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ON *R* SPACES

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Abstract: The main purpose of this paper is to introduce and study some new separation axioms by using γ-open sets.

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1. Introduction

Recent developments in general topology show that separation axioms play a fundemental roles in several situations. In literature, many authors introduced various separation axioms. The aim of this paper is to introduce and study weakly $\gamma - R_0$ and $\gamma - R_0$ separation axioms.

In this paper, spaces X and Y mean topological spaces. Let A be a subset of a space X. For a subset A of X, $\mathrm{cl}(A)$ and $\mathrm{int}(A)$ represent the closure of A and the interior of A, respectively.

A subset G is said to be preopen [13] (resp. semiopen [11]) if $G \subset \mathrm{cl}(\mathrm{int}(G))$ (resp. $G \subset \mathrm{cl}(\mathrm{int}(G))$). The complement of a preopen (resp. semiopen) set is said to be preclosed [9] (resp. semiclosed [4]).

A subset G of a topological space X is said to be γ-open [8] or b-open [1] or sp-open [5] if $G \subset \mathrm{int}(\mathrm{cl}(G)) \cup \mathrm{cl}(\mathrm{int}(G))$. The complement of a γ-open set is called a γ-closed set [8].

Arbitrary union (resp. intersection) of γ-open (resp. γ-closed) sets in X is γ-open (resp. γ-closed).

A subset M of a topological space X is called a γ-neighbourhood of a point x of X if there exists a γ-open set G such that $U \subset M$.

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The union of all γ-open (resp. preopen, semiopen) sets, each contained in a set V in a topological space X is called the γ-interior [8] (rep. preinterior [9], semiinterior [4]) of V and it is denoted by $\gamma\text{-int}\,(V)$ (resp. $\text{pint}\,(V)$, $\text{sint}(V)$).

The intersection of all γ-closed (resp. preclosed, semiclosed) sets, each containing a set V in a topological space X is called the γ-closure [8] (rep. preclosure [9], semiclosure [4]) of V and it is denoted by $\gamma\text{-cl}\,(V)$ (resp. $\text{pcl}\,(V)$, $\text{scl}\,(V)$).

The family of all γ-open sets containing a point $x \in X$ is denoted by $\gamma O(X, x)$. The family of all γ-open (resp. γ-closed) sets of X is denoted by $\gamma O(X)$ (resp. $\gamma C(X)$).

2. Separation Axioms

In this section, the notion of weakly γ-R_0 and γ-R_0 spaces are introduced and basic properties of them are investigated.

Definition 1. A topological space X is said to be weakly pre-R_0 [10] (resp. weakly semi-R_0 [2]) if $\bigcap_{x\in X} \text{pcl}\,(\{x\}) = \emptyset$ (resp. $\bigcap_{x\in X} \text{scl}\,(\{x\}) = \emptyset$).

Definition 2. Let X be a topological space. Then X is called γ-symmetric if $x \in \gamma\text{-cl}\,(\{y\})$, then $y \in \gamma\text{-cl}\,(\{x\})$ for x and y in X.

Definition 3. Let X be a topological space. The net $(x_i)_{i\in I}$ is said to be γ-convergent to a point x [7] if for each γ-open set U containing x there exists an element $i_0 \in I$ such that $i \geq i_0$ implies $x_i \in U$.

Definition 4. Let X be a topological space and $S \subset X$. The γ-kernel of S, denoted by $\gamma\text{-ker}\,(S)$, is defined to be the set $\gamma\text{-ker}\,(S) = \cap\{U \in \gamma O(X) : S \subset U\}$.

Proposition 5. *Let X be a topological space and $S \subset X$. The following properties hold:*

(1) S is γ-closed if and only if $S = \gamma\text{-cl}\,(S)$,

(2) If $M \subset N$, then $\gamma\text{-cl}\,(M) \subset \gamma\text{-cl}\,(N)$,

(3) $\gamma\text{-cl}\,(S)$ is γ-closed,

(4) $\gamma\text{-cl}\,(\gamma\text{-cl}\,(S)) = \gamma\text{-cl}\,(S)$,

(5) $x \in \gamma\text{-cl}\,(S)$ if and only if $S \cap V \neq \emptyset$ for every γ-open set V containing x.

Theorem 6. *Let X be a topological space and $x \in X$. Then $\gamma\text{-ker}\,(S) = \{x \in X : \gamma\text{-cl}\,(\{x\}) \cap S \neq \emptyset\}$.*

Proof. Let $x \in \gamma\text{-ker}(S)$. Suppose that $\gamma\text{-cl}(\{x\}) \cap S = \emptyset$. Thus, $x \notin X \backslash \gamma\text{-cl}(\{x\})$ which is a γ-open set containing S. Since $x \in \gamma\text{-ker}(S)$, this is a contradiction. Hence, $\gamma\text{-cl}(\{x\}) \cap S \neq \emptyset$.

Conversely, suppose that $\gamma\text{-cl}(\{x\}) \cap S \neq \emptyset$ for $x \in X$ and that $x \notin \gamma\text{-ker}(S)$. There exists a γ-open set V containing S and $x \notin V$. Let $y \in \gamma\text{-cl}(\{x\}) \cap S$. Thus, V is a γ-neighborhood of y such that $x \notin V$. This a contradiction. We obtain $x \in \gamma\text{-ker}(S)$. □

Definition 7. Let X be a topological space. Then X is said to be weakly $\gamma\text{-R}_0$ if $\bigcap_{x \in X} \gamma\text{-cl}(\{x\}) = \emptyset$.

Remark 8. The following hold for a topological space X:

$$\text{weakly pre-}R_0 \Rightarrow \text{weakly } \gamma\text{-}R_0 \Leftarrow \text{weakly semi-}R_0$$

These implications are not reversible.

Example 9. Let $p \in X$ and $\tau = \{G \subset X : p \notin G\} \cup \{X\}$. Then X is weakly $\gamma\text{-R}_0$, but it is not weakly pre-R_0.

Example 10. Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, X, \{a, b\}\}$. Then X is weakly $\gamma\text{-R}_0$, but it is not weakly semi-R_0.

Theorem 11. *Let X be atopological space. Then X is weakly γ-R_0 if and only if $\gamma\text{-ker}(\{x\}) \neq X$ for every $x \in X$.*

Proof. ($\Rightarrow$): Let X be weakly $\gamma\text{-R}_0$. Suppose that $\gamma\text{-ker}(\{y\}) = X$ for a point y in X. Then $\{y\} \neq S$ which S is any proper γ-open subset of X. Then $y \in \bigcap_{x \in X} \gamma\text{-cl}(\{x\})$. This is a contradiction.

($\Leftarrow$): Let $\gamma\text{-ker}(\{x\}) \neq X$ for each $x \in X$. If there exists a point y in X such that $y \in \bigcap_{x \in X} \gamma\text{-cl}(\{x\})$, then every γ-open set containing y must contain every point of X. Then X is the unique γ-open set containing y. Thus, $\gamma\text{-ker}(\{y\}) = X$.

This is a contradiction and then X is weakly $\gamma\text{-R}_0$. □

Definition 12. A function $f : X \to Y$ is called γ-closed if image of every γ-closed subset of X is γ-closed in Y.

Theorem 13. *If $f : X \to Y$ is an injective γ-closed function and X is weakly γ-R_0, then Y is weakly γ-R_0.*

Proof. Obvious. □

Definition 14. A topological space X is said to be a pre-R_0 space [3] if every preopen set contains the preclosure of each of its singletons.

Definition 15. A topological space X is said to be a semi-R_0 space [6, 12] if every semiopen set contains the semiclosure of each of its singletons.

Definition 16. A topological space X is said to be a γ-R_0 space if every γ-open set contains the γ-closure of each of its singletons.

Remark 17. We investigate the relationships among pre-R_0 space, γ-R_0 space and semi-R_0 space.

Example 18. Let $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, X, \{b\}, \{d\}, \{b, d\}\}$. Then X is γ-R_0, but it is not pre-R_0.

Example 19. Let $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, X, \{a, b\}\}$. Then X is γ-R_0, but it is not semi-R_0.

Question. Does there exist a space which is pre-R_0 and it is not γ-R_0 and does there exist a space which is semi-R_0 and it is not γ-R_0?

Theorem 20. *Let X be a topological space and $x \in X$. Then $y \in \gamma$-$\ker(\{x\})$ if and only if $x \in \gamma$-$\mathrm{cl}(\{y\})$.*

Proof. Let $y \notin \gamma$-$\ker(\{x\})$. Then there exists a γ-open set V containing x such that $y \notin V$. Hence, $x \notin \gamma$-$\mathrm{cl}(\{y\})$.

Converse is similar. □

Theorem 21. *Let X be a topological space. Then the following statements are equivalent for any points x and y in X:*

(1) γ-$\ker(\{x\}) \neq \gamma$-$\ker(\{y\})$,

(2) γ-$\mathrm{cl}(\{x\}) \neq \gamma$-$\mathrm{cl}(\{y\})$.

Proof. (1) $\Rightarrow$ (2): Let γ-$\ker(\{x\}) \neq \gamma$-$\ker(\{y\})$. Then there exists a point k in X such that $k \in \gamma$-$\ker(\{x\})$ and $k \notin \gamma$-$\ker(\{y\})$. Since $k \in \gamma$-$\ker(\{x\})$, it follows that $\{x\} \cap \gamma$-$\mathrm{cl}(\{k\}) \neq \emptyset$ and then $x \in \gamma$-$\mathrm{cl}(\{k\})$. Since $k \notin \gamma$-$\ker(\{y\})$, then $\{y\} \cap \gamma$-$\mathrm{cl}(\{k\}) = \emptyset$ and since $x \in \gamma$-$\mathrm{cl}(\{k\})$, it follows that γ-$\mathrm{cl}(\{x\}) \subset \gamma$-$\mathrm{cl}(\{k\})$ and $\{y\} \cap \gamma$-$\mathrm{cl}(\{k\}) = \emptyset$. Hence, γ-$\mathrm{cl}(\{x\}) \neq \gamma$-$\mathrm{cl}(\{y\})$. By using γ-$\ker(\{x\}) \neq \gamma$-$\ker(\{y\})$, we obtain γ-$\mathrm{cl}(\{x\}) \neq \gamma$-$\mathrm{cl}(\{y\})$.

(2) $\Rightarrow$ (1): Let γ-$\mathrm{cl}(\{x\}) \neq \gamma$-$\mathrm{cl}(\{y\})$. Then there exists a point k in X such that $k \in \gamma$-$\mathrm{cl}(\{x\})$ and $k \notin \gamma$-$\mathrm{cl}(\{y\})$ and then there exists a γ-open set containing k and x but not y. We have $y \notin \gamma$-$\ker(\{x\})$ and then γ-$\ker(\{x\}) \neq \gamma$-$\ker(\{y\})$. □

Theorem 22. *Let X be a topological space. Then X is γ-R_0 if and only if for every γ-closed set K and $x \notin K$, there exists a γ-open set S such that $K \subset S$ and $x \notin S$.*

Proof. ($\Rightarrow$): Let X be a γ-R_0 space and K be a γ-closed subset such that $x \notin K$. We have $X \backslash K$ is γ-open and $x \in X \backslash K$. Since X is γ-R_0, then γ-$\mathrm{cl}(\{x\}) \subset X \backslash K$. We obtain $K \subset X \backslash \gamma$-$\mathrm{cl}(\{x\})$. Take $S = X \backslash \gamma$-$\mathrm{cl}(\{x\})$. Thus, S is a γ-open set such that $K \subset S$ and $x \notin S$.

($\Leftarrow$): Let S be a γ-open set and $x \in U$. Then $X \backslash S$ is a γ-closed set and $x \notin X \backslash S$. Then there exists a γ-open subset U such that $X \backslash S \subset U$ and $x \notin U$. We obtain $X \backslash U \subset S$ and $x \in X \backslash U$. Since $X \backslash U$ is a γ-closed set, then γ-$\mathrm{cl}(\{x\}) \subset X \backslash U \subset S$. Hence, X is a γ-R_0 space. □

Theorem 23. *Let X be a topological space. Then the following properties are equivalent:*

(1) X is a γ-R_0 space,

(2) $x \in \gamma$-$\mathrm{cl}(\{y\})$ if and only if $y \in \gamma$-$\mathrm{cl}(\{x\})$ for any points x and y in X.

Proof. (1) $\Rightarrow$ (2): Let X be γ-R_0. Let $x \in \gamma$-$\mathrm{cl}(\{y\})$ and S be any γ-open set such that $y \in S$. By (1), $x \in S$. Hence, every γ-open set which contains y contains x and then $y \in \gamma$-$\mathrm{cl}(\{x\})$.

(2) $\Rightarrow$ (1): Let U be a γ-open set and $x \in U$. If $y \notin U$, then $x \notin \gamma$-$\mathrm{cl}(\{y\})$ and hence $y \notin \gamma$-$\mathrm{cl}(\{x\})$. We have γ-$\mathrm{cl}(\{x\}\} \subset U$. Thus, X is γ-R_0. □

Theorem 24. *Let X be a topological space. Then the following properties are equivalent:*

(1) X is a γ-R_0 space,

(2) X is a γ-symmetric.

Theorem 25. *Let X be a topological space. Then the following properties are equivalent:*

(1) X is a γ-R_0 space,

(2) for any nonempty set S and $G \in \gamma O(X)$ such that $S \cap G \neq \emptyset$, there exists $K \in \gamma C(X)$ such that $S \cap K \neq \emptyset$ and $K \subset G$,

(3) $G = U\{K \in \gamma C(X) : K \subset G\}$ for any $G \in \gamma O(X)$,

(4) $K = \cap\{G \in \gamma O(X) : K \subset G\}$ any $K \in \gamma C(X)$,

(5) γ-$\mathrm{cl}(\{x\}) \subset \gamma$-$\mathrm{ker}(\{x\})$ for any $x \in X$.

Proof. (1) $\Rightarrow$ (2): Let S be a nonempty set of X and $G \in \gamma O(X)$ such that $S \cap G \neq \emptyset$. There exists $x \in S \cap G$. Since $x \in G \in \gamma O(X)$, it follows that γ-$\mathrm{cl}(\{x\}) \subset G$. Take $K = \gamma$-$\mathrm{cl}(\{x\})$, then $K \in \gamma C(X)$, $K \subset G$ and $S \cap K \neq \emptyset$.

(2) $\Rightarrow$ (3): Let $G \in \gamma O(X)$. We have $G \supset \cup\{K \in \gamma C(X) : K \subset G\}$. Let x be any point of G. There exists $K \in \gamma C(X)$ such that $x \in K$ and $K \subset G$. Thus, we have $x \in K \subset \cup\{K \in \gamma C(X) : K \subset G\}$ and hence $G = \cup\{K \in \gamma C(X) : K \subset G\}$.

(3) $\Rightarrow$ (4): It is clear.

(4) $\Rightarrow$ (5): Let x be any point of X and $y \notin \gamma\text{-ker}(\{x\})$. There exists $V \in \gamma O(X)$ such that $x \in V$ and $y \notin V$; hence $\gamma\text{-cl}(\{y\}) \cap V = \emptyset$. By (4) $[\cap\{G \in \gamma O(X) : \gamma\text{-cl}(\{y\}) \subset G\}] \cap V = \emptyset$ and there exists $G \in \gamma O(X)$ such that $x \notin G$ and $\gamma\text{-cl}(\{y\}) \subset G$. Hence, $\gamma\text{-cl}(\{x\}) \cap G = \emptyset$ and $y \notin \gamma\text{-cl}(\{x\})$. Thus, $\gamma\text{-cl}(\{x\}) \subset \gamma\text{-ker}(\{x\})$.

(5) $\Rightarrow$ (1): Let $G \in \gamma O(X)$ and $x \in G$. Let $y \in \gamma\text{-ker}(\{x\})$. We have $x \in \gamma\text{-cl}(\{y\})$ and $y \in G$. It follows that $\gamma\text{-ker}(\{x\}) \subset G$. Thus, we obtain $x \in \gamma\text{-cl}(\{x\}) \subset \gamma\text{-ker}(\{x\}) \subset G$. This shows that X is a γ-R_0 space. □

Theorem 26. *Let X be a topological space. Then X is a γ-R_0 space if and only if for any x and y in X, $\gamma\text{-cl}(\{x\}) \neq \gamma\text{-cl}(\{y\})$ implies $\gamma\text{-cl}(\{x\}) \cap \gamma\text{-cl}(\{y\}) = \emptyset$.*

Proof. ($\Rightarrow$): Let X be γ-R_0 and $x, y \in X$ such that $\gamma\text{-cl}(\{x\}) \neq \gamma\text{-cl}(\{y\})$. Then, there exist a $k \in \gamma\text{-cl}(\{x\})$ such that $k \notin \gamma\text{-cl}(\{y\})$ (or $k \in \gamma\text{-cl}(\{y\})$ such that $k \notin \gamma\text{-cl}(\{x\})$) and then there exists $V \in \gamma O(X)$ such that $y \notin V$ and $k \in V$ and hence $x \in V$. Thus, $x \notin \gamma\text{-cl}(\{y\})$ and $x \in X \backslash \gamma\text{-cl}(\{y\}) \in \gamma O(X)$. We have $\gamma\text{-cl}(\{x\}) \subset X \backslash \gamma\text{-cl}(\{y\})$ and $\gamma\text{-cl}(\{x\}) \cap \gamma\text{-cl}(\{y\}) = \emptyset$.

($\Leftarrow$): Let $V \in \gamma O(X)$ and $x \in V$. Let $y \notin V$. We have $y \in X \backslash V$. Then $x \neq y$ and $x \notin \gamma\text{-cl}(\{y\})$. We obtain $\gamma\text{-cl}(\{x\}) \neq \gamma\text{-cl}(\{y\})$ and then $\gamma\text{-cl}(\{x\}) \cap \gamma\text{-cl}(\{y\}) = \emptyset$. Thus, $y \notin \gamma\text{-cl}(\{x\})$ and then $\gamma\text{-cl}(\{x\}) \subset V$. We obtain that X is a γ-R_0 space. □

Theorem 27. *Let X be a topological space. Then the following properties are equivalent:*

(1) X is a γ-R_0 space,

(2) if K is γ-closed, then $K = \gamma\text{-ker}(K)$,

(3) if K is γ-closed and $x \in K$, then $\gamma\text{-ker}(\{x\}) \subset K$,

(4) if $x \in X$, then $\gamma\text{-ker}(\{x\}) \subset \gamma\text{-cl}(\{x\})$.

Proof. (1) $\Rightarrow$ (2): By using Theorem 25, it can be obtained.

(2) $\Rightarrow$ (3): Since $\{x\} \subset K$, it follows that $\gamma\text{-ker}(\{x\}) \subset \gamma\text{-ker}(K) = K$.

(3) $\Rightarrow$ (4): Since $x \in \gamma\text{-cl}(\{x\})$ and $\gamma\text{-cl}(\{x\})$ is γ-closed, it follows that $\gamma\text{-ker}(\{x\}) \subset \gamma\text{-cl}(\{x\})$.

(4) $\Rightarrow$ (1): Let $x \in \gamma\text{-cl}(\{y\})$. By Theorem 20, $y \in \gamma\text{-ker}(\{x\})$. By (4), $y \in \gamma\text{-ker}(\{x\}) \subset \gamma\text{-cl}(\{x\})$. Thus, $x \in \gamma\text{-cl}(\{y\})$ and then $y \in \gamma\text{-cl}(\{x\})$. We obtain the converse similarly. Hence, X is γ-R$_0$. □

Theorem 28. *Let X be a topological space. Then X is a γ-R_0 space if and only if $\gamma\text{-ker}(\{x\}) \neq \gamma\text{-ker}(\{y\}$ for any points x and y in X, implies $\gamma\text{-ker}(\{x\}) \cap \gamma\text{-ker}(\{y\}) = \emptyset$.*

Proof. ($\Rightarrow$): Let X is a γ-R$_0$ space. By Theorem 21, if $\gamma\text{-ker}(\{x\}) \neq \gamma\text{-ker}(\{y\})$ for any points x and y in X, then $\gamma\text{-cl}(\{x\}) \neq \gamma\text{-cl}(\{y\})$. Let $k \in \gamma\text{-ker}(\{x\} \cap \gamma\text{-ker}(\{y\})$. Since $k \in \gamma\text{-ker}(\{x\})$, by Theorem 20, it follows that $x \in \gamma\text{-cl}(\{k\})$. Since $x \in \gamma\text{-cl}(\{k\})$, by Theorem 26 $\gamma\text{-cl}(\{x\}) = \gamma\text{-cl}(\{k\})$. Similarly, we have $\gamma\text{-cl}(\{y\}) = \gamma\text{-cl}(\{k\}) = \gamma\text{-cl}(\{x\})$. This is a contradiction. Thus, $\gamma\text{-ker}(\{x\}) \cap \gamma\text{-ker}(\{y\}) = \emptyset$.

($\Leftarrow$): Let X be a topological space such that if $\gamma\text{-ker}(\{x\}) \neq \gamma\text{-ker}(\{y\})$ for any points x and y in X, then $\gamma\text{-ker}(\{x\}) \cap \gamma\text{-ker}(\{y\}) = \emptyset$. If $\gamma\text{-cl}(\{x\}) \neq \gamma\text{-cl}(\{y\})$, then by Theorem 21, $\gamma\text{-ker}(\{x\}) \neq \gamma\text{-ker}(\{y\})$. Hence $\gamma\text{-ker}(\{x\}) \cap \gamma\text{-ker}(\{y\}) = \emptyset$ and then $\gamma\text{-cl}(\{x\}) \cap \gamma\text{-cl}(\{y\}) = \emptyset$. If $k \in \gamma\text{-cl}(\{x\})$, it follows that $x \in \gamma\text{-ker}(\{k\})$ and hence $\gamma\text{-ker}(\{x\}) \cap \gamma\text{-ker}(\{k\}) \neq \emptyset$. We have $\gamma\text{-ker}(\{x\}) = \gamma\text{-ker}(\{k\})$. Then $k \in \gamma\text{-cl}(\{x\} \cap \gamma\text{-cl}(\{y\})$ implies that $\gamma\text{-ker}(\{x\}) = \gamma\text{-ker}(\{k\}) = \gamma\text{-ker}(\{y\})$. This is a contradiction. Thus, $\gamma\text{-cl}(\{x\}) \cap \gamma\text{-cl}(\{y\}) = \emptyset$. By Theorem 26, we obtain that X is a γ-R$_0$ space. □

Theorem 29. *Let X be a topological space and let $x, y \in X$. If every net in X γ-converging to y γ-converges to x, then $x \in \gamma\text{-cl}(\{y\})$.*

Proof. Let $x_i = y$ for each $i \in I$. Then $(x_i)_{i \in I}$ is a net in $\gamma\text{-cl}(\{y\})$. Since $(x_i)_{i \in I}$ γ-converges to y, then $(x_i)_{i \in I}$ γ-converges to x and then $x \in \gamma\text{-cl}(\{y\})$. □

Theorem 30. *Let X be a topological space. Then the following statements are equivalent :*

(1) X is a γ-R_0 space,

(2) If $x, y \in X$, then $y \in \gamma\text{-cl}(\{x\})$ if and only if every net in X γ-converging to y γ-converges to x.

Proof. (1) $\Rightarrow$ (2): Let $x, y \in X$ such that $y \in \gamma\text{-cl}(\{x\})$. Let $(x_i)_{i \in I}$ be a net in X such that $(x_i)_{i \in I}$ γ-converges to y. Since $y \in \gamma\text{-cl}(\{x\})$, by Theorem 26, it follows that $\gamma\text{-cl}(\{x\}) = \gamma\text{-cl}(\{y\})$. Thus, $x \in \gamma\text{-cl}(\{y\})$ and then $(x_i)_{i \in I}$ γ-converges to x.

Conversely, let x, $y \in X$ such that every net in X γ-converging to y γ-converges to x. By Theorem 29 $x \in \gamma\text{-cl}(\{y\})$. By Theorem 26, $\gamma\text{-cl}(\{x\}) = \gamma\text{-cl}(\{y\})$. Hence, $y \in \gamma\text{-cl}(\{x\})$.

(2) $\Rightarrow$ (1): Let x and y be any two points of X such that $\gamma\text{-cl}(\{x\}) \cap \gamma\text{-cl}(\{y\}) \neq \emptyset$. Let $k \in \gamma\text{-cl}(\{x\}) \cap \gamma\text{-cl}(\{y\})$. Then there exists a net $(x_i)_{i\in I}$ in $\gamma\text{-cl}(\{x\})$ such that $(x_i)_{i\in I}$ γ-converges to k. Since $k \in \gamma\text{-cl}(\{y\})$, it follows that $(x_i)_{i\in I}$ γ-converges to y. Then $y \in \gamma\text{-cl}(\{x\})$. By using similar way, we obtain $x \in \gamma\text{-cl}(\{y\})$. Hence, $\gamma\text{-cl}(\{x\}) = \gamma\text{-cl}(\{y\})$ and then X is γ-R_0 by Theorem 26. □

Theorem 31. *For a topological space X, the following properties are equivalent:*

(1) X is a γ-R_0 space,

(2) $\gamma\text{-cl}(\{x\}) = \gamma\text{-ker}(\{x\})$ for all $x \in X$.

Proof. (1) $\Rightarrow$ (2): Let X be a γ-R_0 space. By Theorem 25, $\gamma\text{-cl}(\{x\}) \subset \gamma\text{-ker}(\{x\})$ for each $x \in X$. Let $y \in \gamma\text{-ker}(\{x\})$. By Theorem 26, we have $x \in \gamma\text{-cl}(\{y\})$ and then $\gamma\text{-cl}(\{x\}) = \gamma\text{-cl}(\{y\})$. Hence, $y \in \gamma\text{-cl}(\{x\})$ and then $\gamma\text{-ker}(\{x\}) \subset \gamma\text{-cl}(\{x\})$. We obtain $\gamma\text{-cl}(\{x\}) = \gamma\text{-ker}(\{x\})$.

(2) $\Rightarrow$ (1): By using Theorem 25, it can be obtained easily. □

Definition 32. A topological space X is called γ-T_0 [7] if for any distinct pair of points in X, there exists a γ-open set containing one of the points but not the other.

Definition 33. A topological space X is called γ-T_1 [7] if for any distinct pair of points x and y in X, there exists a γ-open U in X containing x but not y and a γ-open set V in X containing y but not x.

Theorem 34. *Let X be a topological space. Then X is γ-T_1 if and only if the singletons are γ-closed sets.*

Theorem 35. *Let X be a topological space. Then X is γ-T_1 if and only if it is a γ-T_0 and γ-R_0 space.*

Proof. Let X be a γ-T_1 space. By the definition of γ-T_1 space, it is a γ-T_0 and γ-R_0 space.

Conversely, let X be a γ-T_0 and γ-R_0 space. Let x, y be any two distinct points of X. Since X is γ-T_0, then there exists a γ-open set U such that $x \in U$ and $y \notin U$ or there exists a γ-open set V such that $y \in V$ and $x \notin V$. Let $x \in U$ and $y \notin U$. Since X is γ-R_0, then $\gamma\text{-cl}(\{x\}) \subset U$. We have $y \notin U$ and then $y \notin \gamma\text{-cl}(\{x\})$. We obtain $y \in X \backslash \gamma\text{-cl}(\{x\})$. Take $S = X \backslash \gamma\text{-cl}(\{x\})$. Thus, U

and S are γ-open sets containing x and y, respectively, such that $y \notin U$ and $x \notin S$. Hence, X is γ-T_1. □

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