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GEOMETRIC APPROACH TO ISO-TAXICAB INNER-PRODUCT

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Abstract: In Euclidean geometry, the inner-product, the norm and the geometrical meaning of inner-product are all well known. In the case of non-Euclidean geometries (spherical and hyperbolic geometries), we usually have difficulties. There are, now, two new non-Euclidean geometries – taxicab and iso-taxicab geometry.

The taxicab geometry is defined in 1975. It is, as mentioned by E.F. Krause, easy to understand and has many application in human life.

The iso-taxicab geometry is defined in 1989 by K.O. Sowell. The inner-product, the norm and the geometrical meaning of inner-product are given by the Ekici, Kocayusufoglu, Akca. The inner-product and the norm of iso-taxicab Geometry are defined by the author.

The aim of this paper is to give the geometrical meaning of iso-taxicab inner-pruduct.

AMS Subject Classification: 51E25, 51F99
Key Words: iso-taxicab geometry, inner-product, norm

1. Introduction

Iso-taxicab geometry is a new geometry. It is defined in 1989 by K.O. Sowell. It is a non-Euclidean geometry. As it is mentioned in [6] that in iso-taxicab geometry three axes occur at the origin: the x-axis, the y-axis and the y'-axis. This latter axis forms an angle of 60° with the x-axis and with the y-axis. But, the points will still be named by ordered pairs of real numbers with respect

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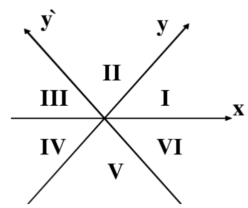

Figure 1:

to the x-axis and the y-axis. These three axes separate the plane into six regions, called hextants. These hextants will be numbered from I to VI in a counterclockwise direction beginning with the hextant where the coordinates of the points are both positive (Figure 1).

At any point in the plane three lines may be drawn parallel to the axes which separate the plane into six regions. Two points, then, may have $I-IV$ or $II-V$ or $III-VI$ orientation to one another. With these orientations, three distance functions arises:

$$d_I(A,B)=\begin{cases} (i) & |x_1-x_2|+|y_1-y_2| \quad , \quad I-IV \text{ orientation}\,, \\ (ii) & |y_1-y_2| \quad , \quad II-V \text{ orientation}\,, \\ (iii) & |x_1-x_2| \quad , \quad III-VI \text{ orientation}\,. \end{cases}$$

If the points lie on a line parallel to x-axis, the formula (iii) holds; parallel to y or y'-axes, the formula (ii) holds [6].

Definitions for inner-product and norm play an important role for any geometry. It is well known that the geometrical approach of inner-product in Euclidean geometry is

$$<\alpha,\beta>_E=\|\alpha\|_E\cdot\|\beta\|_E\cdot\cos_E(\theta)\,.$$

In taxicab geometry, it is proved in [1] that

$$<\alpha,\beta>_T=\|\alpha\|_T\cdot\|\beta\|_T\cdot\cos_T(\theta)-R_T\,,$$

where

$$R_T=\begin{cases} 2\,|a_2b_1| & , \quad I,I; \quad III,III\,, \\ -2\,|a_2b_1| & , \quad I,III; \quad II,IV\,, \\ 2\,|a_1b_2| & , \quad II,II; \quad IV,IV\,, \\ -2\,|a_1b_2| & , \quad I,III; \quad II,IV\,, \\ 0 & , \quad I,II\,, \quad I,IV; \quad II,III\,, \quad III,IV; \end{cases}$$

is called taxicab constant.

Here, in this paper, we will prove similar result for iso-taxicab geometry. Let us first give the reduction and substraction formulas in iso-taxicab geometry given in [4] which we will use them to give the geometrical approach.

2. Reduction Formulas

$\cos_I(\frac{\pi_I}{3}-\theta)$	$=$	$\sin_I(\theta)$
$\cos_I(\frac{\pi_I}{3}+\theta)$	$=$	$-\sin_I(\theta)$
$\cos_I(\frac{2\pi_I}{3}-\theta)$	$=$	$-1+\sin_I(\theta)$
$\cos_I(\frac{2\pi_I}{3}+\theta)$	$=$	-1
$\cos_I(\pi_I-\theta)$	$=$	-1
$\cos_I(\pi_I+\theta)$	$=$	$-1+\sin_I(\theta)$
$\cos_I(\frac{4\pi_I}{3}-\theta)$	$=$	$-\sin_I(\theta)$
$\cos_I(\frac{4\pi_I}{3}+\theta)$	$=$	$\sin_I(\theta)$
$\cos_I(\frac{5\pi_I}{3}-\theta)$	$=$	$1-\sin_I(\theta)$
$\cos_I(\frac{5\pi_I}{3}+\theta)$	$=$	1
$\cos_I(2\pi_I-\theta)$	$=$	1
$\cos_I(2\pi_I+\theta)$	$=$	$\cos_I(\theta)$

$\sin_I(\frac{\pi_I}{3}-\theta)$	$=$	$\cos_I(\theta)$
$\sin_I(\frac{\pi_I}{3}+\theta)$	$=$	1
$\sin_I(\frac{2\pi_I}{3}-\theta)$	$=$	1
$\sin_I(\frac{2\pi_I}{3}+\theta)$	$=$	$1-\sin_I(\theta)$
$\sin_I(\pi_I-\theta)$	$=$	$\sin_I(\theta)$
$\sin_I(\pi_I+\theta)$	$=$	$-\sin_I(\theta)$
$\sin_I(\frac{4\pi_I}{3}-\theta)$	$=$	$-1+\sin_I(\theta)$
$\sin_I(\frac{4\pi_I}{3}+\theta)$	$=$	-1
$\sin_I(\frac{5\pi_I}{3}-\theta)$	$=$	-1
$\sin_I(\frac{5\pi_I}{3}+\theta)$	$=$	$-1+\sin_I(\theta)$
$\sin_I(2\pi_I-\theta)$	$=$	$-\sin_I(\theta)$
$\sin_I(2\pi_I+\theta)$	$=$	$\sin_I(\theta)$

3. Substraction Formulas

There are 36 cases.

	$\boldsymbol{\theta_1}$	$\boldsymbol{\theta_2}$	$\boldsymbol{\theta_3}$	$\cos_I(\boldsymbol{\theta_1}-\boldsymbol{\theta_2})$
1	I	I	I	$1+\cos_I(\theta_1)-\cos_I(\theta_2)$
2	II	II	I	$1+\cos_I(\theta_1)-\cos_I(\theta_2)$
3	III	III	I	$1+\sin_I(\theta_1)-\sin_I(\theta_2)$
4	IV	IV	I	$1-\cos_I(\theta_1)+\cos_I(\theta_2)$
5	V	V	I	$1-\cos_I(\theta_1)+\cos_I(\theta_2)$
6	VI	VI	I	$1-\sin_I(\theta_1)+\sin_I(\theta_2)$
7	II	I	I	$1+\cos_I(\theta_1)-\cos_I(\theta_2)$
8	II	I	II	$1+\cos_I(\theta_1)-\cos_I(\theta_2)$

	θ_1	θ_2	θ_3	$\cos_I(\theta_1 - \theta_2)$
9	III	I	II	$-1+\sin_I(\theta_1)-\cos_I(\theta_2)$
10	III	I	III	-1
11	III	II	I	$-1+\sin_I(\theta_1)-\cos_I(\theta_2)$
12	III	II	II	$-1+\sin_I(\theta_1)-\cos_I(\theta_2)$
13	IV	I	III	-1
14	IV	I	IV	$-1+\cos_I(\theta_1)+\cos_I(\theta_2)$
15	IV	II	II	$-1+\sin_I(\theta_1)-\cos_I(\theta_2)$
16	IV	II	III	-1
17	IV	III	I	$1+\sin_I(\theta_1)-\sin_I(\theta_2)$
18	IV	III	II	$1+\sin_I(\theta_1)-\sin_I(\theta_2)$
19	V	I	IV	$-1+\cos_I(\theta_1)+\cos_I(\theta_2)$
20	V	I	V	$-1+\cos_I(\theta_1)+\cos_I(\theta_2)$
21	V	II	III	-1
22	V	II	IV	$-1-\cos_I(\theta_1)+\cos_I(\theta_2)$
23	V	III	II	$-\cos_I(\theta_1)-\sin_I(\theta_2)$
24	V	III	III	-1
25	V	IV	I	$1-\cos_I(\theta_1)+\cos_I(\theta_2)$
26	V	IV	II	$1-\cos_I(\theta_1)+\cos_I(\theta_2)$
27	VI	I	V	$-1+\sin_I(\theta_1)-\cos_I(\theta_2)$
28	VI	I	VI	1
29	VI	II	IV	$1+\sin_I(\theta_1)+\cos_I(\theta_2)$
30	VI	II	V	$1+\sin_I(\theta_1)+\cos_I(\theta_2)$
31	VI	III	III	-1
32	VI	III	IV	$-1+\sin_I(\theta_1)+\sin_I(\theta_2)$
33	VI	IV	II	$-1-\sin_I(\theta_1)+\cos_I(\theta_2)$
34	VI	IV	III	-1
35	VI	V	I	$-1-\sin_I(\theta_1)+\cos_I(\theta_2)$
36	VI	V	II	$-1-\sin_I(\theta_1)+\cos_I(\theta_2)$

Let us now give the inner product and gometrical meaning of iso-taxicab inner-product. Notice that if α, β are in the same quadrant, then there are 6 cases. If α, β are in different quadrant, then there are $\binom{6}{2} = 15$ cases. In total, we have 21 cases.

4. Iso-Taxicab Inner-Product and Iso-taxicab Constant

Here, we give the iso-taxicab inner product and iso-taxicab constant.

	θ_1	θ_2	$<\alpha,\beta>_I$	R_I
1	I	I	$\|a_1b_1\|+\|a_2b_2\|$	$2\|a_2b_1\|$
2	II	II	$\|a_2b_2\|$	$\|a_1b_2\|-\|a_2b_1\|$
3	III	III	$\|a_1b_1\|$	$\|a_1b_2\|-\|a_2b_1\|$
4	IV	IV	$\|a_1b_1\|+\|a_2b_2\|$	$2\|a_2b_1\|$
5	V	V	$\|a_2b_2\|$	$\|a_1b_2\|-\|a_2b_1\|$
6	VI	VI	$\|a_1b_1\|$	$\|a_1b_2\|-\|a_2b_1\|$
7	II	I	$-\|a_1b_1\|+\|a_2b_2\|$	$-\|a_2b_1\|$
8	III	I	$-\|a_1b_1\|$	$-\|a_2b_1\|$
9	III	II	$\|a_1b_1\|+\|a_2b_2\|$	$-\|a_2b_1\|$
10	IV	I	$-\|a_1b_1\|-\|a_2b_2\|$	$-2\|a_2b_1\|$
11	IV	II	$-\|a_2b_2\|$	$-\|a_2b_1\|$
12	IV	III	$\|a_1b_1\|-\|a_2b_2\|$	$-\|a_2b_1\|$
13	V	I	$\|a_1b_1\|-\|a_2b_2\|$	$\|a_2b_1\|$
14	V	II	$-\|a_2b_2\|$	0
15	V	III	$-\|a_1b_1\|-\|a_2b_2\|$	0
16	V	IV	$-\|a_1b_1\|+\|a_2b_2\|$	$-\|a_2b_1\|$
17	VI	I	$\|a_1b_1\|$	$\|a_2b_1\|$
18	VI	II	$-\|a_1b_1\|-\|a_2b_2\|$	$\|a_2b_1\|$
19	VI	III	$-\|a_1b_1\|$	0
20	VI	IV	$-\|a_1b_1\|$	$-\|a_2b_1\|$
21	VI	V	$\|a_1b_1\|+\|a_2b_2\|$	$-\|a_2b_1\|$

5. Geometrical Approach

Let $\alpha,\beta \in R^2$ be any two vectors and θ be the angle between them. The geometrical approach of iso-taxicab inner product is

$$<\alpha,\beta>_I = \|\alpha\|_I \cdot \|\beta\|_I \cdot \cos_I(\theta) - R_I \,.$$

Since the trigonometric equalities are different in iso-taxicab geometry, we need R_I to make the geometrical meaning of iso-taxicab inner product as in Euclidean geometry. Let us prove this for one case.

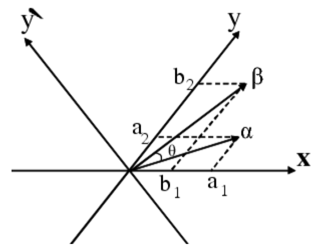

Figure 2:

Case 1. Let $\alpha = (a_1, a_2) \in I$ and $\beta = (b_1, b_2) \in I$ as in Figure 2. We have

$$< \alpha, \beta >_I = |a_1 b_1| + |a_2 b_2| \ .$$

Notice, on the other hand, that

$$\|\alpha\|_I = |a_1| + |a_2| \, , \quad \|\beta\|_I = |b_1| + |b_2| \ ,$$

and

$$\cos_I(\theta) = \cos_I(\boldsymbol{\theta_1} - \boldsymbol{\theta_2}) = 1 + \cos_I(\theta_1) - \cos_I(\theta_2) = 1 + \frac{|b_1|}{\|\beta\|_I} - \frac{|a_1|}{\|\alpha\|_I} \, .$$

Hence,

$$\|\alpha\|_I \ \|\beta\|_I \cos_I(\theta)$$
$$= \|\alpha\|_I \ \|\beta\|_I \ (1 + \frac{|b_1|}{\|\beta\|_I} - \frac{|a_1|}{\|\alpha\|_I}) = |a_1 b_1| + |a_2 b_2| + 2 \, |a_2 b_1| \ .$$

Thus,

$$< \alpha, \beta >_I = \|\alpha\|_I \cdot \|\beta\|_I \cdot \cos_I(\theta) - R_I$$

as claimed.

Case 2. Let $\alpha = (a_1, a_2) \in II$ and $\beta = (b_1, b_2) \in II$ as in Figure 3. We have

$$< \alpha, \beta >_I = |a_2 b_2| \ .$$

On the other hand,

$$\|\alpha\|_I = |a_2| \ , \quad \|\beta\|_I = |b_2| \ ,$$

and

$$\cos_I(\theta) = \cos_I(\boldsymbol{\theta_1} - \boldsymbol{\theta_2}) = 1 + \cos_I(\theta_1) - \cos_I(\theta_2) = 1 - \frac{|b_1|}{\|\beta\|_I} + \frac{|a_1|}{\|\alpha\|_I} \, .$$

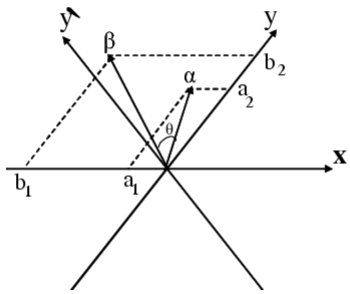

Figure 3:

Hence,

$$\|\alpha\|_I \ \|\beta\|_I \cos_I(\theta) = \|\alpha\|_I \ \|\beta\|_I \ (1 - \frac{|b_1|}{\|\beta\|_I} + \frac{|a_1|}{\|\alpha\|_I}) = |a_2 b_2| + |a_1 b_2| - |a_2 b_1|$$

Thus,

$$< \alpha, \beta >_I = \|\alpha\|_I \cdot \|\beta\|_I \cdot \cos_I(\theta) - R_I$$

as claimed.

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