9.5* to find the invariant density of the acim for a Markov switching position dependent random map.

Example 6.1. Consider the Markov switching position dependent random map

$$T = \{\tau_1, \tau_2; p_1, p_2; W\},$$

where $\tau_1, \tau_2 : [0,1] \to [0,1]$ are defined by (see [3])

$$\tau_1(x) = 6x^3 - 9x^2 + 8x \pmod 1,$$

$$\tau_2(x) = \begin{cases} x^2 + 3x & , 0 \le x < \frac{-3}{2} + \frac{1}{2}\sqrt{13}, \\ \frac{x - \frac{3}{4}}{\frac{9}{4} - \frac{1}{2}\sqrt{13}} + 1 & , \frac{-3}{2} + \frac{1}{2}\sqrt{13} \le x < \frac{3}{4}, \\ 4x - 3 & , \frac{3}{4} \le x \le 1\,, \end{cases}$$

and the position dependent switching matrix W,

$$W = \begin{bmatrix} W_{1,1}(x) & W_{1,2}(x) \\ W_{2,1}(x) & W_{2,2}(x) \end{bmatrix}$$

is defined by

$$W_{1,1}(x) = \begin{cases} .8 & , 0 \le x < \frac{1}{2}, \\ .2 & , \frac{1}{2} \le x \le 1, \end{cases} \qquad W_{1,2}(x) = \begin{cases} .2 & , 0 \le x < \frac{1}{2}, \\ .8 & , \frac{1}{2} \le x \le 1\,, \end{cases}$$

$$W_{2,1}(x) = \begin{cases} .5 & , 0 \le x < \frac{1}{2}, \\ .2 & , \frac{1}{2} \le x \le 1\,, \end{cases} \qquad W_{2,2}(x) = \begin{cases} .5 & , 0 \le x < \frac{1}{2}, \\ .8 & , \frac{1}{2} \le x \le 1. \end{cases}$$

Notice that $\inf_x |\tau_1'(x)| = 3.5$ and $\inf_x |\tau_2'(x)| = \frac{1}{\frac{9}{4} - \frac{1}{2}\sqrt{13}} = 2.236014146$. We have, for $x \in [0, \frac{1}{2}), \alpha_1 + \alpha_2 = .45714 + .44723 = .90437 < 1$ and for $x \in [\frac{1}{2}, 1], \alpha_1 + \alpha_2 = .71556 + .17889 = .89445 < 1$. Thus, by 3.2, the random map T has an acim. Now we want to approximate the invariant density of the acim using our method described in Section 4. We have a *Maple* program (*Maple 9.5*) that gives, for any positive integer n, the transition matrices

$$\widehat{M}_n(k) = \left(\frac{\lambda(I_i \cap \tau_k^{-1}(I_j))}{\lambda(I_i)} \right)_{1 \le i,j \le n}, \quad k = 1, 2\,,$$

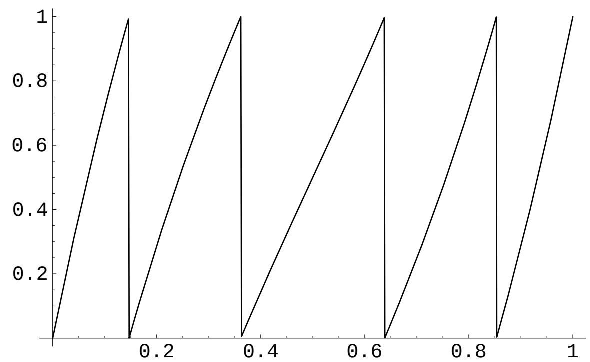

Figure 3: The graph of τ_1

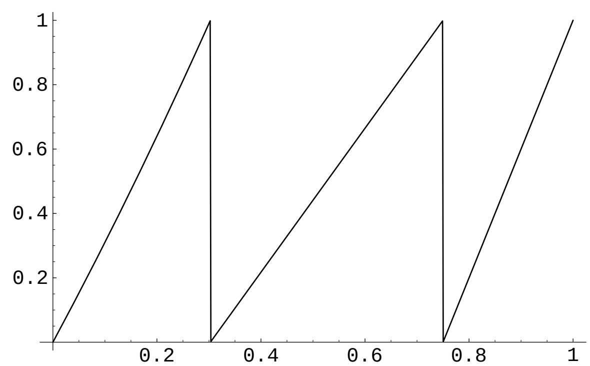

Figure 4: The graph of τ_2

and the $2 \times n$ by $2 \times n$ stochastic matrix

$$\hat{S_n} = \left[\begin{array}{cc} \widehat{M}_n(1)\text{diag}[Q^{(n)}(W_{1,1})] & \widehat{M}_n(1)\text{diag}[Q^{(n)}(W_{1,2})] \\ \widehat{M}_n(2)\text{diag}[Q^{(n)}(W_{2,1})] & \widehat{M}_n(2)\text{diag}[Q^{(n)}(W_{2,2})] \end{array} \right],$$

and finally the left eigenvector of the matrix $\widehat{S_n}$.

Here is a typical example for $n = 8$:

$\widehat{M}_n(1)$

$$= \begin{bmatrix} .12726 & .13199 & .13711 & .14266 & .14867 & .15518 & .15712 & 0 \\ .17840 & .18752 & .19728 & .20792 & .05368 & 0 & .00520 & .17000 \\ .10656 & 0 & 0 & 0 & .16536 & .23088 & .24272 & .25448 \\ .15880 & .27464 & .28144 & .28512 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & .28512 & .28144 & .27464 & .15880 \\ .25448 & .24272 & .23088 & .16536 & 0 & 0 & 0 & .10656 \\ .17000 & .00520 & 0 & .05368 & .20792 & .19728 & .18752 & .17840 \\ 0 & .15712 & .15520 & .14864 & .14264 & .13712 & .13200 & .12728 \end{bmatrix},$$

$$\widehat{M_n}(2)$$

$$= \begin{bmatrix} .32883 & .32028 & .31241 & .03848 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & .26648 & .29816 & .29176 & .14360 & 0 \\ .44720 & .13040 & 0 & 0 & 0 & 0 & .14216 & .28008 \\ 0 & .31688 & .44720 & .23592 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & .21128 & .44720 & .34152 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & .10568 & .44720 & .44712 \\ .25000 & .25000 & .25000 & .25000 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & .25000 & .25000 & .25000 & .25000 \end{bmatrix}.$$

Solving $x \cdot S_n = x$ we get,

$$x = [1.1698, 1.2088, 1.2280, 1.3348, .41270, .41454, .41760, .42095, \\ .64043, .65901, .66676, .76380, 1.6495, 1.6587, 1.6701, 1.6844]$$

and hence the approximate normalized density

$$d_n = [.90544, .93413, .94738, 1.0493, 1.0309, 1.0364, 1.0439, 1.0525].$$

For $n = 16$,

$$d_n = [.89878, .90659, .91053, .91503, .94037, .98519, .98728, 1.1539, \\ 1.0198, 1.0257, 1.0296, 1.0325, 1.0373, 1.0491, 1.0526, 1.0560]. \quad (6.1)$$

For $n = 32$,

$$d_n = [.89625, .90175, .90429, .90745, .90950, .91370, .91559, .91859, \\ .92179, .95654, .98114, .98259, .99025, .99300, 1.1465, 1.1661, \\ 1.0189, 1.0201, 1.0220, 1.0233, 1.0255, 1.0286, 1.0351, 1.0348, \\ 1.0383, 1.0405, 1.0456, 1.0478, 1.0506, 1.0525, 1.0566, 1.0565]. \quad (6.2)$$

For $n = 64$,

$$d_n = [.89415, .89704, .89923, .90345, .90524, .90860, .90712, .90985, .91032, .91079, .91305, .91524, .91688, .91813, .91844, .92024, .92227, .92469, .94149, .96039, .96891, .98172, .98906, .98813, .98891, .99078, .99344, .99414, 1.1225, 1.1633, 1.1666, 1.1657, 1.0123, 1.0182, 1.0173, 1.0208, 1.0226, 1.0253, 1.0238, 1.0258, 1.0241, 1.0280, 1.0272, 1.0302, 1.0332, 1.0340, 1.0329, 1.0337, 1.0366, 1.0428, 1.0389, 1.0422, 1.0440, 1.0471, 1.0449, 1.0472, 1.0494, 1.0532, 1.0515, 1.0523, 1.0544, 1.0584, 1.0557, 1.0585]. \quad (6.3)$$

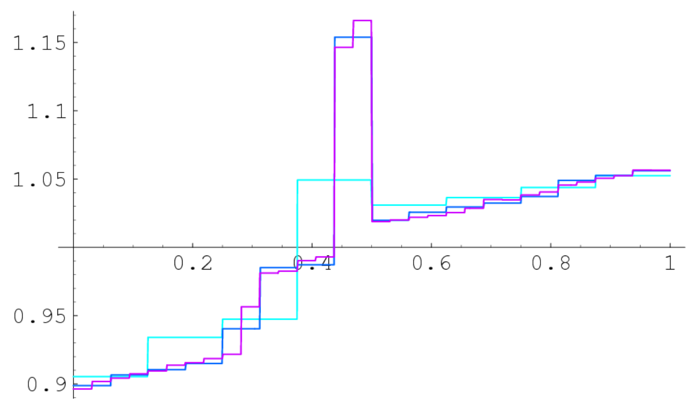

Figure 5: The graph of approximate densities for $n = 8, n = 16$ and $n = 64$

In the Figure 5, the red graph is the approximate density corresponding to partition points 8, the green graph is the approximate density corresponding to partition points 16 and the blue one is the approximate density corresponding to partition points 64.

Now, we plot(see Figure 6) errors to see the convergence rate of our method. In the x direction we consider number of partition points n and in the y direction we consider the difference of the L^1 norm with partition points n and L^1 norm with partition points $2n$.

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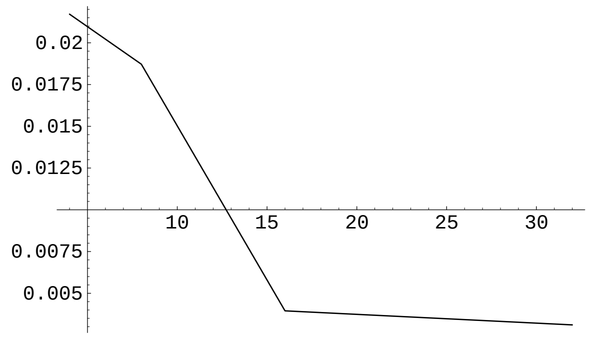

Figure 6: The graph of errors

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