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ON SOME NONLINEAR ELLIPTIC-PARABOLIC EQUATIONS OF SECOND ORDER

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In honour of Professor Phillipe Benilan.

Abstract: We study a nonlinear elliptic-parabolic PDE in one dimensional space with the form: $b(u)_t - a(u, \varphi(u)_x)_x = f$. Using the nonlinear semi-groups theory in Banach spaces, we establish existence and uniqueness of mild solutions of the associated Cauchy problem under general assumptions on the data. We prove with additional assumptions, that mild solutions are entropy solutions.

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1. Introduction

We consider the Cauchy problem:

$$\begin{cases} b(u)_t - a(u, \varphi(u)_x)_x = f & \text{in} \quad \mathbb{Q} =]0, T[\times\mathbb{R}\,, \\ b(u(0,.)) = v_0 & \text{on} \quad \mathbb{R}, \end{cases} \tag{EP}$$

where $f \in L^1(\mathbb{Q}), v_0 \in L^1(\mathbb{R}), a : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous and nondecreasing with respect to the second variable φ, and $b : \mathbb{R} \to \mathbb{R}$ are continuous, nondecreasing and b surjective.

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Whenever u takes a value such that $b(u)$ is constant, (EP) degenerates into an elliptic problem of the form:

$$\begin{cases} -a(u,\varphi(u)_x)_x = f \text{ in } \mathbb{Q} =]0,T[\times\mathbb{R}\,, \\ b(u(0,.)) = v_0 \text{ on } \mathbb{R}. \end{cases} \tag{1}$$

Take $b = id$, on each part where u takes a value such that $\varphi(u)$ is constant, (EP) degenerates to a scalar conservation law of the form:

$$\begin{cases} u_t - a(u,0)_x = f \text{ in } \mathbb{Q} =]0,T[\times\mathbb{R}\,, \\ u(0,.) = u_0 \text{ on } \ \mathbb{R}. \end{cases} \tag{2}$$

It is clear that we include in (EP), some first order hyperbolic problems, for which (even under assumptions of regularity on data) there is no hope of getting classical global solutions.

Using nonlinear semi-groups theory in L^1, we deduce the results for the "evolution problem" (EP) of the properties of the " stationary problem":

$$b(u) - a(u,\varphi(u)_x)_x = f \text{ on } \mathbb{R}. \tag{SP}$$

The study of (SP) has been presented in a first paper [12]; we will quickly recall the results in Section 2, where we will develop the concept of mild solution of (EP).

Under assumptions on the data a, b and φ which we will specify below, there are existence and uniqueness of a mild solution of (EP) for any $(v_0, f) \in L^1(\mathbb{R}) \times L^1(Q)$.

When, moreover, $v_0 \in L^\infty(\mathbb{R})$ and $\int_0^T \|f(t,.)\|_\infty < \infty$, mild solutions are bounded.

Under complementary assumptions on the functions a, b and φ, we show in Section 2 and Section 3 that the mild solutions are in $\mathrm{Lip}(0,T;L^1(\mathbb{R})) \cap L^\infty(Q)$ and are entropy solutions of problem (EP).

2. Mild Solution

In all this paper, a, b and φ are given functions such that:

$$a : \mathbb{R}\times\mathbb{R}\to\mathbb{R},\ \varphi : \mathbb{R}\to\mathbb{R},\ b : \mathbb{R}\to\mathbb{R} \ \text{ continuous and verify:} \tag{1.1}$$

$$\begin{cases} a(k,\xi) \text{ is nondecreasing in } \xi\,, \\ b(k) \text{ and } \varphi(k) \text{ are nondecreasing with } b \text{ surjective.} \end{cases} \tag{1.2}$$

We denote by

$$H(k) = a(k, 0) \text{ for } k \in \mathbb{R} \text{ and } h = a(u, \varphi(u)_x)_x. \tag{1.3}$$

Our main assumption is the coerciveness of a with respect to ξ, for k bounded; more precisely:

$$\lim_{|\xi|\to\infty} \inf_{|k|<\mathbb{R}} \mid a(k,\xi) \mid = +\infty\,, \quad \forall \mathbb{R} > 0. \tag{H$_1$}$$

Let us recall for completeness, our results for the stationnary problem (see [12]).

Definition 1. Let $f \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$; a weak solution of (SP) is a measurable function $u \in L^\infty(\mathbb{R})$ such that $\varphi(u) \in W^{1,\infty}(\mathbb{R})$ and

$$b(u) - a(u, \varphi(u)_x)_x = f \text{ in } D'(\mathbb{R}).$$

Definition 2. Let $f \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$; an entropy solution of (SP) is a weak solution u satisfying:

(i) there exists $h \in C(\mathbb{R})$ such that $h = a(u, \varphi(u)_x)$ a.e on $\mathbb{R}$.

(ii) $\forall \xi \in D^+(\mathbb{R}), k \in \mathbb{R}$, the following entropy inequality

$$(a) \qquad \int_{\mathbb{R}} \operatorname{sign}_0(b(u) - b(k)) \left\{\xi_x(H(k) - h) + \xi(f - b(u))\right\} dx \geq 0\,,$$

$$(b) \qquad \int_{\mathbb{R}} \operatorname{sign}_0(b(k) - b(u)) \left\{\xi_x(H(k) - h) + \xi(f - b(u))\right\} dx \leq 0\,.$$

In [12] we proved under general assumptions on the data that (SP) admit a unique entropy solution which permit us to define the $L^1(\mathbb{R})$ operator A_b associated with the evolution problem (EP) by $A_b b(u) = -a(u, \varphi(u)_x)_x$ satisfying:

$$\begin{cases} v \in A_b \Longleftrightarrow b(u) \in L^1(\mathbb{R}), v \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}) \\ \text{and u is entropy solution of (3) with } f = v + b(u). \end{cases}$$

We showed for this operator, the following result that we present as lemma.

Lemma 3. *Under assumptions (1.1), (1.2) and* (H_1), *the operator* A_b *defined below satisfied:*

1. A_b *is T-accretive in* $L^1(\mathbb{R})$.
2. *For any* $\lambda > 0$, $R(I + \lambda A_b)$ *is dense in* $L^1(\mathbb{R})$.
3. $D(A_b)$ *is dense in* $L^1(\mathbb{R})$.

As usual in the theory of evolution equations governed by accretive operators, we consider an approximation of (EP) by an implicit time discretization (also used by Alt and Luckhaus [1])

$$\left\{\begin{array}{l} \dfrac{b(u_i)-b(u_{i-1})}{t_i-t_{i-1}} = a(u_i,\varphi(u_i)_x)_x + f_i\,, \\ u_i \in L^\infty(\mathbb{R}), b(u_i)\in L^1(\mathbb{R}), \varphi(u_i)\in W^{1,\infty}(\mathbb{R})\,, \\ \text{for} \ \ i=1,...,n, \end{array}\right. \tag{PDE}$$

where

$$\left\{\begin{array}{l} t_0=0<t_1<...<t_n\le T, t_{i-1}-t_i\le \epsilon, T-t_n\le\epsilon\,, \\ f_1,...,f_n\in L^\infty(\mathbb{R}), \sum\displaystyle\int_{t_i}^{t_{i-1}} \|f(t)-f_i\|\,dt\le\epsilon\,, \\ u_0\in L^\infty(\mathbb{R}), \|v_0-b(u_0)\|\le\epsilon, \ \ \text{for} \ \ \epsilon>0. \end{array}\right. \tag{DE}$$

This method is actually the method of nonlinear semigroup theory. Naturally, we are led to give the following definition according to [6] (see also [9], [3]).

Definition 4. A mild solution of (EP) is a mesurable function $u:\mathbb{Q}\to\mathbb{R}$ satisfying $v=b(u)\in C([0,T];L^1(\mathbb{R})), v(0)=v_0$ and, $\forall\epsilon>0$, there exist $(t_0,...,t_n,f_1,...,f_n,u_0,...,u_n)$ satisfying (DE) and (PDE), such that $\|v(t)-b(u_i)\|_1\le\epsilon \ \ \forall t\in]t_{i-1},t_i], \ i=1,...,n.$

Using nonlinear semigroup theory (cf. [4], [3]) by interpreting problem (EP) in the form of evolution equation in $L^1(\mathbb{R})$

$$\left\{\begin{array}{l} \dfrac{dv}{dt}+A_b v\ni f \ \ \text{in} \ \ [0,T]\ , \\ v(0)=v_0\,, \end{array}\right. \tag{CP}$$

with $v=b(u)$; one deduces immediately from the preceding lemma, the following theorem.

Theorem 5. *Under assumptions (1.1), (1.2) and* (H_1), $\forall\ v_0=b(u_0)\in L^1(\mathbb{R})$, $f\in L^1(\mathbb{Q})$, *there exists a unique mild solution* u *of* (EP) *which is characterized by the property:*

$$\left\{\begin{array}{l} b(u(0,.))=v_0 \ \textit{and}\ \forall\underline{u}\in D(A), \ \ \xi\in D([0,T]), \ \xi\ge 0, \\ \exists\alpha\in L^\infty(\mathbb{Q}), \alpha\in \operatorname{sign}(b(u)-b(\underline{u})) \ \textit{a.e.}\ (t,x)\in Q \ \textit{such that} \\ \displaystyle\iint_{\mathbb{Q}} \alpha\left\{(b(u)-b(\underline{u}))\xi_t+(f-A_{b,a,\varphi}b(\underline{u})\xi)\right\}dxdt\ge 0; \end{array}\right. \tag{1.4}$$

moreover, if u_1, u_2 *are mild solutions with respect to* $(v_{0,1};f_1)$, $(v_{0,2};f_2)$, *one has*

$$\max_{t\in[0,T]}\int_{\mathbb{R}}[b(u_1(t))-b(u_2(t))]^+dx \\ \leq \int_{\mathbb{R}}(v_{0,1}-v_{0,2})^+dx+\iint_{\mathbb{Q}}(f_1-f_2)^+dxdt\,, \quad (1.5)$$

in particular,

$$v_{0,1}\leq v_{0,2} \text{ a.e. on } \mathbb{R}, f_1\leq f_2 \text{ on } \mathbb{Q}\Longrightarrow b(u_1)\leq b(u_2) \text{ a.e. on } \mathbb{Q}. \quad (1.6)$$

If v_0 and f verify

$$v_0\in L^1(\mathbb{R})\cap L^\infty(\mathbb{R}), f\in L^1(Q) \text{ and } \int_0^T \|f(t,.)\|_\infty<\infty, \quad (1.7)$$

then, the mild solution is bounded; more precisely, one has the following estimation.

Proposition 6. *Let us suppose (1.7) is satisfied and u being the mild solution of (EP), then $u\in L^\infty(Q)$ and*

$$\|b(u)\|_{L^\infty(Q)}\leq\|v_0\|_{L^\infty(\mathbb{R})}+\int_0^T\|f(t,.)\|_{L^\infty(\mathbb{R})}\,dt. \quad (1.8)$$

Proof. For the proof of this proposition, we interprete the problem (EP) in the form of the evolution equation (CP) in $L^1(\mathbb{R})$ with $v=b(u)$; and, in this case, the result follow immediately by using Proposition 1 – Proposition 4 of [5]. □

3. Entropy Solutions

In this section, we reinforce the assumptions on the data.

First of all, we intoduce the generalized domain of operator A_b defined by

$$\widehat{D}(A_b)=\left\{\begin{array}{l} v_0\in L^1(\mathbb{R});\ \text{there exist } v_n=b(u_n)\in D(A_b),\\ \text{such that } v_n\longrightarrow v_0 \text{ in } L^1(\mathbb{R}),\ A_bv_n \text{ is bounded in } L^1(\mathbb{R}).\end{array}\right.$$

If $f\in BV(0,T;L^1(\mathbb{R}))$ and $v_0\in\widehat{D}(A_b)$, then according to nonlinear semigroup theory (see [3]), the mild solution u of (EP) verify $b(u)\in \mathrm{Lip}([0,T];L^1(\mathbb{R}))$.

Now let us introduce the concept of weak and entropy solutions of the evolution problem.

Definition 7. Let $f \in L^2(0,T;H^{-1}_{loc}(\mathbb{R}))$ and $v_0 \in L^1(\mathbb{R})$. A weak solution of problem (EP) is a function u such that $\varphi(u) \in L^2(0,T;H^1_{loc}(\mathbb{R}))$, $b(u) \in L^1_{loc}(Q)$ and verify:

i) $b(u)_t \in L^2(0,T;H^{-1}_{loc}(\mathbb{R})), h = a(u,\varphi(u)_x) \in L^2_{loc}(Q)$.

ii) $b(u)_t - h_x = f$ in $D'(Q)$ and $b(u(0)) = v_0$.

The last condition must be understood in the following sense:

$$\int_0^T \langle b(u)_t, \xi\rangle \, dt = -\int_Q b(u)\xi_t dt - \int_{\mathbb{R}} v_0 \xi(0) dx$$

for any $\xi \in L^2(0,T;D(\mathbb{R})) \cap W^{1,1}(0,T;L^\infty(\mathbb{R}))$, such that $\xi(T) = 0$, where $\langle , \rangle$ represent the duality product between $H^{-1}(\mathbb{R})$ and $H^1(\mathbb{R})$.

Definition 8. Let $f \in L^2(0,T;H^{-1}_{loc}(\mathbb{R})) \cap L^1(Q)$ and $v_0 \in L^1(\mathbb{R})$. An entropy solution of (EP) is a weak solution u satisfying:

$$(i) \qquad \left\{ \begin{array}{l} \displaystyle\int_Q \operatorname{sign}_0(u-k)\left\{\xi_x(h - H(k)) - (b(u) - b(k))\xi_t - f\xi\right\} dxdt \\ \displaystyle - \int_{\mathbb{R}} (b(u_0) - b(k))^+ \xi(0) dx \leq 0\,, \end{array} \right.$$

$$(ii) \qquad \left\{ \begin{array}{l} \displaystyle\int_Q \operatorname{sign}_0(k-u)\left\{\xi_x(h - H(k)) - (b(u) - b(k))\xi_t - f\xi\right\} dxdt \\ \displaystyle - \int_{\mathbb{R}} (b(u_0) - b(k))^- \xi(0) dx \geq 0\,, \end{array} \right.$$

$\forall \xi \in D^+([0,T] \times \mathbb{R})$, $k \in \mathbb{R}$, and $\xi(T) = 0$.

We suppose that

$$\left\{ \begin{array}{l} v_0 \in \widehat{D}(A_b) \cap L^\infty(\mathbb{R}), f \in BV(0,T;L^1(\mathbb{R})), \text{ and} \\ \displaystyle\int_0^T \|f(t,.)\|_{L^\infty(\mathbb{R})} \, dt < +\infty, \end{array} \right. \tag{H$_2$}$$

so that the mild solution u of (EP) verify $b(u) \in Lip(0,T;L^1(\mathbb{R})) \cap L^\infty(Q)$.

We make the following complementary assumptions:

$$b + \varphi \text{ is one to one.} \tag{H$_3$}$$

$$\left\{ \begin{array}{l} \text{There exist } \widetilde{a} : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R} \text{ continuous, nondecreasing} \\ \text{with respect to the second variable and such that} \\ \widetilde{a}(b(u), \varphi(u)_x) = a(u, \varphi(u)_x). \end{array} \right. \tag{H$_4$}$$

Theorem 9. *Let (H_1), (H_2), and (H_3) or (H_4) hold, then the mild solution u of (EP) is an entropy solution.*

Proof. First Case. Suppose (H_1), (H_2), and (H_3) hold; by definition of the generalized domain, one can always choose $v_\epsilon(0) \in D(A_b)$ with $A_b v_\epsilon(0)$ bounded in $L^1(\mathbb{R})$. We can also choose f_ϵ bounded in $BV(0,T;L^1(\mathbb{R}))$; by accretiveness of A_b, it follow that $A_b v_\epsilon$ is bounded in $L^\infty(0,T;L^1(\mathbb{R}))$ and then, we deduce that $h_\epsilon = a(u_\epsilon, \varphi(u_\epsilon)_x)$ is bounded in $L^\infty(0,T;BV(\mathbb{R}))$ according to the coerciveness assumption (H_1); using again (H_1), we obtain that $\varphi(u_\epsilon)_x$ is also bounded in $L^\infty(0,T;L^\infty(\mathbb{R}))$.

Since $b(u_\epsilon)$ is bounded in $L^\infty(Q)$ and b is surjective then u_ϵ is also bounded in $L^\infty(Q)$.

Since $b(u_\epsilon) \longrightarrow v$ in $C([0,T];L^1(\mathbb{R}))$, $\varphi \circ b^{-1}$ is a maximal monotone graph in $\mathbb{R}$ and $b+\varphi$ is one to one, there exist $u \in b^{-1}(v)$ and a subsequence (u_{ϵ_k}) of (u_ϵ) such that:

$$\begin{cases} u_{\epsilon_k} \longrightarrow u & \text{in } L^2_{loc}(Q) \text{ and a.e.}, \\ b(u_{\epsilon_k}) \longrightarrow b(u) & \text{in } L^2_{loc}(Q) \text{ and a.e.}, \\ \varphi(u_{\epsilon_k}) \longrightarrow \varphi(u) & \text{in } L^2_{loc}(Q) \text{ and a.e.}, \\ \varphi(u_{\epsilon_k})_x \rightharpoonup \varphi(u)_x & \text{weakly in } L^2_{loc}(Q). \end{cases} \tag{2.1}$$

Moreover, $(h_\epsilon(t))$ is bounded in $BV(\mathbb{R})$; according to Weil Theorem (see [13]), $(h_\epsilon(t))$ is relatively compact in $L^1_{loc}(\mathbb{R})$ uniformly for any t bounded.

Since $a(k,.)$ is nondeceasing, we may deduce that there exists a subsequence of (u_ϵ) denoted again by (u_{ϵ_k}) verifying (2.1) and such that:

$$h_{\epsilon_k} = a(u_{\epsilon_k}, \varphi(u_{\epsilon_k})_x) \longrightarrow h = a(u, \varphi(u)_x) \text{ in } L^1_{loc}(Q).$$

Let us check now that $b(u)$ verifies the inequalities (i) and (ii) of Definition 6. $b(u)$ is a uniform limit for $t \in [0,T]$, of a sequence of step functions $b(u_\epsilon)$ defined by:

$$0 \le t_0 < t_1 < ... < t_n \le T, \quad t_0 \le \epsilon, \quad t_i - t_{i-1} \le \epsilon, \quad T - t_n \le \epsilon,$$

$$b(u_\epsilon(t)) = b(u_\epsilon^i) \text{ on }]t_{i-1}, t_i], \quad b(u_\epsilon(t)) = b(u_\epsilon^0) \text{ for } t \le t_0,$$

$$b(u_\epsilon^0) \in D(A_b) \text{ with } \frac{b(u_\epsilon^i) - b(u_\epsilon^{i-1})}{t_i - t_{i-1}} + A_b b(u_\epsilon^i) = f.$$

Consequently, according to the definition of A_b, we have:

$$\forall \xi \in D^+(\mathbb{R}), \ \forall k \in \mathbb{R}: \qquad \int_{\mathbb{R}} \operatorname{sign}_0(b(u_\epsilon^i) - b(k))$$

$$\times \left\{ - \left[h_\epsilon(t_i) - H(k) \right] \xi_x + \left[f - \frac{b(u_\epsilon^i) - b(u_\epsilon^{i-1})}{t_i - t_{i-1}} \right] \xi \right\} dx \geq 0 \,,$$

from which we deduce:

$$\int_{\mathbb{R}} \operatorname{sign}_0(b(u_\epsilon^i) - b(k)) \left\{ - \left[h_\epsilon(t_i) - H(k) \right] \xi_x + f\xi \right\} dx \\ \geq \int_{\mathbb{R}} \operatorname{sign}_0(b(u_\epsilon^i) - b(k)) \frac{b(u_\epsilon^i) - b(u_\epsilon^{i-1})}{t_i - t_{i-1}} \xi dx.$$

Let us integrate this inequality on $]t_{i-1}, t_i]$:

$$\int_{\mathbb{R}} \int_{t_{i-1}}^{t_i} \operatorname{sign}_0(b(u_\epsilon^i) - b(k)) \left\{ - \left[h_\epsilon(t_i) - H(k) \right] \xi_x + f\xi \right\} dxdt \\ \geq \int_{\mathbb{R}} \operatorname{sign}_0(b(u_\epsilon^i) - b(k)) b(u_\epsilon^i) - b(u_\epsilon^{i-1}) \xi dx. \quad (2.2)$$

Using the following inequality:

$$\int_{\mathbb{R}} \operatorname{sign}_0(b(u_\epsilon^i) - b(k)) b(u_\epsilon^i) - b(u_\epsilon^{i-1}) \xi dx \\ \geq \int_{\mathbb{R}} \left| b(u_\epsilon^i) - b(k) \right| \xi dx - \int_{\mathbb{R}} \left| b(u_\epsilon^{i-1}) - b(k) \right| \xi dx \,,$$

we obtain

$$\int_{\mathbb{R}} \int_{t_{i-1}}^{t_i} \operatorname{sign}_0(b(u_\epsilon^i) - b(k)) \left\{ - \left[h_\epsilon(t_i) - H(k) \right] \xi_x + f\xi \right\} dxdt \\ \geq \int_{\mathbb{R}} \left| b(u_\epsilon^i) - b(k) \right| \xi dx - \int_{\mathbb{R}} \left| b(u_\epsilon^{i-1}) - b(k) \right| \xi dx, \quad (2.3)$$

$\forall s, t$ such that $0 \leq s < t \leq T$, there exists j, i such that $t \in]t_{j-1}, t_j]$, $s \in]t_{i-1}, t_i]$ and $i + 1 < j$. Then let us make the sum of the integrals obtained of $i + 1$ to j:

$$\int_{\mathbb{R}} \int_{t_{i+1}}^{t_j} \operatorname{sign}_0(b(u_\epsilon) - b(k)) \left\{ - \left[h_\epsilon(\tau) - H(k) \right] \xi_x + f\xi \right\} dxd\tau \\ \geq \int_{\mathbb{R}} |b(u_\epsilon(t)) - b(k)| \xi dx - \int_{\mathbb{R}} |b(u_\epsilon(s)) - b(k)| \xi dx. \quad (2.4)$$

Taking $\epsilon \longrightarrow 0$ in (2.4); then there exist $\alpha \in L^\infty(Q)$, $\alpha \in \operatorname{sign}(b(u) - b(k))$, a.e $(t, x) \in Q$, such that:

$$\int_{\mathbb{R}}\int_s^t \alpha\left\{-\left[h-H(k)\right]\xi_x + f\xi\right\}dxd\tau$$
$$\geq \int_{\mathbb{R}} \left|b(u(t))-b(k)\right|\xi dx - \int_{\mathbb{R}} \left|b(u(s))-b(k)\right|\xi dx\,,$$

which is equivalent to (i) and (ii) (see [4], [13]).

Second Case. Suppose that (H_1), (H_2) and (H_4) hold then as in the proof of first case, we have

$$\begin{cases} b(u_{\epsilon_k}) \longrightarrow b(u) & \text{in } L^2_{loc}(Q) \text{ and a.e.}\,, \\ \varphi(u_{\epsilon_k}) \longrightarrow \varphi(u) & \text{in } L^2_{loc}(Q) \text{ and a.e.}\,, \\ \varphi(u_{\epsilon_k})_x \rightharpoonup \varphi(u)_x & \text{weakly in } L^2_{loc}(Q). \end{cases} \tag{2.5}$$

Consequently using the monotonicity of $\widetilde{a}(k,.)$, (2-5) and Proposition 2.5 of [8], one obtains:

$$h_{\epsilon_k} = a(u_{\epsilon_k}, \varphi(u_{\epsilon_k})_x) \longrightarrow h = a(u, \varphi(u)_x) \text{ in } L^1_{loc}(Q).$$

We conclude in the same way as the proof of the first case that u is entropy solution. □

4. Remarks

The concept of uniqueness considered here is the uniqueness of $b(u)$; on the other hand, if b is one to one, the uniqueness of $b(u)$ is equivalent to that of u.

The assumption (H_3) is interesting because it does not necessarily require b or φ to be one to one and enables us to obtain a result of existence of entropy solutions without Alt and Luckhaus type condition. Assumption (H_3) is not necessary if we have Alt and Luckhaus type condition on the nonlinearity a (assumption (H_4)).

We do not know if, under only assumptions (H_1) and (H_2), we can get existence of entropy solutions.

It should be noted that we study in this moment the question in which case, the mild solution obtained is renormalized entropy solution without Alt and Luckhaus structure condition by using a technique of perturbation according to Ammar and Wittbold [2], the difficulty here is that the question of continuous dependence of the mild solution with respect to the data in the general case which we consider is not acquired because of the strong (triple) nonlinearity of the problem.

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