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SMOOTHING METHOD FOR A MINIMAX PROBLEM IN PORTFOLIO SELECTION

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Abstract: In this paper, a minimax model for portfolio selection is presented. By maximum entropy technique, the nonsmooth minimax problem is approximated by a smooth problem. Based on KKT optimality conditions, this smooth optimization problem with equality constraints is transformed into a system of smooth equations. Then the classical Newton method is applied to solving it. The numerical test shows the valid of this method.

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1. Introduction

Portfolio selection is to seek an optimal allocation of wealth among a basket of assets. The mean – variance formulation proposed by Markowitez plays an important role in portfolio selection [7]. Based on this formulation, we try to maximize expected return and to minimize the relevant variance. But arguments have been made that the mean-variance model is available if the investor's utility is quadratic or the joint distribution of return is normal. However, these conditions are not satisfied in most applications. In the last fifty

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years, some other measures of risk have been proposed in literatures. Instead of using variance, Markowitz [8] developed another measure, called semi-variance, which reflects the investors' psychology. For general believing that the investor expects his return higher than the mean. While in the standard mean-variance model, risk is defined as the fluctuation from the mean, including both higher and lower. Konno and Yamazaki [4] proposed absolute deviation method to measure risk. Furthermore, Young [12] introduced a linear program method to maximize the minimum return over time periods. A minimax risk function is introduced by Cai et al [1]. By making use of the special structure of the risk function, a simple analytical solution scheme is obtained for the efficient frontier of the portfolio optimization problem without solving any optimization problem. Moreover, Teo and Yang [10] developed an alternative minimax risk function in portfolio optimization. Such risk function is defined as the average of maximum individual risks over a number of last time periods. The corresponding portfolio optimization problem is formulated as a bi-objective linear programming problem. For a given weight on the return, the scalar linear program can be solved easily. More recently, Deng, Li and Wang [2] proposed a minimax principal to choose portfolio in a market without frictions. In this minimax principal, one maximizes the worst (minimally) possible expected rates of returns on portfolio. Li and Wang [6] constructed a minimax model in a market without frictions (such as tax and transaction cost). Wang, Yamamoto and Yu [11] developed a minimax model in a frictional market, and obtained some properties of the efficient frontier, where it is supposed that the frictions are smooth.

In this paper, we concern with a minimax model in a general frictional market. In applications, tax and transaction cost are sometimes described as nonsmooth functions, which cannot be given the efficient frontier and an obvious analysis. The present paper is organized as follows: In the reminder of this section, some preliminaries are reviewed. In Section 2, the nonsmooth model of minimax problem is derived from portfolio selection. In Section 3, an algorithm for solving our minimax problem is developed. In Section 4, a numerical example is listed.

The general form of the constrained minimax problem is

$$\begin{aligned} &\max \min_{1\le i\le m} f_i(x)\,, \\ &\text{s.t.}\;\; g_j(x) \ge 0, \quad j = 1, \dots, p\,, \\ &\qquad h_k(x) = 0, \quad k = 1, \dots, q\,, \end{aligned} \tag{P$_1$}$$

where $x \in \Re^n$, $f, g, h : \Re^n \to \Re$ are continuously differentiable.

Recently, the methods for solving (P_1) have been the topic of intense researches, seeing for instance [3, 5, 13]. However these methods are sometimes too complicated and cannot be applied directly.

2. minimax Model for Portfolio Selection

The minimax problem in portfolio in a market without frictions was proposed by Li and Wang[6]. Its objective is to maximize the minimally possible expected rates of returns on portfolio. The minimax problem is given as follows

$$\begin{aligned} &\max_x \min_y \ (1-\omega)\sum_{i=1}^n x_i y_i - \omega \sum_{i=1}^n \sigma_{ij} x_i x_j \,, \\ &\text{s.t.} \quad \sum_{i=1}^n x_i = 1 \,, \\ &\qquad y_{i\,\min} \le y_i \le y_{i\,\max}, \quad i = 1, \ldots, n \,, \end{aligned} \tag{P$_2$}$$

where $x, y \in \Re^n$, x_i is percentage of asset owned by investor i after the exchange, y_i the rate of return on asset, σ_{ij} the covariance of x_i and x_j, $y_{i\,\min}$ and $y_{i\,\max}$ the lower bounded and upper bounded of y_i respectively. $0 \le \omega \le 1$ means risk adverse, the higher ω is, the higher adverse degree is. However, in a frictional market, we should make some modification, in other words, we take tax and transaction cost into consideration.

We consider a specific model, which can also be applied in various areas, and give a practical algorithm in next section. For simplicity, we construct a model from portfolio selection as the following

$$\begin{aligned} &\max_x \min_y a \sum_{i=1}^n x_i y_i + b \sum_{i=1}^n |x_i - c_i| + F(x) \,, \\ &\text{s.t.} \quad \sum_{i=1}^n x_i = 1 \,, \\ &\qquad y_{i\,\min} \le y_i \le y_{i\,\max}, \quad i = 1, \ldots, n \,, \end{aligned} \tag{P$_3$}$$

where $x, y \in \Re^n$, $y_{i\,\min}$ and $y_{i\,\max}$ are defined in (P_2), a, b and c_i, $i = 1, \ldots, n$ are non-zero constants, F is quadratic.

3. Algorithm and Related Theorems

To solve (P$_3$), we firstly fix $x \in \Re^n$, and only consider the inner minimizing problem with y as variable

$$\begin{aligned} &\min_y a \sum_{i=1}^{n} x_i y_i + b \sum_{i=1}^{n} |x_i - c_i| + F(x)\,, \\ &\text{s.t.} \quad y_{i\min} \le y_i \le y_{i\max}, \quad i = 1, \ldots, n\,. \end{aligned} \tag{P$_4$}$$

Theorem 3.1. *Problem* $\min\limits_{x \in X} \sum\limits_{i=1}^{n} f_i(x_i) + A$ *shares the same optimal solution with problems of* $\min\limits_{x \in X} f_i(x_i)$, $i = 1, \ldots, n$.

Proof. The proof is trivial. □

According to Theorem 3.1, we can consider the following problems instead of (P$_4$)

$$\begin{aligned} &\min_y x_i y_i\,, \\ &\text{s.t.} \quad y_{i\min} \le y_i \le y_{i\max}\,, \quad i = 1, \ldots, n\,. \end{aligned} \tag{P$_{5i}$}$$

Theorem 3.2. *The unique optimal solution of* (P$_{5\text{i}}$) *has the form*

$$y_i^* = y_{i\min} \max\{0, \operatorname{sign} x_i\} + y_{i\max} \max\{0, -\operatorname{sign} x_i\}, i = 1, \ldots, n\,, \tag{3.1}$$

where sign *is a signal function, if* $x \ge 0$, $\operatorname{sign} x = 1$*; otherwise* $\operatorname{sign} x = -1$.

Proof. Noting that if x_i is fixed, the objective of (P_{5i}) is linear with respect to y_i. However, its monotone depends on the sign of x_i. In other words, if $x_i \ge 0$, it is non-decreasing with respect to y_i, otherwise, decreasing. Thus we conclude that, when $x_i \ge 0$, $y_{i\min}$, the lower bounded of y_i, admits the optimal solution. Otherwise the upper bounded $y_{i\max}$ is the optimal solution. Hence, we obtain (3.1). The proof is complete. □

According to Theorem 3.2, the optimal value of (P$_{5\text{i}}$) is expressed as

$$\begin{aligned} opt_i^* &= x_i y_{i\min} \max\{0, \operatorname{sign} x_i\} + x_i y_{i\max} \max\{0, -\operatorname{sign} x_i\} \\ &= y_{i\min} \max\{0, x_i\} - y_{i\max} \max\{0, -x_i\}\,, \quad i = 1, \ldots, n\,. \end{aligned} \tag{3.2}$$

By virtue of (3.2), problem (P$_3$) is transformed into

$$\begin{aligned} &\max a \sum_{i=1}^{n} opt_i^* + b \sum_{i=1}^{n} |x_i - c_i| + F(x)\,, \\ &\text{s.t.} \quad \sum_{i=1}^{n} x_i = 1\,. \end{aligned} \tag{P$_6$}$$

Evidently, the objective function of (P_6) is nonsmooth. In what follows, we try to construct an approximative function which is smooth to instead the objective functions in (P_6). For this purpose, the maximum entropy function is a good choice. The maximum entropy function is defined as

$$U_\rho(x) = \rho^{-1} \ln \sum_{i=1}^{m} \exp[\rho g_i(x)]\,, \tag{3.3}$$

where $\rho \geq 0$. It was shown [5] that $\lim_{\rho\to+\infty} U_\rho(x) = \max_{1\leq i\leq m} g_i(x)$ and $0 \leq U_\rho(x) - U(x) \leq \rho^{-1}\ln m$.

According to (3.3), we can obtain the following proposition immediately

Proposition 3.1. *The approximative functions for the nonsmooth problems in* (P_6) *are given below*

$$\bar{opt}_i = \rho^{-1} y_{i\,\min} \ln(e^{\rho x_i}+1) - \rho^{-1} y_{i\,\max} \ln(e^{-\rho x_i}+1), \quad i = 1,\dots,n\,, \tag{3.4}$$

$$\begin{aligned} |x_i - c_i| &= \max\{0, x_i - c_i\} + \max\{0, c_i - x_i\} \\ &= \rho^{-1}\ln(e^{\rho(x_i-c_i)}+1) + \rho^{-1}\ln(e^{\rho(c_i-x_i)}+1), \quad i = 1,\dots,n\,, \end{aligned} \tag{3.5}$$

where $\rho \to +\infty$.

By (3.4) and (3.5), (P_6) is equivalent to the following problem

$$\begin{aligned} &\max a \sum_{i=1}^{n} [\rho^{-1} y_{i\,\min}\ln(e^{\rho x_i}+1) - \rho^{-1} y_{i\,\max}\ln(e^{-\rho x_i}+1) \\ &\quad + b\sum_{i=1}^{n}[\rho^{-1}\ln(e^{\rho(x_i-c_i)}+1) + \rho^{-1}\ln(e^{\rho(c_i-x_i)}+1)] + F(x)\,, \\ &\text{s.t.} \quad \sum_{i=1}^{n} x_i = 1\,. \end{aligned} \tag{P$_7$}$$

Thus we can solve (P_7) instead of (P_6). Problem (P_7) approximates (P_6) with a certain accuracy. Hence, we can solve (P_7) to obtain an approximative solution for (P_6).

Theorem 3.3. *The objective of* (P_7) *approximates the objective function of* (P_6) *by accuracy*

$$|\, a\sum_{i=1}^{n}[\rho^{-1} y_{i\,\min}\ln(e^{\rho x_i}+1) - \rho^{-1} y_{i\,\max}\ln(e^{-\rho x_i}+1)]$$

$$+ b\sum_{i=1}^{n}[\rho^{-1}\ln(e^{\rho(x_i-c_i)}+1)+\rho^{-1}\ln(e^{\rho(c_i-x_i)}+1)]+F(x)$$

$$-[a\sum_{i=1}^{n}(y_{i\min}\max\{0,x_i\}-y_{i\max}\max\{0,-x_i\})+b\sum_{i=1}^{n}\mid x_i-c_i\mid+F(x)]\mid$$

$$\le \rho^{-1}\ln 2\sum_{i=1}^{n}(ay_{i\min}+ay_{i\max}+2b)$$

Proof. By calculating, it is obtained that

$$\mid a\sum_{i=1}^{n}[\rho^{-1}y_{i\min}\ln(e^{\rho x_i}+1)-\rho^{-1}y_{i\max}\ln(e^{-\rho x_i}+1)]$$

$$+ b\sum_{i=1}^{n}[\rho^{-1}\ln(e^{\rho(x_i-c_i)}+1)+\rho^{-1}\ln(e^{\rho(c_i-x_i)}+1)]+F(x)$$

$$-[a\sum_{i=1}^{n}(y_{i\min}\max\{0,x_i\}-y_{i\max}\max\{0,-x_i\})+b\sum_{i=1}^{n}\mid x_i-c_i\mid+F(x)]\mid$$

$$=\mid a\sum_{i=1}^{n}[\rho^{-1}y_{i\min}\ln(e^{\rho x_i}+1)-y_{i\min}\max\{0,x_i\}]$$

$$+ a\sum_{i=1}^{n}[y_{i\max}\max\{0,-x_i\}-\rho^{-1}y_{i\max}\ln(e^{-\rho x_i}+1)]$$

$$+ b\sum_{i=1}^{n}[\rho^{-1}\ln(e^{\rho(x_i-c_i)}+1)-\max\{0,x_i-c_i\}]$$

$$+ b\sum_{i=1}^{n}[\rho^{-1}\ln(e^{\rho(c_i-x_i)}+1)-\max\{0,c_i-x_i\}]\mid \quad (\text{by } |a+b|\le|a|+|b|)$$

$$\le a\sum_{i=1}^{n}\mid \rho^{-1}y_{i\min}\ln(e^{\rho x_i}+1)-y_{i\min}\max\{0,x_i\}\mid+a\sum_{i=1}^{n}\mid y_{i\max}\max\{0,-x_i\}$$

$$-\rho^{-1}y_{i\max}\ln(e^{-\rho x_i}+1)\mid+b\sum_{i=1}^{n}\mid \rho^{-1}\ln(e^{\rho(x_i-c_i)}+1)-\max\{0,x_i-c_i\}\mid$$

$$+ b\sum_{i=1}^{n}\mid \rho^{-1}\ln(e^{\rho(c_i-x_i)}+1)-\max\{0,c_i-x_i\}\mid$$

$$(\text{by } 0\le U_\rho(x)-U(x)\le\rho^{-1}\ln m)$$

$$\leq a\sum_{i=1}^{n} y_{i\,\min}\rho^{-1}\ln 2 + a\sum_{i=1}^{n} y_{i\,\max}\rho^{-1}\ln 2 + b\sum_{i=1}^{n}\rho^{-1}\ln 2 + b\sum_{i=1}^{n}\rho^{-1}\ln 2$$
$$= \rho^{-1}\ln 2\sum_{i=1}^{n}(a y_{i\,\min} + a y_{i\,\max} + 2b)\,.$$

This completes the proof of the theorem. □

We know that the Karush-Kuhn-Tucker (KKT) optimality condition for (P_7) has the form

$$\begin{cases} a[(y_{i\,\min} - y_{i\,\max})(1+e^{-\rho x_i})^{-1} + y_{i\,\max}] \\ +b[2(1+e^{-\rho(x_i-c_i)})^{-1} - 1] + \bigtriangledown F(x) - \lambda = 0, \quad i = 1,\dots,n\,, \\ \sum\limits_{i=1}^{n} x_i - 1 = 0\,, \end{cases} \tag{3.6}$$

where λ is Lagrange multiplier, $\bigtriangledown F(x) = \partial F(x)/\partial x$. Under some constraint qualifications, problem (P_7) is equivalently to the system (3.6). Hence, we can apply a Newton method to solving the system (3.6).

Consider the following nonlinear equations

$$H(x) = 0\,, \tag{3.7}$$

where $H(x) : \Re^n \to \Re$ is locally Lipschitzian. The Newton method for solving the equations (3.7) is given by

$$x^{k+1} = x^k - J_k^{-1} H(x^k)\,, \tag{3.8}$$

where J_k is Jacobian of H at x^k [9].

We now give the algorithm of smoothing Newton method for solving (3.6).

Algorithm 3.1. *Step 0.* Set $k = 0$, and choose an initial point x^0. Give ρ and termination accuracy ε.

Step 1. Calculate the value of the function in the left side of (3.6) and J_k^{-1} at x^k, where

$$J_k = \begin{vmatrix} \nabla_1 + \frac{\partial^2 F(x)}{\partial x_1^2} & \frac{\partial^2 F(x)}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 F(x)}{\partial x_1 \partial x_n} & -1 \\ \frac{\partial^2 F(x)}{\partial x_1 \partial x_2} & \nabla_2 + \frac{\partial^2 F(x)}{\partial x_2^2} & \cdots & \frac{\partial^2 F(x)}{\partial x_2 \partial x_n} & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{\partial^2 F(x)}{\partial x_1 \partial x_n} & \frac{\partial^2 F(x)}{\partial x_2 \partial x_n} & \cdots & \nabla_n + \frac{\partial^2 F(x)}{\partial x_n^2} & -1 \\ 1 & 1 & \cdots & 1 & 0 \end{vmatrix},$$

with $\nabla_i = -\frac{a\rho(y_{i\,\min} - y_{i\,\max})}{e^{\rho x_i}(1+e^{\rho x_i})^2} - \frac{2b\rho}{e^{-\rho(x_i-c_i)}(1+e^{-\rho(x_i-c_i)})^2}$, $(i = 1,\dots,n)$.

Step 2. Compute x^{k+1} by (3.8).

Step 3. If $|x^{k+1} - x^k| \le \varepsilon$, then stop. Otherwise, set $k = k + 1$ and go to Step 1.

4. Numerical Example

We present the result of a numerical test. It was implemented in *Matlab* with single precision on a microcomputer. ρ is given by $10E+5$, and the termination accuracy fixed at $\varepsilon \le 10^{-20}$. Suppose there existing four assets in a frictional market. Let

$$a = 0.32, \quad b = 0.12, \quad c = (0,0,0,0)\,,$$

$$y_{i\,\min} = (0.134,\ 0.106,\ 0.042,\ 0.030), \quad y_{i\,\max} = (0.189,\ 0.142,\ 0.067,\ 0.043)\,,$$

and

$$F(x) = -(19.18x_1^2 + 4.2x_2^2 + 0.76x_3^2 + 0.11x_4^2 + 6.96x_1x_2 + 1.16x_1x_3 \\ - 0.5x_1x_4 + 0.5x_2x_3 + 0.07x_2x_4 + 0.14x_3x_4)\,.$$

The the solution is $x^* = (0.0521, -0.0372,\ 0.0119,\ 0.9732)$.

5. Conclusions

In this paper, we consider a specific minimax problem arising from portfolio selection, and propose a practical method. For solving the nonsmooth minimax problem, we utilize maximum entropy technique to approximate it. Under some constraint qualifications, we can solve a smooth problem based on its KKT optimality condition. Newton method is applied to searching for the optimal solution of a system of smooth equations. Finally we give a numerical test which shows the valid of our method.

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