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SOME NOTES ON GENERALIZED LIE IDEALS

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Abstract: In [3], I.N. Herstein has proved if R is a prime ring and T is a Lie ideal of R such that $[T,T] \subset Z$ then $T \subset Z$. In the first part of this note the above theorem is generalized for $(\sigma,\tau)-$left Lie ideal U of prime ring. In the second part, some results given for one sided $(\sigma,\tau)-$left Lie ideals of prime rings.

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1. Introduction

Let R be a ring and σ, τ be two mappings from R into itself. For any two elements $x, y \in R$, we denote $[x,y] = xy - yx$ and $[x,y]_{\sigma,\tau} = x\sigma(y) - \tau(y)x$. An additive mapping $d : R \to R$ is called a derivation if $d(xy) = d(x)y + xd(y)$ holds for pairs $x, y \in R$. We use the identities $[x, yz] = [x,y]z + x[y,z]$. A derivation d is inner if there exists an $a \in R$ such that $D(x) = [a,x]$ holds for all $x \in R$. Recall that a ring is prime if $aRb = \{0\}$ implies that $a = 0$ or $b = 0$. For subsets $A, B \subset R$, let $[A,B]$ ($[A,B]_{\sigma,\tau}$) be the additive subgroup generated by all $[a,b]$ ($[a,b]_{\sigma,\tau}$) for all $a \in A$ and $b \in B$. We recall that a Lie ideal, L is an additive

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subgroup of R such that $[R,L] \subset L$. We first introduce the generalized Lie ideal in [5] as following. Let U be an additive subgroup of R, $\sigma,\tau : R \to R$ two mappings. Then: (i) U is a (σ,τ)-right Lie ideal of R if $[U,R]_{\sigma,\tau} \subset U$; (ii) U is a (σ,τ)-left Lie ideal of R if $[R,U]_{\sigma,\tau} \subset U$; (iii) U is both a (σ,τ)-right Lie ideal and (σ,τ)-left Lie ideal of R then U is a $(\sigma,\tau)-$Lie ideal of R. Every Lie ideal of R is a $(1,1)-$left Lie ideal of R, where $1 : R \to R$ is the identity map. As an example, let I be the set of integers,

$$R = \left\{ \begin{pmatrix} x & y \\ z & t \end{pmatrix} \mid x,y,z,t \in I \right\}, \quad U = \left\{ \begin{pmatrix} x & y \\ 0 & x \end{pmatrix} \mid x,y \in I \right\} \subset R,$$

and $\sigma,\tau : R \to R$ the mappings defined by $\tau(x) = axa$, $\sigma(x) = bxb^{-1}$, where

$$a = \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \in R.$$

Then U is a (σ,τ) left Lie ideal but not a Lie ideal of R.

In [3], I. N. Herstein has proved if R is prime ring and T is a Lie ideal of R such that $[T,T] \subset Z$ then $T \subset Z$. This result is generalized for (σ,τ)-right Lie ideals [5]. In the first part of this paper we shall generalize the above theorem for (σ,τ)-left Lie ideal U of prime ring. In the second part, some results will be given for one-sided $(\sigma,\tau)-$Lie ideals of prime rings.

Throughout the present paper R will be a prime ring of with characteristic not two and $\sigma,\tau \in AutR$. We set $C_{\sigma,\tau} = \{c \in R \mid c\sigma(x) = \tau(x)c, \text{ for all } x \in R\}$ and call $(\sigma,\tau)-$center of R. Furthermore we shall use the following indentities:

$$[xy,z]_{\sigma,\tau} = x[y,z]_{\sigma,\tau} + [x,\tau(z)]y\,, \tag{1}$$

$$[xy,z]_{\sigma,\tau} = x[y,\sigma(z)] + [x,z]_{\sigma,\tau}y\,, \tag{2}$$

$$[x,yz]_{\sigma,\tau} = \tau(y)[x,z]_{\sigma,\tau} + [x,y]_{\sigma,\tau}\sigma(z)\,. \tag{3}$$

2. Results

Theorem 1. *Let R be a prime ring of characteristic not two, $(0) \neq U$, $(\sigma,\tau)-$left Lie ideal of R and $a \in R$. If $[U,a]_{\sigma,\tau} = 0$ then $a = 0$ or $\sigma(u) + \tau(u) \in Z$, for all $u \in U$.*

Proof. For any $x \in R, u \in U$, using the identity [1], we have $[\tau(u)x,u]_{\sigma,\tau} = \tau(u)[x,u]_{\sigma,\tau} \in U$. Hence

$$0 = [\tau(u)[x,u]_{\sigma,\tau},a]_{\sigma,\tau} = \tau(u)[[x,u]_{\sigma,\tau},a]_{\sigma,\tau} + [\tau(u),\tau(a)][x,u]_{\sigma,\tau}$$

and so

$$\tau([u,a])[x,u]_{\sigma,\tau} = 0, \quad \forall x \in R, u \in U. \tag{4}$$

Replacing x by $xy, y \in R$ in [4] and using the identity [1] and the equation [4], we have

$$0 = \tau([u,a])[xy,u]_{\sigma,\tau} = \tau([u,a])x[y,\sigma(u)] + \tau([u,a])[x,u]_{\sigma,\tau}y$$

yields that

$$\tau([u,a])x[y,\sigma(u)] = 0, \quad \forall x \in R, u \in U. \tag{5}$$

Since R is prime ring, [5] implies that for all $u \in U$,

$$[u,a] = 0 \quad \text{or} \quad u \in Z.$$

We set $K = \{u \in U \mid u \in Z\}$ and $L = \{u \in U \mid [u,a] = 0\}$. Clearly each of L and K is additive subgroup of U. Morever, U is the set-theoretic union of L and K. But a group cannot be the set-theoretic union of two proper subgroups, hence $K = U$ or $L = U$. In the first case $U \subset Z$ which forces $\sigma(u) + \tau(u) \in Z$, for all $u \in U$. In the latter case, $[U,a] = 0$. By [1, Lemma 6] we get $a \in Z$ or $\sigma(u) + \tau(u) \in Z$, for all $u \in U$. □

Corollary 1. *Let R be a prime ring of characteristic not two, $(0) \neq U$, $(\sigma,\tau)-$left Lie ideal of R. If $[U,U]_{\sigma,\tau} = 0$ then $\sigma(u) + \tau(u) \in Z$, for all $u \in U$.*

Lemma 1. *Let R be a prime ring of characteristic not two, $a \in R$ and d the additive mapping on R defined by $d(x) = [x,a]_{\sigma,\tau}$. If $d^2(R) = 0$ then $a \in Z$.*

Proof. For any $x, y \in R$, using the identity (2), we have

$$d(xy) = [xy,a]_{\sigma,\tau} = x[y,\sigma(a)] + [x,a]_{\sigma,\tau}y.$$

That is

$$d(xy) = x[y,\sigma(a)] + d(x)y, \quad \forall x,y \in R. \tag{6}$$

By the hypothesis,

$$\begin{aligned} 0 &= d^2(xy) = d(d(xy)) = d(d(x)y + x[y,\sigma(a)]) \\ &= d^2(x)y + 2d(x)[y,\sigma(a)] + x[[y,\sigma(a)],\sigma(a)]. \end{aligned}$$

Therefore

$$2d(x)[y,\sigma(a)] + x[[y,\sigma(a)],\sigma(a)] =, \quad \forall x,y \in R. \tag{7}$$

Taking $d(x)$ instead of x in [7] and using the hypothesis, we obtain

$$d(x)[[y,\sigma(a)],\sigma(a)] = 0, \forall x,y \in R.$$

It follows from [7, Lemma 1(i)] that

$$a \in Z \quad \text{or} \quad [[y, \sigma(a)], \sigma(a)] = 0, \forall y \in R.$$

Assume that $[[y, \sigma(a)], \sigma(a)] = 0,$ for all $y \in R$. Let us consider the following map on R. $I_{\sigma(a)} = [x, \sigma(a)]$ is an inner derivation determined by $\sigma(a)$. It is easy to see that $I^2_{\sigma(a)}(R) = 0$. By [8, Theorem 1] we have $I_{\sigma(a)} = 0$, and so $a \in Z$. □

The following result is generalization of Lemma 3 in [3] mentioned in the introduction.

Theorem 2. *Let R be a prime ring of characteristic not two, $(0) \neq U$, $(\sigma, \tau)-$left Lie ideal of R. If $[U, U]_{\sigma,\tau} \subset C_{\sigma,\tau}$ then $U \subset Z$.*

Proof. Since U is $(\sigma, \tau)-$left Lie ideal, for any $x \in R, u \in U, \tau(u)[x, u]_{\sigma,\tau} \in U$. Using the identity (1), we have

$$[\tau(u)[x, u]_{\sigma,\tau}, u]_{\sigma,\tau} = \tau(u)[[x, u]_{\sigma,\tau}, u]_{\sigma,\tau} \in C_{\sigma,\tau}. \tag{8}$$

Since $[[x, u]_{\sigma,\tau}, u]_{\sigma,\tau} \in C_{\sigma,\tau}$ by [6, Lemma 6], the equation (8) implies that

$$[[x, u]_{\sigma,\tau}, u]_{\sigma,\tau} = 0 \quad \text{or} \quad u \in Z, \forall x \in R, u \in U. \tag{9}$$

If $[[x, u]_{\sigma,\tau}, u]_{\sigma,\tau} = 0$, for all $x \in R$, we define the mapping on R by $d(x) = [x, u]_{\sigma,\tau}$. It follows that $d^2(R) = 0$. Appliying Lemma 1, we have $u \in Z$. Therefore [9] implies that $U \subset Z$. □

Theorem 3. *Let R be a prime ring of characteristic not two, $(0) \neq U$, $(\sigma, \tau)-$left Lie ideal of $R, a, b \in R$ and $f : R \to R$ a map defined by $f(x) = xa - bx$. If $f(U) = 0$ then $\sigma(u) + \tau(u) \in Z$, for all $u \in U$ or $f = 0$.*

Proof. Suppose that a or b in Z. Since $f(U) = 0$ and $U \neq (0)$, we have $U(a - b) = 0$ and so by [7, Lemma 1(iii)], we get $a = b$ or $U \subset Z$. If $a = b$ then f becomes the inner derivation determined by a. Hence, since $f(U) = 0$, by [1, Lemma 6] we have $f = 0$ or $\sigma(u) + \tau(u) \in Z$, for all $u \in U$. This complates the proof. □

Now assume that neither a nor b in Z. From the definition of U, using the identity (1), we get for any $x \in R, u \in U, \tau(u)[x, u]_{\sigma,\tau} \in U$. Appliying the hypothesis

$$0 = \tau(u)[x, u]_{\sigma,\tau}a - b\tau(u)[x, u]_{\sigma,\tau}$$

and so

$$[\tau(u), b][x, u]_{\sigma,\tau} = 0, \quad \forall x \in R, u \in U. \tag{10}$$

Replacing x by $xy, y \in R$ in (10) and using the identity (2), the equation (10), one obtains

$$[\tau(u), b]x[y, \sigma(u)] = 0, \quad \forall x, y \in R, u \in U.$$

Since R is prime ring, it follows either $u \in Z$ or $[\tau(u), b] = 0$, for all $u \in U$. By a standart argument one of these must hold for all $u \in U$. If $u \in Z$ then $[\tau(u), b] = 0$, for all $u \in U$. Therefore we have

$$[U, \tau^{-1}(b)] = 0.$$

Appliying [1, Lemma 6] and nothing that $b \notin Z$, one obtains $\sigma(u) + \tau(u) \in Z$, for all $u \in U$.

Theorem 4. *Let R be a prime ring of characteristic not two, $(0) \neq U$, $(\sigma,\tau)-$right Lie ideal of $R, a, b \in R$ and $f : R \rightarrow R$, $f(x) = xa - bx$. If $f(U) = 0$ then $U \subset C_{\sigma,\tau}$ or $f = 0$.*

Proof. If a or b in Z, then from the hypothesis $U(a-b) = 0$. Since $U \neq (0)$, it follows from [2, Lemma 3(ii)] that $U \subset C_{\sigma,\tau}$ or $a = b$. If $a = b$ then f becomes the inner derivation determined by a (or b). By [2, Theorem 4] we have $f = 0$ or $U \subset C_{\sigma,\tau}$. This complates the proof. □

Now assume that neither a nor b in Z. Since U is $(\sigma,\tau)-$ right Lie ideal, for any $y, z \in R$, $u \in U, [u, yz]_{\sigma,\tau} \in U$. Using the identity (3) and the hypothesis, we obtain

$$\begin{aligned} 0 &= [u, yz]_{\sigma,\tau} a - b[u, yz]_{\sigma,\tau} \\ &= \tau(y)[u, z]_{\sigma,\tau} a + [u, y]_{\sigma,\tau}\sigma(z)a - b\tau(y)[u, z]_{\sigma,\tau} - b[u, y]_{\sigma,\tau}\sigma(z)\,, \end{aligned}$$

and so

$$[\tau(y), b][u, z]_{\sigma,\tau} + [u, y]_{\sigma,\tau}[\sigma(z), a] = 0, \quad \forall x, y, z \in R, u \in U.$$

Substituting z by $\sigma^{-1}(a)$ in this equation, we get

$$[\tau(y), b][u, \sigma^{-1}(a)]_{\sigma,\tau} = 0, \quad \forall y \in R, u \in U. \tag{11}$$

Replacing y by $yx, x \in R$ in (11) and using this, we get

$$[\tau(x), b]\tau(y)[u, \sigma^{-1}(a)]_{\sigma,\tau} = 0, \quad \forall x, y \in R, u \in U. \tag{12}$$

Since R is prime ring and $b \notin Z$, (12) implies that

$$[U, \sigma^{-1}(a)]_{\sigma,\tau} = 0.$$

By [2, Lemma 2], we get $U \subset C_{\sigma,\tau}$.

Theorem 5. *Let R be a prime ring of characteristic not two, $(0) \neq U$, $(\sigma,\tau)-$right Lie ideal of R and $a \in R$. If $[U, a]_{\sigma,\tau} = 0$ then $a \in Z$ or $U \subset C_{\sigma,\tau}$.*

Proof. Let us consider the following map on R, $f(x) = [x,a]_{\sigma,\tau} = x\sigma(a) - \tau(a)x$. It is easy to see, by hypothesis, $f(U) = 0$. By Theorem 4, we get $f = 0$ or $U \subset C_{\sigma,\tau}$. If $f = 0$ then use the identity (1), we have

$$0 = f(xy) = [xy,a]_{\sigma,\tau} = x[y,a]_{\sigma,\tau} + [x,\tau(a)]y$$

and so

$$[x,\tau(a)]y = 0, \quad \forall x,y \in R.$$

Since R is prime ring, we obtain that $a \in Z$. □

Corollary 2. *Let R be a prime ring of characteristic not two, $(0) \neq U$, $(\sigma,\tau)-$right Lie ideal of R and $a \in R$. If $[U,U]_{\sigma,\tau} = 0$ then $U \subset Z$ or $U \subset C_{\sigma,\tau}$.*

Theorem 6. *Let R be a prime ring of characteristic not two and f,g be nonzero derivations of R such that $uf(x) = g(x)u$, for all $x \in R, u \in U$.*

(i) If U is $(\sigma,\tau)-$left Lie ideal of R then $\sigma(u) + \tau(u) \in Z$, for all $u \in U$.

(ii) If $(\sigma,\tau)-$right Lie ideal of R then $U \subset Z$ or $U \subset C_{\sigma,\tau}$.

Proof. Taking xy instead of x in the hypothesis, one obtains

$$0 = uf(xy) - g(xy)u = uf(x)y + uxf(y) - g(x)yu - xg(y)u$$

and so

$$g(x)[u,y] + [u,x]f(y) = 0, \quad \forall x,y \in R, u \in U. \tag{13}$$

Replacing y by $yu, u \in U$ in (13) and using (13), we obtain

$$\begin{aligned} 0 &= g(x)[u,y]u + [u,x]f(y)u + [u,x]yf(u) \\ &= (g(x)[u,y] + [u,x]f(y))u + [u,x]yf(u)\,, \end{aligned}$$

and so

$$[u,x]Rf(u) = 0, \quad \forall x \in R, u \in U. \tag{14}$$

Since R is prime ring, (14) implies that

$$u \in Z \quad \text{or} \quad f(u) = 0.$$

Now let us define the set $K = \{u \in U \mid u \in Z\}$ and $L = \{u \in U \mid f(u) = 0\}$. Clearly each of L and K is additive subgroup of U. Morever, U is the set-theoretic union of L and K. By Brauer's Trick, we must have $U = K$ or $U = L$.

(i) Let assume that U is a $(\sigma,\tau)-$left Lie ideal. In the former case $U \subset Z$ and so we have

$$0 = uf(x) - g(x)u = u(f(x) - g(x)).$$

Since R is prime ring, $U \neq (0)$, it implies that $f(x) = g(x)$, for all $x \in R$. Hence, from the hypothesis we get

$$0 = uf(x) - f(x)u = [u, f(x)], \quad \forall x \in R, u \in U.$$

By [3, Theorem] we conclude that $U \subset Z$, and so $\sigma(u) + \tau(u) \in Z$, for all $u \in U$.

In the latter case, $f(U) = 0$ then $\sigma(u) + \tau(u) \in Z$, for all $u \in U$, by [6, Lemma 3].

(ii) Let assume that U is a $(\sigma,\tau)-$right Lie ideal. If $U = K$ then it is obvious that $U \subset Z$. If $U = L$ then $U \subset C_{\sigma,\tau}$ by [9, Lemma 2] or R is commutative. If R is commutative, then $U \subset Z$. Thus we complate the proof of the theorem.

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