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ON SOME DIFFERENCE SEQUENCE SPACES GENERATED BY INFINITE MATRICES

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Abstract: The sequence spaces $\left(\hat{A},p,\Delta\right)_0$, $\left(\hat{A},p,\Delta\right)$ and $\left(\hat{A},p,\Delta\right)_\infty$ were studied Solak [16]. The main purpose of the present paper is to introduce the spaces $\left(\hat{A},p,\Delta^r\right)_0$, $\left(\hat{A},p,\Delta^r\right)$ and $\left(\hat{A},p,\Delta^r\right)_\infty$ consisting of all sequences whose differences are in the spaces $\left(\hat{A},p,\Delta\right)_0$, $\left(\hat{A},p,\Delta\right)$ and $\left(\hat{A},p,\Delta\right)_\infty$, respectively, and to fill up the gap in the existing literature. Also, we investigate some properties of these spaces. Our results are more general than some theorems of Nanda [13] and Solak [16].

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1. Introduction

Let s be the linear space of all real or complex sequences and let l_∞, c and c_0 denote the Banach spaces of bounded, convergent and null sequences $x = (x_k)$, respectively, normed as usual by $\|x\|_\infty = \sup_{k\geq 0} |x_k|$. Some definitions and conventions are made in this section and some necessary explanations will be given. The shift operator D is defined on s by $(Dx)_n = (x_{n+1})$. A Banach limit L is defined on l_∞, as a non-negative linear functional, such that $L(Dx) = L(x)$ and $L(e) = 1$ [2, p. 32], where $e = (1, 1, \cdots)$. A sequence $x \in l_\infty$ is said to be

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almost convergent if all Banach limits of x coincide [8]. Let $\hat{c}$ denote the set of all almost convergent sequences. Lorentz [8] proved that

$$\hat{c} = \left\{ x : \lim_{m\to\infty} \frac{1}{m+1} \sum_{i=0}^{m} x_{n+i} \text{ exists uniformly in } n \right\}.$$

Several authors including Lorentz [8], King [6] Nanda [11] and Solak [3, 15] have studied almost convergent sequences.

Let $\hat{c}_0$ denote the set of all sequences which are almost convergent to zero. In [12] the spaces $\hat{c}$ and $\hat{c}_0$ were extended to $\hat{c}\,(p)$ and $\hat{c}_0\,(p)$ in the same manner as l_∞, c and c_0 were extended to $l_\infty\,(p)$, $c\,(p)$ and $c_0\,(p)$ respectively [9, 10].

Kızmaz [7] defined the sequence spaces

$$X\,(\Delta) = \{x : \Delta x \in X\}\ ,$$

where $\Delta x = (\Delta x_k) = (x_k - x_{k+1})\,,\ X = l_\infty$, c, or c_0 and showed that these are Banach spaces with norm

$$\|x\|_\Delta = |x_1| + \|\Delta x\|_\infty\,.$$

In [1] the spaces $l_\infty\,(\Delta)$, $c\,(\Delta)$ and $c_0\,(\Delta)$ were extended to $\Delta l_\infty\,(p)$, $\Delta c\,(p)$ and $\Delta c_0\,(p)$. It may be noted here that the spaces

$$\Delta X\,(p) = \{x : \Delta x \in X\,(p)\}\ ,$$

where $X = l_\infty,\ c$, or c_0.

Recently, Et and Çolak [4] generalized the above sequence spaces to the following sequence spaces

$$l_\infty\,(\Delta^r) = \{x : \Delta^r x \in l_\infty\}\,,\quad c\,(\Delta^r) = \{x : \Delta^r x \in c\}\,,$$
$$c_0\,(\Delta^r) = \{x : \Delta^r x \in c_0\}\,,$$

and showed that these spaces are Banach with the norm

$$\|x\|_\Delta = \sum_{i=1}^{r} |x_i| + \|\Delta^r x\|_\infty\,,$$

where r is a positive integer, $\Delta^0 x = (x_k)$, $\Delta^r x = (\Delta^r x_k) = \left(\Delta^{r-1} x_k - \Delta^{r-1} x_{k+1}\right)$ and

$$\Delta^r x_k = \sum_{v=0}^{r} (-1)^v \begin{pmatrix} r \\ v \end{pmatrix} x_{k+v}\ .$$

For convenience Et and Başarır [5] denoted these spaces by $\Delta^r l_\infty$, $\Delta^r c$, $\Delta^r c_0$, and call the constituent sequences Δ^r-bounded, Δ^r-convergent and Δ^r-null sequences, respectively. It is easy to see that $\Delta^r x = \Delta(\Delta^{r-1}x)$ is a bounded linear operator [5]. It is trivial that $c_0(\Delta^r) \subset c_0(\Delta^{r+1})$, $c(\Delta^r) \subset c(\Delta^{r+1})$, $l_\infty(\Delta^r) \subset l_\infty(\Delta^{r+1})$, and $c_0(\Delta^r) \subset c(\Delta^r) \subset l_\infty(\Delta^r)$ are satisfied and strict, [4].

Let $A = (a_{nk})$ be an infinite matrix of nonnegative real numbers and (p_k) be a bounded sequence of positive real numbers (these assumptions are made throughout). We write $B_{mn}(x) = \sum\limits_k a_{mk}x_{k+n}$ if the series converges for each m and n. Here and afterwards summation without limits runs from 1 to ∞.

Nanda [13] defined the sequence spaces

$$\left(\hat{A}, p\right)_0 = \{x : |B_{mn}(x)|^{p_m} \to 0 \text{ as } m \to \infty \text{ uniformly in } n\},$$

$$\left(\hat{A}, p\right) = \Big\{x : |B_{mn}(x - le)|^{p_m} \to 0 \text{ as } m \to \infty \text{ for some } l \text{ uniformly in } n\Big\},$$

$$\left(\hat{A}, p\right)_\infty = \left\{x : \sup_{m,n} |B_{mn}(x)|^{p_m} < \infty\right\}.$$

Finally, the sequence spaces $\left(\hat{A}, p, \Delta\right)_0$, $\left(\hat{A}, p, \Delta\right)$, $\left(\hat{A}, p, \Delta\right)_\infty$ have been defined by Solak [16] as follows

$$\left(\hat{A}, p, \Delta\right)_0 = \{x : |\Delta B_{mn}(x)|^{p_m} \to 0 \text{ as } m \to \infty \text{ uniformly in } n\},$$

$$\left(\hat{A}, p, \Delta\right) = \Big\{x : |\Delta B_{mn}(x - le)|^{p_m} \to 0 \text{ as } m \to \infty \text{ for some } l \text{ uniformly in } n\Big\},$$

$$\left(\hat{A}, p, \Delta\right)_\infty = \left\{x : \sup_{m,n} |\Delta B_{mn}(x)|^{p_m} < \infty\right\},$$

where $\Delta B_{mn}(x) = \sum\limits_k \Delta a_{mk}x_{k+n}$, $(\Delta a_{mk} = a_{mk} - a_{m+1,k})$.

The main object of this paper is to introduce new sequence spaces $\left(\hat{A}, p, \Delta^r\right)_0$, $\left(\hat{A}, p, \Delta^r\right)$, $\left(\hat{A}, p, \Delta^r\right)_\infty$ and investigate some properties these spaces.

Now we define

$$\left(\hat{A}, p, \Delta^r\right)_0 = \{x : |\Delta^r B_{mn}(x)|^{p_m} \to 0 \text{ as } m \to \infty \text{ uniformly in } n\},$$

$$\left(\hat{A}, p, \Delta^r\right) = \left\{x : \left|\Delta^r B_{mn}(x - le)\right|^{p_m} \to 0 \text{ as } m \to \infty \text{ uniformly in } n\right\},$$

$$\left(\hat{A}, p, \Delta^r\right)_\infty = \left\{x : \sup_{m,n} \left|\Delta^r B_{mn}(x)\right|^{p_m} < \infty\right\},$$

where $\Delta^r B_{mn}(x) = \sum_k \Delta^r a_{mk} x_{k+n}$, $\left(\Delta^r a_{mk} = \sum_{v=0}^{r} (-1)^v \binom{r}{v} a_{m+v,k}\right)$.

Note that if $r = 0$ and $A = (c, 1)$ then $\left(\hat{A}, p, \Delta^r\right)_0 = \hat{c}_0(p)$, $\left(\hat{A}, p, \Delta^r\right) = \hat{c}(p)$ and $\left(\hat{A}, p, \Delta^r\right)_\infty = \hat{m}(p)$. The space $\hat{m}(p)$ was introduces and discussed in [14], but it may be remarked here that if $p_m = p$ for all m, then $\hat{m}(p) = l_\infty$.

2. Linear Topological Structure of These Spaces

We again assume thorough that $0 < p_m \leq 1$, for if $0 < p_m < \infty$ and $\sup p_m < \infty$ then $0 < \frac{p_m}{\sup p_m} \leq 1$ and we can, without loss of generality, replace p_m by $\frac{p_m}{\sup p_m}$.

If $p \in l_\infty$, then $\left(\hat{A}, p, \Delta^r\right)_0$, $\left(\hat{A}, p, \Delta^r\right)$ and $\left(\hat{A}, p, \Delta^r\right)_\infty$ are linear spaces over the complex field $\mathbb{C}$. It is clear that $\left(\hat{A}, p, \Delta^r\right)_0 \subset \left(\hat{A}, p, \Delta^r\right)_0$ and $\left(\hat{A}, p, \Delta^r\right)_0 \subset \left(\hat{A}, p, \Delta^r\right)_\infty$. But we have been able to prove $\left(\hat{A}, p, \Delta^r\right) \subset \left(\hat{A}, p, \Delta^r\right)_\infty$ only for a special case. We have the following result.

Theorem 1. *$\left(\hat{A}, p, \Delta^r\right) \subset \left(\hat{A}, p, \Delta^r\right)_\infty$ if*

$$\sup_m \left|\sum_k \Delta^r a_{mk}\right|^{p_m} < \infty. \tag{2.1}$$

Proof. Suppose that $x \in \left(\hat{A}, p, \Delta^r\right)$ and (2.1) holds. We have

$$\begin{aligned} \left|\Delta^r B_{mn}(x)\right|^{p_m} &= \left|\Delta^r B_{mn}(x - le + le)\right|^{p_m} \leq \left|\Delta^r B_{mn}(x - le)\right|^{p_m} \\ &+ \left|l \sum_k \Delta^r a_{mk}\right|^{p_m} \leq \left|\Delta^r B_{mn}(x - le)\right|^{p_m} + \sup_m |l|^{p_m} \left|\sum_k \Delta^r a_{mk}\right|^{p_m}. \end{aligned} \tag{2.2}$$

Therefore $x \in \left(\hat{A}, p, \Delta^r\right)_\infty$ and this completes the proof. □

Theorem 2. *$\left(\hat{A}, p, \Delta^r\right)_0$ is a linear topological space paranormed by g defined by*

$$g(x) = \sup_{m,n} \left|\Delta^r B_{mn}(x)\right|^{p_m}.$$

$\left(\hat{A}, p, \Delta^r\right)_\infty$ is paranormed by g if $\inf p_m > 0$. If (2.1) holds, then $\left(\hat{A}, p, \Delta^r\right)$ is paranormed with the same paranorm g. $\left(\hat{A}, p, \Delta^r\right)_0$ and $\left(\hat{A}, p, \Delta^r\right)_\infty$ are complete in their paranorm topologies. $\left(\hat{A}, p, \Delta^r\right)$ is complete if

$$\left|\sum_k \Delta^r a_{mk}\right|^{p_m} \to 0 \text{ as } m \to \infty. \tag{2.3}$$

Proof. This is routine verification and can be obtained by using standard techniques. If (2.1) holds, then by Theorem (1), $\left(\hat{A}, p, \Delta^r\right) \subset \left(\hat{A}, p, \Delta^r\right)_\infty$ and hence $\left(\hat{A}, p, \Delta^r\right)$ has the same paranorm g. If (2.3) holds, then (2.1) also holds and it follows from the inequality (2.2) that $\left(\hat{A}, p, \Delta^r\right) = \left(\hat{A}, p, \Delta^r\right)_0$ and therefore the completeness of $\left(\hat{A}, p, \Delta^r\right)$ follows from the completeness of $\left(\hat{A}, p, \Delta^r\right)_0$. □

Theorem 3. *If $0 < p_m < q_m \leq 1$, then $\left(\hat{A}, q, \Delta^r\right)_\infty$ is a closed subspace of $\left(\hat{A}, p, \Delta^r\right)_\infty$.*

Proof. Let $x \in \left(\hat{A}, q, \Delta^r\right)_\infty$. Then there exists a constant H such that $\left|\Delta^r B_{mn}(x)\right|^{q_m} \leq H$ for all m, n. This implies that $\left|\Delta^r B_{mn}(x)\right|^{p_m} \leq H$ for all m, n. Thus $x \in \left(\hat{A}, p, \Delta^r\right)_\infty$. To show that $\left(\hat{A}, q, \Delta^r\right)_\infty$ is closed, suppose that $x^i \in \left(\hat{A}, q, \Delta^r\right)_\infty$, $x^i \to x$ and $x \in \left(\hat{A}, p, \Delta^r\right)_\infty$. Then for every ε, $0 < \varepsilon < 1$, there exist for all m, n.

$$\left|\Delta^r B_{mn}\left(x^i - x\right)\right|^{p_m} < \varepsilon \quad \text{for} \quad i > N.$$

Now

$$\left|\Delta^r B_{mn}\left(x^i - x\right)\right|^{q_m} < \left|\Delta^r B_{mn}\left(x^i - x\right)\right|^{p_m} < \varepsilon \text{ for } i > N.$$

Therefore $x \in \left(\hat{A}, q, \Delta^r\right)_\infty$ and this completes the proof. □

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