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# EXAMPLE OF STRICTLY CONVEX METRIC SPACES WITH NOT CONVEX BALLS

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**Abstract:** The paper introduces a notion of strictly convex metric space, a generalization of a well known concept of strictly convex Banach space. We show that there exist strictly convex metric spaces in which closed balls are not convex. The specific example is constructed.

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**Key Words:** convex set, strictly convex Banach space, strictly convex metric space

## 1. Introduction

The notion of convexity plays a very important role in many results in mathematics. Usually we apply the notion of convexity to the subsets of vector spaces; in this case we say that a set $A$ is convex if for arbitrary $x, y \in A$ set $A$ contains a closed segment $[x, y] = \{z | z = tx + (1 - t)y, t \in [0, 1]\}$. This is so called Minkowski definition (see L. Blumenthal [1], p. 40). In vector spaces closed balls are convex and intersection of convex sets is again a convex set. When one considers generalisations of convexity in non-vector spaces, one usually wants

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these two properties to be preserved. Thus, there is a natural possibility, for example in metric space, to consider a class of sets that contains all closed balls and is closed with respect to set intersection, and define all sets from this class to be convex. This idea have been used by several mathematicians, e.g. W. Takahashi [14], J.P. Penot [13], W.A. Kirk [9], [10], V.L. Klee [11], M.R. Taskovic [15], A. Liepiņš [12], T.H. Kim and W.A. Kirk [8], C.D. Horvath [6] and others.

In this paper we want to demonstrate the difficulties one encounters attempting to generalise the notion of convexity to metric spaces by the means of metric relations and without the explicit selection of class of convex sets. It turns out, that in such case it is a real possibility that there will exist metric spaces in which closed balls are not convex or intersection of convex sets is not convex. To illustrate this we will consider the notion of strictly convex metric space (which is a generalisation of strictly convex normed vector space) in which intersection of convex sets is indeed convex, but we can show an example of such space in which not all closed balls are convex. Thus, normed vector spaces apparently possess some specific property not covered by the notion of strict convexity.

## 2. Definitions and Some Properties

Now we consider the class of strictly convex spaces introduced by Clarkson and by Krein (by V.I. Istratescu [7]).

**Definition 2.1.** A Banach space $X$ is said to be *strictly convex* (or rotund) if the boundary of its unit ball contains no line segments.

Many equivalent characterizations of strict convexity are known. For example, M.M. Day ([4], p. 111) used a following definition: a unit ball $B$ in the Banach space is rotund if every open segment in $B$ is disjoint from boundary of $B$ (see also other properties [4], p. 112).

V.I. Istratescu [7] has proved that in Banach space $X$ the following conditions are equivalent:

1. $X$ is a strictly convex space;

2. $\forall x, y \in B(0,1),\ x \neq y: \quad \|x+y\| < 2$ (with $B(a,r)$ we denote closed ball with center $a$ and radius $r$);

3. $\forall x, y \in X: \|x+y\| = \|x\| + \|y\| \Longrightarrow ((\exists \lambda > 0 : x = \lambda y) \bigvee (x = 0) \bigvee (y = 0))$.

Now we consider situation in metric spaces.

Let $(X, d)$ be a metric space with metric $d$.

**Definition 2.2.** (see I. Galiņa [5], I. Bula [2]) A set $K \subset X$ is said to be *convex* if for each $x, y \in K$ and for each $t \in [0,1]$ there exists $z \in K$ that satisfies $d(x,z) = td(x,y)$ and $d(z,y) = (1-t)d(x,y)$ and the set

$$\{z \in K \mid d(x,z) = td(x,y),\ d(z,y) = (1-t)d(x,y),\ t \in [0,1]\}$$

is said to be a *segment* joining $x$ and $y$ (or shortly segment $xy$).

We note that by means of this Definition 2.2 closed balls may be non-convex sets and intersection of convex sets may be a non-convex set (see, for example, situations in discrete metric space and in space $\mathbf{R^2}$ with maximum metric $d(x,y) = \max\{|x_i - y_i|\,|i = 1,2\}$, I. Galiņa [5]). Therefore we define strictly convex metric space in following way.

**Definition 2.3.** (see I. Galiņa [5], I. Bula [2]) A metric space $(X,d)$ is said to be *strictly convex* if for each $x, y \in X$ and for each $t \in [0,1]$ there exists a *unique* $z \in X$ that satisfies

$$d(x,z) = td(x,y) \ \text{ and } \ d(z,y) = (1-t)d(x,y).$$

This is not new original definition; we can find it in bibliography, for example, in W. Takahashi [14]. R.J. Bumcrot [3] has shown that $X$ is strictly convex if and only if metric convexity is equivalent to convexity in $X$ (a subset $A \subset X$ is metric convex if any two points $a, b \in A$ are contained in a subset of $A$ that is isometric to be the real interval $[0, \|b-a\|]$). The fact that there exists "unique metric segment" is equivalent to usual strict convexity in normed space has already been shown in 1967 (R.J. Bumcrot [3]). In 1992 the author of this article has proved (I. Galiņa [5]) that the following conditions are equivalent in Banach space $X$

1. $\forall x, y \in X\colon\ \|x+y\| = \|x\| + \|y\| \Longrightarrow ((\exists \lambda > 0\colon x = \lambda y) \bigvee (x = 0) \bigvee (y = 0))$;

2. $\forall x, y \in X\ \forall t \in [0,1]\ \exists! z \in X\colon \|x-z\| = t\|x-y\|,\ \|z-y\| = (1-t)\|x-y\|$.

Since the first condition is equivalent with concept of strictly convex Banach space, we conclude that strictly convex Banach space indeed is strictly convex metric space in particular case.

It is easy to prove that intersection of convex sets (in the sence of Definition 2.2) is convex set in strictly convex metric space (I. Galiņa [5]). But there exist strictly convex metric spaces in which closed balls are not convex.

## 3. Example of Strictly Convex Metric Space with

## Not Convex Closed Balls

The set $X$ consists of points of an unbounded plane sector with an angle stictly less than 2 radians. Let one side of this sector lies on the axis of polar coordinates and the origin of angle coincides with the origin of axis. We choose two arbitrary points $x=(r_1,\varphi_1)$ and $y=(r_2,\varphi_2)$ (in polar coordinates, i.e. $(r,\varphi)$ means, that $r$ is radius and $\varphi$ is angle (see Figure 1)).

The distance between $x$ and $y$ is defined by equality

$$d(x,y) := |\varphi_1-\varphi_2|\min\{r_1,r_2\}+|r_1-r_2|.$$

In Figure 1 the distance between $x$ and $y$ is equal to the length of thick curve.

Now we prove that this distance is metric in set $X$. The first two axioms of metric

1) $\forall x,y\in X:\ d(x,y)\geq 0 \quad \text{and} \quad d(x,y)=0 \ \Leftrightarrow\ x=y,$
2) $\forall x,y\in X:\ d(x,y)=d(y,x)$

obviously are satisfied. Is it true that

$$\forall x,y,z\in X:\ d(x,y)\leq d(x,z)+d(z,y)\ ?$$

Since length of a circular arc grows with the increase of its radius, then triangle inequality is true if coordinates $(r_3,\varphi_3)$ of $z$ are such that

$$r_3\geq \min\{r_1;r_2\}.$$

We note that $d(x,y)=d(x,z)+d(z,y)$ only if $r_3=\min\{r_1;r_3\}$ and $\min\{\varphi_1;\varphi_2\}\leq\varphi_3\leq\max\{\varphi_1;\varphi_2\}$; $\varphi_3=\max\{\varphi_1;\varphi_2\}$ and $\min\{r_1;r_2\}\leq r_3\leq\max\{r_1;r_2\}$, i.e. $z$ belongs to segment joining points $x$ and $y$ (in Figure 1 segment $xy$ is represented by thick curve).

Let $r_3<\min\{r_1;r_2\}$. We consider two possible cases.

*Case A.* $\min\{\varphi_1;\varphi_2\}\leq\varphi_3\leq\max\{\varphi_1;\varphi_2\}$. Let $\varphi_1\leq\varphi_2$ and $r_1\leq r_2$ (another cases are similar), then we have $\varphi_1\leq\varphi_3\leq\varphi_2$.

We can prove that the following inequality holds

$$\begin{aligned} d(x,y)=(\varphi_2-\varphi_1)r_1+(r_2-r_1)\leq d(x,z)+d(z,y))\\ =(\varphi_3-\varphi_1)r_3+(r_1-r_3)+(\varphi_2-\varphi_3)r_3+(r_2-r_3).\end{aligned}$$

Since $r_2-r_3=(r_2-r_1)+(r_1-r_3)$ then

$$\begin{aligned} 0\leq -(\varphi_2-\varphi_1)r_1+\varphi_3r_3-\varphi_1r_3+\varphi_2r_3-\varphi_3r_3+2(r_1-r_3)\\ =(\varphi_2-\varphi_1)(r_3-r_1)+2(r_1-r_3)=(r_1-r_3)(2-(\varphi_2-\varphi_1)).\end{aligned}$$

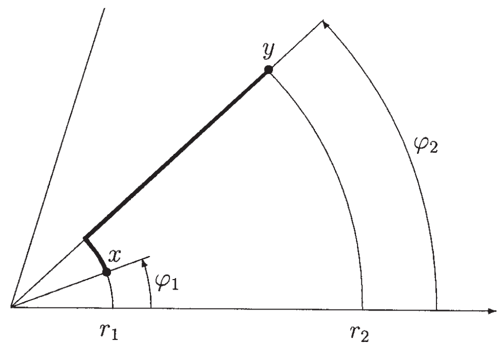


Figure 1:

The last inequality is always true, at the same time it is strict inequality because $r_1 > r_3$ and $\varphi_2 - \varphi_1 < 2$ (we have assumed that $X$ is strictly smaller as 2 radian).

*Case B.* $\varphi_3 < \min\{\varphi_1; \varphi_2\}$ or $\varphi_3 > \max\{\varphi_1; \varphi_2\}$.

We consider situation $\varphi_3 > \varphi_2 \geq \varphi_3$ and $r_1 \leq r_2$, another sitations are similar. In this situation we can prove that

$$(\varphi_2 - \varphi_1)r_1 + (r_2 - r_1) \leq (\varphi_3 - \varphi_1)r_3 + (r_1 - r_3) + (\varphi_3 - \varphi_2)r_3 + (r_2 - r_3).$$

Since $r_2 - r_3 = (r_2 - r_1) + (r_1 - r_3)$ and $\varphi_3 - \varphi_1 = (\varphi_3 - \varphi_2) + (\varphi_2 - \varphi_1)$ then

$$\begin{aligned}
&0 \leq (\varphi_3 - \varphi_2)r_3 + (\varphi_2 - \varphi_2 - \varphi_1)r_3 + (\varphi_3 - \varphi_2)r_3 \\
&- (\varphi_2 - \varphi_1)r_1 + 2(r_1 - r_3) = 2(\varphi_3 - \varphi_2)r_3 + (\varphi_2 - \varphi_1)(r_3 - r_1) + 2(r_1 - r_3) \\
&\qquad = 2(\varphi_3 - \varphi_2)r_3 + (r_1 - r_3)(2 - (\varphi_2 - \varphi_1)).
\end{aligned}$$

The last ineqality is always true and it is strict inequality because $\varphi_3 > \varphi_2$, $r_1 > r_3$ and $\varphi_2 - \varphi_1 < 2$ ($\varphi_1$ and $\varphi_2 < 2$).

Therefore $(X, d)$ is a metric space.

The space $(X, d)$ is strictly convex metric space, i.e. for every $x, y \in X$ and for every $t \in [0; 1]$ there exists a unique $z \in X$ such that $d(x, z) = td(x, y)$ and $d(z, y) = (1 - t)d(x, y)$ – this $z$ belongs to segment joining points $x$ and $y$, the uniqueness follows from the proof of triangle ineqality.

The closed ball $B(A, \rho) = \{x \in X | d(A, x) \leq \rho\}$ in the space $(X, d)$ is a convex set, for example, if $A = (0, 0)$. But an arbitrary choosen ball is not necessarily a convex set. For example, we consider ball with center $A = (r_0, \varphi_0) \in X$ and radius $\rho$ ($r_0 > \rho > 0$) such that boundary points of ball $K = (r_0 - \rho, \varphi_0)$ and $L = (r_0, \varphi_0 + \frac{\rho}{r_0})$ belong to $X$ (see Figure 2). In this case ball $B(A, \rho)$ is not a convex set – the segment joining points $K$ and $L$ does

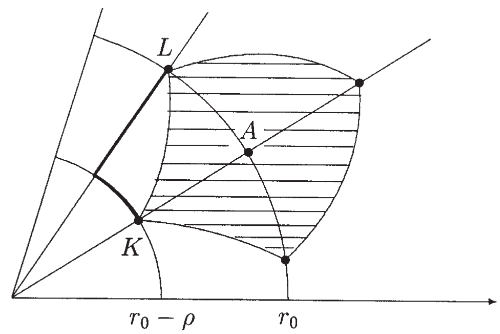


Figure 2:

not belong to ball $B(A,\rho)$ (in Figure 2 segment $KL$ is drawn by thick curve and ball $B(A,\rho)$ is represented by the striped area). Since $r_0 > \rho > 0$ then there exist points from segment $KL$ that belong to the same arc as point $K$, i.e. coordinates of these points are $(r_0-\rho;\alpha)$, where $\varphi_0 < \alpha < \varphi_0 + \frac{\rho}{r_0}$, and the distances between there points and center $A$ are

$$d((r_0,\varphi_0),(r_0-\rho,\alpha))$$
$$|\varphi_0-\alpha|(r_0-\rho)+|r_0-(r_0-\rho)| = |\varphi_0-\alpha|(r_0-\rho)+\rho > \rho.$$

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