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ON AN INVERSE PROBLEM FOR SEMILINEAR PARABOLIC EQUATION

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Abstract: In this study, inverse problems have been investigated for determination of the function which is not related to the one of space coordinates of semilinear parabolic equation in the right hand side. Moreover, theorems which are related to existence, uniqueness and stability of solution of considered problem have been proved.

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1. Introduction

Constructed problems to find one or a few of coefficients for differential equations with partial derivative is called coefficient inverse problems. If searched coefficient or coefficients depend on variable in the number of $n \geq 2$ then this kind of inverse problems are called coefficient inverse problems with several dimensionals.

Uniqueness problem of solution is so important in the coefficient inverse problems. Uniqueness theorems of solution guarantee the result of suitable physical experiments for this kind of problems. Moreover, if solution of inverse problem is searched in a compact set then also uniqueness theorems guarantee the stability of solution. For this reason, investigation of conditional stability for inverse problems that is evaluated of continuity module of inverse operator is important.

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2. Uniqueness Theorems and Stability

Let us assume the following notations:

Denote by D', bounded region of R^{n-1} and $D = D' \times (r_1, r_2)$, where r_1, r_2 are any real numbers, $x' = (x_1, ..., x_{n-1})$, $x = (x', x_n)$ are respectively any points of the regions D' and D; and denote $Q = D \times (0, T]$, $Q' = D' \times (0, T]$, $S = \partial D \times [0, T]$, $0 < T =$const.

Let us consider the following problem to determinate of the pair $\{f(x', t), u(x, t)\}$:

$$u_t - Lu = f(x', t) g(u), \quad (x, t) \in Q\,, \tag{1}$$

$$u(x, 0) = \varphi(x), \quad x \in \overline{D} = D \cup \partial D\,,\ u(x, t) = \psi(x, t)\,,\ (x, t) \in S\,, \tag{2}$$

$$u(x', r_0, t) = h(x', t)\,, \quad r_0 \in (r_1, r_2)\,,\ (x', t) \in Q'\,, \tag{3}$$

where

$$u_t = \frac{\partial u}{\partial t}\,, \quad u_{x_i x_j} = \frac{\partial^2 u}{\partial x_i \partial x_j}\,,$$

$$Lu = \sum_{i,j=1}^{n} a_{ij}(x, t)\, u_{x_i x_j} + \sum_{i=1}^{n} b_i(x, t) u_{x_i} + c(x, t) u\,.$$

As known that this kind of problems are not well established problems in the sense of Adamar and have been studyed for example in [5], [4], [7], [8], [9], [1], etc. If $f(x', t)$ is a given function in the equation (1) then the condition (3) will not be given as naturally.

Investigation of existence of the solution of the problem (1)-(2) is obvious as generally. For example; it has been observed in [6], [2], [3], etc. We shall assume the following assumptions related to the data of problem:

For any real vector $\nu = (\nu_1, ..., \nu_n)$ and any arbitrary

$$(x, t) \in \overline{Q}\,, \quad m_0 \sum_{i=1}^{n} \nu_i^2 \le \sum_{i,j=1}^{n} a_{ij}(x, t) \nu_i \nu_j \le m_1 \sum_{i=1}^{n} \nu_i^2, \quad 0 < m_0 < m_1;$$

are satisfied the following hypotheses:

1) $a_{ij}(x, t), \quad b_i(x, t),\ c(x, t) \in C^{\alpha, \alpha/2}(\overline{Q}), \quad i, j = \overline{1, n}.$

2) $g(\cdot) \in Lip_{(loc)}(R^1),\ |g(\cdot)| \ge m_2 > 0.$

3) $\varphi(x) \in C^{2+\alpha}(\overline{D}),\ \psi(x, t) \in C^{2+\alpha, 1+\alpha/2}(S), \varphi(x) = \psi(x, 0),\ x \in \partial D.$

4) $h(x', t) \in C^{2+\alpha, 1+\alpha/2}(\overline{Q}'), \quad h(x', 0) = \varphi(x', r_0),\ x' \in \overline{D}'.$

5) $g(\varphi(x', r_0))\,[\psi_t(x,0) - L\,u|_{t=0}] = g(\varphi(x))\Big[h_t(x',0) - Lu|_{x_n=r_0,t=0}\Big]$, $x' \in \partial D'$, $x \in \partial D$.

Definitions of the spaces $C^{l+\alpha,\,(l+\alpha)/2}(\cdot)$, $l = 0,1,2,\ 0 < \alpha < 1$ and norms in these spaces are given in [6], p. 16.

Definition 1. If the following conditions are satisfied, the pair (f, u) is called solution of problem (1)-(3):

1) $f(x',t) \in C(\overline{Q}')$.

2) $u(x,t) \in C^{2,1}(\overline{Q})$.

3) The functions f and u satify equalities (1)-(3).

Let us denote

$$K = \big\{(f,u)\big|f(x',t) \in C^{\alpha,\alpha/2}(\overline{Q}'),\ u(x,t) \in C^{2+\alpha,1+\alpha/2}(\overline{Q})\big\}.$$

Theorem 1. *Let us say that the conditions 1)-4) are satisfied. Then if the solution belongs to set K of the problem (1)-(3) exists, this solutions is unique and it has the following evaluations:*

$$\|u-\overline{u}\|_0 + \big\|f-\overline{f}\big\|_0 \leq M_1\left[\|g-\overline{g}\|_0 + \|\varphi-\overline{\varphi}\|_2 + \big\|\psi-\overline{\psi}\big\|_{2,1} + \big\|h-\overline{h}\big\|_{0,1}\right], \quad (4)$$

where $\|\cdot\|_\ell = \|\cdot\|_{C^\ell}$. The pair of $\big\{\overline{f}(x',t), \overline{u}(x,t)\big\}$ belongs to set K and (1)-(3) which satisfies the conditions 2)-4) and has the data $\overline{g}$, $\overline{\varphi}$, $\overline{\psi}$, $\overline{h}$, is a solution of the problem (1)-(3). The number $M_1 > 0$ is a number which depends upon the set of K and the data (especially, we will denote the constants which depend on the set of K and the data of problem by M_i and denote the constants depending on only the data of problem by N_i).

Proof. If we write $x_n = r_0$ by considering the property of Theorem 1 then we get the following expression from the equation (1) for the function $f(x',t)$:

$$f(x',t) = \big[h_t(x',t) - L\,u|_{x_n=r_0}\big]/g(h(x',t)), \quad (x',t) \in \overline{Q}'. \quad (5)$$

Let us define the function

$$\begin{aligned} &p(x,t) \in C^{2+\alpha,1+\alpha/2}(\overline{Q}), \quad p(x,0) = \varphi(x), \quad x \in \overline{D}, \\ &p(x,t) = \psi(x,t), \quad (x,t) \in S, \end{aligned} \quad (6)$$

in [2], p. 87. We put

$$z(x,t) = u(x,t) - \overline{u}(x,t), \qquad \lambda(x',t) = f(x',t) - \overline{f}(x',t),$$

$$\delta_1(u) = g(u) - \overline{g}(u), \qquad \delta_2(x) = \varphi(x) - \overline{\varphi}(x),$$
$$\delta_3(x,t) = \psi(x,t) - \overline{\psi}(x,t), \qquad \delta_4(x',t) = h(x',t) - \overline{h}(x',t),$$
$$\delta_5(x,t) = p(x,t) - \overline{p}(x,t).$$

It is seen easily that the pair of the functions $\{\lambda(x',t), \upsilon(x,t) = z(x,t) -\delta_5(x,t)\}$ satisfies the following system:

$$\upsilon_t - L\upsilon = \lambda\,(x',t)g(u) + F(x,t)\,, \quad (x,t) \in Q\,, \tag{7}$$
$$\upsilon(x,0),\ x \in \overline{D}\ \upsilon(x,t) = 0, \quad (x,t) \in S, \tag{8}$$
$$\lambda(x',t) = \left[\delta_4(x',t) - L\,z|_{x_n=r_0}\right]/g(h(x',t)) + H(x',t), \quad (x',t) \in \overline{Q}', \tag{9}$$

where $F(x,t) = \overline{f}(x',t)[g(u) - \overline{g}(\overline{u})] - \delta_{5t}(x,t) + L\delta_5(x,t)$,

$$H(x',t) = \left[(g(h) - \overline{g}(\overline{h}))L\,\overline{u}|_{x_n=r_0} - \overline{h}_t\,(x',t)\right]/\left[g(h)\overline{g}(\overline{h})\right].$$

We get based on the conditions of the theorem that the coefficients of the equation (7) and the function in the right hand side satisfy the condition of Hölder. Therefore, classical solution of the problem (7)-(8) exists and it has the following representation [6], p. 468:

$$\upsilon(x,t) = \int_0^t \int_D G(x,t;\xi,\tau)\left[\lambda\,(\xi',\tau)g(u) + F(\xi,\tau)\right] d\xi d\tau, \tag{10}$$

where $\xi' = (\xi_1, ..., \xi_{n-1})$, $\xi = (\xi', \xi_n)$, $d\xi = d\xi_1 ... d\xi_n$, $G(\cdot)$ is a Green function of the problem (7), (8) and it satisfies the following inequalities [6], Chapter IV:

$$|G(x,t;\xi,\tau)| \le N_1(t-\tau)^{-n/2}\exp(-N_2|x-\xi|^2/(t-\tau)),$$
$$\int_\Omega \left|D_x^k G\,(x,t;\xi,\tau))\right| d\xi \le N_3(t-\tau)^{-(k-\alpha)/2}, \quad k = 0,1,2, \tag{11}$$

and $D_x^k\cdot$ represents the derivatives of the k-th order with respect to the variables $x_i\,(i = \overline{1,n})$. If we consider the $\upsilon(x,t) = z(x,t) - \delta_5(x,t)$, then we get from (10) that

$$z(x,t) = \delta_5(x,t) + \int_0^t \int_D G(x,t;\xi,\tau)\cdot\left[\lambda(\xi',\tau)g(u) + F(\xi,\tau)\right] d\xi\, d\tau\,. \tag{12}$$

Let $\chi \equiv \|u-\overline{u}\|_0 + \left\|f-\overline{f}\right\|_0$. From the conditions of theorem and the definition of the set K, if we consider the inequality (11) then we get from (12) and (9) that

$$|z(x,t)| \leq M_2\Big[\|\delta_1\|_0 + \|\delta_5\|_{2,1}\Big] + M_3\chi t, \quad (x,t)\in\overline{Q}\,, \tag{13}$$

$$\left|\lambda(x',t)\right| \leq M_4\Big[\|\delta_1\|_0 + \|\delta_4\|_{0.1} + \|\delta_5\|_{2,1}\Big] + M_5\chi t^{\alpha/2}, \quad (x',t)\in\overline{Q}'\,. \tag{14}$$

The inequality (13), (14) are satisfied for all $(x,t)\in\overline{Q}$. For this reason, these inequalities will be satisfied for maximal values of the expressions in the right hand side. Therefore

$$\chi \leq M_6\Big[\,\|\delta_1\|_0 + \|\delta_4\|_{0,1} + \|\delta_5\|_{2,1}\Big] + M_7\,\chi t^{\alpha/2}\,. \tag{15}$$

Let us choose the number T_1 $(0<T_1\leq T)$ such that $M_7 T_1^{\alpha/2}<1$. Then we get from (15) that the evaluation (4) is satisfied for the solution of problem(1)-(3) for $(x,t)\in\overline{D}\times[0,T_1]$.

If we examine the problem (1)-(3) in the regions of $\overline{D}\times[T_1,2T_1]$, $\overline{D}\times[2T_1,3T_1]$, we get that evaluation(4) is satisfied in regions of $\overline{D}\times[0,T]$ after the finite step. In the evaluation (4) if we get that

$$g(u)=\overline{g}(u), \quad \varphi(x)=\overline{\varphi}(x), \quad \psi(x,t)=\overline{\psi}(x,t), \quad h(x,t)=\overline{h}(x',t)\,,$$

then we had been obtained the uniqueness of the solution of the problem (1)-(3). Existence of the solution of the problem (1)-(3) is proved by the succesive approximations in the sense of concept given Definition 1. Given algorithm in the following is used to find the pair of functions $\left\{f^{(s)}(x',t),u^{(s)}(x,t)\right\}$, $s=0,1,2,...$:

$$u_t^{(s+1)} - Lu^{(s+1)} = f^{(s)}(x',t)g(u^{(s)}), \quad (x,t)\in Q\,, \tag{16}$$

$$u^{(s+1)}(x,0)=\varphi(x),\ x\in\overline{D};\ \ u^{(s+1)}(x,t)=\psi(x,t), \quad (x,t)\in S\,, \tag{17}$$

$$f^{(s+1)}(x',t) = \left[h_t(x',t) - Lu^{(s+1)}\Big|_{x_n=r_0}\right]/g(u^{(s+1)}(x',r_0,t)), \qquad (x',t)\in\overline{Q}'\,. \tag{18}$$

Theorem 2. *Let the conditions 1)-5) be satisfied, where $\partial D\in C^{2+\alpha}$. Then the problem (1)-(3) has at least one solution in the sense of Definition 1.*

Proof. It can be easily tested that if we choose it as

$$u^{(0)}(x,t) \in C^{2+\alpha,\,1+\alpha/2}(\overline{Q}), \quad f^{(0)}(x',t) \in C^{\alpha,\alpha/2}(\overline{Q}'),$$

then for all $s = 1, 2, \ldots u^{(s)}(x,t) \in C^{2+\alpha,1+\alpha/2}(\overline{Q})$ [6], p. 364, according to Theorem 2. Then we get from (18) that $f^{(s)}(x',t) \in C^{\alpha,\alpha/2}(\overline{Q}')$. In accordance with Theorem 2. Let us show that the sequences

$$\left\{f^{(s)}(x',t)\right\}, \left\{D_x^k u^{(s)}(x,t)\right\}, \quad k = 0, 1, 2\,,$$

are uniformly bounded. By using the function $p(x,t)$ defined with the expression (6) and Green function [6], p. 468, of solution, in order to determine of solution $u^{(s+1)}(x,t)$ from (16) and (17) we get the expression

$$u^{(s+1)}(x,t) = p(x,t) + \int_0^t \int_D G(x,t;\xi,\tau)\Big[f^{(s)}(\xi',\tau)g(u^{(s)}) + Lp - p_\tau\Big]d\xi d\tau\,. \quad (19)$$

Similarly, to the proof of Theorem 1, using the inequality (11) and the conditions of Theorem 2, we obtain

$$\left|D_x^k u^{(s+1)}(x,t)\right| \le N_4\|p\|_{2,1} + N_5 t^{(2+\alpha-k)/2} \cdot \left|f^{(s)}(x',t)\right|, \quad k = 0,\, 1,\, 2\,,$$

$$\left|f^{(s+1)}(x',t)\right| \le N_6\|h\|_{0,1} + N_7 t^{\alpha/2}\sum_{k=0}^{2}\left|D_x^k u^{(s+1)}(x,t)\right|,$$

or from (18) and (19)

$$\gamma^{(s+1)} \le N_8\Big[\|h\|_{0,1} + \|p\|_{2,1}\Big] + N_9 t^{\alpha/2}\gamma^{(s)}\,.$$

Here $\gamma^{(s)} \equiv \sum_{k=0}^{2}\left\|D_x^k u^{(s)}\right\|_0 + \left\|f^{(s)}\right\|_0$.

From the last inequality;

$$\gamma^{(s+1)} \le N_8\Big[\|h\|_{0,1} + \|p\|_{2,1}\Big]\,(1-q^s)/(1-q) + q^s\gamma^{(0)}, \quad q = N_9 t^{\alpha/2}\,.$$

Let us choose the number T_2 $(0 < T_2 \le T)$ such that $N_9 T_2^{\alpha/2} < 1$. Then we get that the sequences $\left\{f^{(s)}\right\}, \left\{D_x^k u^{(s)}\right\}, \quad k = 0,\, 1,\, 2$ are uniformly bounded for$x \in \overline{D}, \quad t \in [0, T_2]$ for each s = 0, 1, 2,... (in the sense of the norm C). As in the proof of Theorem 1, it is shown that the sequences $\left\{f^{(s)}\right\}$, $\left\{D_x^k u^{(s)}\right\}$, $k = 0, 1,\ 2$ are uniformly bounded for all $t \in [0, T]$.

Equicontinuity of the sequences $\left\{D_x^k u^{(s)}\right\}$, $k = 0,\ 1,\ 2$ is obtained from the following inequalities:

$$\left|D_x^k u^{(s+1)}(x,t) - D_x^k u^{(s+1)}(\overline{x},\overline{t})\right| \le \left|D_x^k u^{(s+1)}(x,t) - D_x^k u^{(s+1)}(\overline{x},t)\right|$$

$$+ \left|D_x^k u^{(s+1)}(\overline{x},t) - D_x^k u^{(s+1)}(\overline{x},\overline{t})\right| \le \left|D_x^k p(x,t) - D_x^k p(\overline{x},t)\right|$$

$$+ \left|D_x^k p(\overline{x},t) - D_x^k p(\overline{x},\overline{t})\right| + \int\limits_0^t \int\limits_D \left|D_x^k G(x,t;\xi,\tau) - D_x^k G(\overline{x},t;\xi,\tau)\right|$$

$$\times \left|F^{(s)}(\xi,\tau)\right| d\xi d\tau + \int\limits_0^{\overline{t}} \int\limits_D \left|D_x^k G(\overline{x},t;\xi,\tau) - D_x^k G(\overline{x},\overline{t},\xi,\tau)\right|$$

$$\times \left|F^{(s)}(\xi,\tau)\right| d\xi d\tau + \int\limits_{\overline{t}}^t \int\limits_D \left|D_x^k G(\overline{x},t;\xi,\tau)\right| \left|F^{(s)}(\xi,\tau)\right| d\xi d\tau$$

(where $F^{(s)}(x,t) = f^{(s)}(x',t) g(u^{(s)}) + Lp - p_{\ t})$. Moreover, uniformly bounded of the sequence $\left\{f^{(s)}(x,t)\right\}$, to be continous and bounded of data, inequality (11) and the satisfying the inequality

$$\left|D_x^k G(x,t;\xi,\tau) - D_{\overline{x}}^k G(\overline{x},t;\xi,\tau)\right|$$
$$\le N_{10} |x - \overline{x}|^{\alpha} (t-\tau)^{-(n+2+\alpha)/2} \exp(-N_{11}|x-\xi|^2/(t-\tau)),$$

$$\left|D_x^k G(x,t;\xi,\tau) - D_x^k G(x,\overline{t};\xi,\tau)\right|$$
$$\le N_{12} \left|t - \overline{t}\right|^{(2+\alpha-k)/2} (\overline{t}-\tau)^{-(n+2+\alpha)/2} \exp(-N_{13}|x-\xi|^2/(t-\tau))$$

for equicontinuity of the sequences $\left\{D_x^k u^{(s)}\right\}$, $k = 0,\ 1,\ 2$ have been kept in view.

If we use uniformly boundedness and equicontinuity of the sequence $\left\{D_x^k u^{(s)}\right\}$, $k = 0,\ 1,\ 2$, also continuity and boundedness of data then equicontinuity of the sequence $\left\{f^{(s)}(x',t)\right\}$, is easily obtained from the following inequality:

$$\left|f^{(s)}(x',t) - f^{(s)}(\overline{x}',\overline{t})\right| \le \left|f^{(s)}(x',t) - f^{(s)}(\overline{x}',t)\right| + \left|f^{(s)}(\overline{x}',t) - f^{(s)}(\overline{x'},\overline{t})\right|$$

$$\leq \left[\left|h_t(x',t)-h_t(\overline{x}',t)\right|+\left|L(u^{(s)}(x,t)-u^{(s)}(\overline{x},t))\right|_{x_n=r_0}\right] / \left|g(u^{(s)}(x',r_0,t))\right|$$
$$+\left[\left|h_t(\overline{x}',t)\right|+\left|Lu^{(s)}(\overline{x},t)\right|_{\overline{x_n}=r_0}\right]\frac{\left|g(u^{(s)}(x',r_0,t))-g(u^{(s)}(\overline{x}',r_0,t))\right|}{\left|g(u^{(s)}(x',r_0,t))g(u^{(s)}(\overline{x}',r_0,t))\right|}$$
$$+\left[\left|h_t(\overline{x}',t)-h_t(\overline{x}',\overline{t})\right|+\left|L(u^{(s)}(\overline{x},t)-u^{(s)}(\overline{x},\overline{t}))\right|_{\overline{x}_n=r_0}|\right]$$
$$/\left|g(u^{(s)}(\overline{x}',r_0,t))\right|+\left[\left|h_t(\overline{x}',\overline{t})\right|+\left|Lu^{(s)}(\overline{x},\overline{t})\right|_{x_n=r_0}|\right]\cdot\left|g(u^{(s)}(\overline{x}',r_0,t))\right.$$
$$\left.-\frac{g(u^{(s)}(\overline{x}',r_0,\overline{t}))\right|}{\left|g(u^{(s)}(\overline{x}',r_0,t))\cdot g(u^{(s)}(\overline{x}',r_0,\overline{t}))\right|}.$$

Equicontinuity and uniformly boundedness of the sequence $\left\{u_t^{(s)}\right\}$ is obtained from (16).

We get from Arzela Theorem [2], p. 84 that convergent subsequences to the functions $\{u_t^*\}$, $\left\{D_x^k u^*\right\}$, $k=0,1,\,2$, $\{f^*\}$ of the sequences $\left\{u_t^{(s)}\right\}$, $\left\{D_x^k u^{(s)}\right\}$, $k=0,1,\,2$, $\left\{f^{(s)}\right\}$ exist respectively. Therefore convergent subsequences to the functions $u^*(x,t)\in C^{2,1}(\overline{Q})$, $f^*(x',t)\in C(\overline{Q}')$ exist. Then if we pass to the limit in the expressions (16)-(18) when $s\to\infty$, we show easily that the pair of $\{f^*(x',t),\ u^*(x,t)\}$ satisfies the conditions (1)-(3). So we had been shown the existence of solution of the problem (1)-(3) in the sense of Definition 1.

Remark. As similar to the problem (1)-(3), to determined of the functions $\{f_k(x',t),\ \ u_k(x,t),\ k=\overline{1,m}\}$ established problem for the parabolic equation system

$$u_{kt}-\Delta u_k=f_k(x',t)g_k(u_1,\cdots,u_m),\quad (x,t)\in Q\,,$$
$$u_k(x,0)=\varphi_k(x),\quad x\in\overline{D},\quad u_k(x,t)=\psi_k(x,t),\quad (x,t)\in S\,,$$
$$u_k(x',r_0,t)=h_k(x',t),\quad (x',t)\in Q'\,,$$

can be considered, where $\Delta\cdot$ is a Laplace operator.

Results which are similar to obtained results in above can be obtained.

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