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UNIQUENESS, STABILITY AND CONTINUOUS DEPENDENCE OF SOLUTIONS OF PARABOLIC SYSTEM

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Abstract: In this paper, we prove the uniqueness and stability of solutions and the continuous dependence on parameter variables for parabolic problem

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{\partial}{\partial x}(\alpha(u)\frac{\partial u}{\partial x}) + f(x,t,u), & (x,t) \in \Omega \times I, \\ u(x,0) = \Phi(x), & x \in \Omega, \\ u(0,t) = p(t), \quad u(l,t) = q(t), & t \in I. \end{cases}$$

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1. Introduction

In the study of heat diffusivity of the Arctic ice, spatially one-dimensional nonlinear parabolic equations need dealing with, i.e. the following problem:

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{\partial}{\partial x}(\alpha(u)\frac{\partial u}{\partial x}) + f(x,t,u), & (x,t) \in \Omega \times I, \\ u(x,0) = \Phi(x), & x \in \Omega, \\ u(0,t) = p(t), \quad u(l,t) = q(t) & t \in I, \end{cases} \tag{1}$$

where $\alpha(u) = a - bu$ is the heat diffusivity, $a, b \in R_+$; $I = [0,T]$ is time and $\Omega = [0,l] \subset R$ is the smooth domain; f, Φ, p, q are the smooth functions.

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Differential theory for parabolic equations is an object of longstanding interest. For the recent developments in parabolic theory we refer to (see [3], [5], [1], [2], [4], and so on). However, $\alpha(u)$ included in all the above papers are constant functions or not explicitly dependent on u, which certainly restricts the range of applications of solution theory. In this paper, based on the sea ice model, we employ the integration to deal with the parabolic equations with $\alpha(u)$ linear dependent on u.

The paper is organized as follows. In Section 2, we introduce two lemmas whose results will be used in the proof of theorems. Next, we give the uniqueness, stability and continuous dependence by the method to integration for problem (1). We draw conclusions in Section 3.

2. Main Results

Throughout this paper we assume $u \in C^2(\Omega \times I)$ and $u(x,t)$ is bounded for every $(x,t) \in \Omega \times I$; $\|\cdot\|$ denotes the norm in R; f is Lipschitizian in u, namely,

$$\|(f(x,t,u_1) - f(x,t,u_2))\| \leq M \|u_1 - u_2\|, \quad M > 0.$$

Lemma 1. *For $t \in I$, $u \in C^2(\Omega \times I)$, it is obvious to obtain the following results:*

(i) let $J(t) = \int_\Omega u^2(x,t)dx$, then $J'(t) = 2\int_\Omega u\frac{\partial u}{\partial t}dx$;

(ii) assume $\alpha(u) = a - bu, a, b \in R_+$, then $\left\|\frac{\partial^2 \alpha(x,t)}{\partial x^2}\right\| < L_1$, $L_1 \in R_+$.

Theorem 2. *For fixed $a, b \in R_+$, $\alpha(u) = a - bu$, assume that (1) exists solutions, then the solution is unique.*

Proof. Let $u_1(x,t)$ and $u_2(x,t)$ be the solutions of (1), then it is obvious that $\overline{u}$=$u_1 - u_2$ satisfies the following equations:

$$\frac{\partial \overline{u}}{\partial t} = \frac{\partial}{\partial x}\left(\alpha(u_1)\frac{\partial u_1}{\partial x} - \alpha(u_2)\frac{\partial u_2}{\partial x}\right) + f(x,t,u_1) - f(x,t,u_2),$$
$$(x,t) \in \Omega \times I, \quad (2)$$

with the initial value

$$\overline{u}(x,0) = 0, \quad x \in \Omega, \quad (3)$$

and the boundary conditions

$$\overline{u}(0,t) = 0, \quad \overline{u}(l,t) = 0, \quad t \in I. \quad (4)$$

Let $J(t)=\int_\Omega \overline{u}^2(x,t)dx$, according to Lemma 1, (2), (3) and (4), we have

$$J'(t)=2\int_\Omega \overline{u}\left(\frac{\partial}{\partial x}\left(\alpha(u_1)\frac{\partial u_1}{\partial x}-\alpha(u_2)\frac{\partial u_2}{\partial x}\right)+f(x,t,u_1)-f(x,t,u_2)\right)dx$$
$$=-2\int_\Omega\left(\alpha(u_1)\frac{\partial u_1}{\partial x}-\alpha(u_2)\frac{\partial u_2}{\partial x}\right)\frac{\partial \overline{u}}{\partial x}dx+2\int_\Omega\left(f(x,t,u_1)-f(x,t,u_2)\right)\overline{u}dx.$$

According to $\|f(x,t,u_1)-f(x,t,u_2)\|\le M\|u_1-u_2\|=M\|\overline{u}\|$ and $\alpha(u)=a-bu$, we get

$$J'(t)\le -2\int_\Omega\left(\alpha(u_1)\frac{\partial u_1}{\partial x}-\alpha(u_2)\frac{\partial u_2}{\partial x}\right)\frac{\partial \overline{u}}{\partial x}dx+2\int_\Omega M\|\overline{u}\|\|\overline{u}\|dx$$
$$=-2a\int_\Omega\left(\frac{\partial \overline{u}}{\partial x}\right)^2dx+2b\int_\Omega\left(u_1\frac{\partial u_1}{\partial x}-u_2\frac{\partial u_2}{\partial x}\right)\frac{\partial \overline{u}}{\partial x}dx+2M\int_\Omega \overline{u}^2dx$$
$$=2a\int_\Omega \overline{u}\frac{\partial^2\overline{u}}{\partial x^2}dx-b\int_\Omega \overline{u}\frac{\partial^2\overline{u}}{\partial x^2}\left(u_1+u_2\right)dx+2MJ(t).$$

Based on the integration by parts, $J'(t)$ can be rewritten by

$$J'(t)=\int_\Omega\left(\alpha(u_1)+\alpha(u_2)\right)\overline{u}\frac{\partial^2\overline{u}}{\partial x^2}dx+2MJ(t)$$
$$=-\int_\Omega\left(\alpha(u_1)+\alpha(u_2)\right)\left(\frac{\partial \overline{u}}{\partial x}\right)^2dx-\int_\Omega\frac{\partial\left(\alpha(u_1)+\alpha(u_2)\right)}{\partial x}\frac{\partial \overline{u}}{\partial x}\overline{u}dx+2MJ(t).$$

Due to the positive heat diffusivity, we arrive at that

$$\int_\Omega\left(\alpha(u_1)+\alpha(u_2)\right)\left(\frac{\partial \overline{u}}{\partial x}\right)^2dx\ge 0.$$

Therefore,

$$J'(t)\le-\int_\Omega\frac{\partial\left(\alpha(u_1)+\alpha(u_2)\right)}{\partial x}\frac{\partial \overline{u}}{\partial x}\overline{u}dx+2MJ(t)$$
$$=\frac{1}{2}\int_\Omega\frac{\partial^2\left(\alpha(u_1)+\alpha(u_2)\right)}{\partial x^2}\overline{u}^2dx+2MJ(t)$$
$$\le\frac{1}{2}\int_\Omega\left(\left\|\frac{\partial^2\alpha(u_1)}{\partial x^2}\right\|+\left\|\frac{\partial^2\alpha(u_2)}{\partial x^2}\right\|\right)\overline{u}^2dx+2MJ(t).$$

In terms of Lemma 1, let $L=L_1+2M$, then we have

$$J'(t)\le L\int_\Omega \overline{u}^2dx=LJ(t).$$

Therefore, $J'(t) \leq LJ(t)$, i.e. $J'(t) - LJ(t) \leq 0$. We multiply both sides of the inequality by e^{-Lt}:

$$J'(t)e^{-Lt} - LJ(t)e^{-Lt} = \frac{d}{dt}\left(J(t)e^{-Lt}\right) \leq 0.$$

Integrating the above inequality on $[0,t]$, $t \in I$, then

$$J(t) \leq J(0)e^{Lt} = 0.$$

Considering that $J(t) \geq 0$, we have

$$J(t) = 0.$$

Hence $\overline{u}^2(x,t) = 0$, i.e. $u_1 = u_2$. That is to say that, the solution of (1) is unique for fixed $a, b \in R_+$. □

Theorem 3. *For $t \in I$, $\alpha(u) = a - bu$, a, $b \in R_+$, let $u_1(x,t)$ and $u_2(x,t)$ be the solutions of (1) corresponding to different initial values $\Phi_1(x)$ and $\Phi_2(x)$, respectively. The solution of (1) is stable, i.e. $\|\Phi_1(x) - \Phi_2(x)\| < \delta$, then $\|u_1 - u_2\| < \varepsilon$.*

Proof. Similar to the proof of Theorem 2, let $L = L_1 + 2M$, we obtain $J(t) \leq J(0)e^{Lt} < e^{Lt}\int_\Omega \delta^2 dx \leq e^{LT}\int_\Omega \delta^2 dx$, which yields $\|u_1 - u_2\| < \varepsilon$. □

Lemma 4. *For $t \in I$, $u \in C^2(\Omega \times I)$, let $\alpha_1 = a_1 - b_1 u$, $\alpha_2 = a_2 - b_2 u$ and $u_1(x,t)$, $u_2(x,t)$ be the solutions of (1) corresponding to $\alpha = \alpha_1$ and $\alpha = \alpha_2$, respectively, $\overline{u} = u_1 - u_2$, then:*

(i) $\int_\Omega \left\|\overline{u}(\frac{\partial^2 u_1}{\partial x^2} + \frac{\partial^2 u_2}{\partial x^2})\right\| dx < L_2$,

(ii) $\int_\Omega \left\|\frac{\partial^2 \overline{u}}{\partial x^2}\frac{u_1^2 + u_2^2}{2}\right\| dx < L_3$,

(iii) $\int_\Omega \left\|\overline{u}^2 \frac{\partial^2 \overline{u}}{\partial x^2}\right\| dx < L_4$,

where a_1, b_1, a_2, b_2, L_2, L_3, $L_4 \in R_+$.

Proof. $u_1(x,t)$ and $u_2(x,t)$ are the solutions of (1), so $u_1(x,t)$ and $u_2(x,t)$ are bounded. Because $u \in C^2(\Omega \times I)$ and $\left\|\overline{u}(\frac{\partial^2 u_1}{\partial x^2} + \frac{\partial^2 u_2}{\partial x^2})\right\| \leq \|\overline{u}\| \cdot (\left\|\frac{\partial^2 u_1}{\partial x^2}\right\| + \left\|\frac{\partial^2 u_2}{\partial x^2}\right\|)$, there exists a constant $\frac{L_2}{l} \in R_+$ such that $\left\|u(\frac{\partial^2 u_1}{\partial x^2} + \frac{\partial^2 u_2}{\partial x^2})\right\| \leq \frac{L_2}{l}$. Therefore, we prove that (i) holds. A similar process is considered, one can obtain (ii) and (iii). □

Theorem 5. *Let $\alpha_1 = a_1 - b_1 u$, $\alpha_2 = a_2 - b_2 u$ and $u_1(x,t)$, $u_2(x,t)$ be the solutions of (1) corresponding to $\alpha = \alpha_1$ and $\alpha = \alpha_2$, respectively. Assume $\|a_1 - a_2\| < \delta$, $\|b_1 - b_2\| < \eta$, $a_1, a_2, b_1, b_2 \in R_+$, then $\|u_1 - u_2\| < \varepsilon$.*

Proof. $\overline{u} = u_1 - u_2$ satisfies the above (3), (4) and the following equation:

$$\frac{\partial \overline{u}}{\partial t} = \frac{\partial}{\partial x}\left(\alpha_1 \frac{\partial u_1}{\partial x} - \alpha_2 \frac{\partial u_2}{\partial x}\right) + f(x,t,u_1) - f(x,t,u_2), \quad (x,t) \in \Omega \times I. \quad (5)$$

Let $J(t) = \int_\Omega \overline{u}^2(x,t)dx$, then, we get

$$\begin{aligned} J'(t) = 2\int_\Omega \overline{u}\left(\frac{\partial}{\partial x}\left(\alpha_1 \frac{\partial u_1}{\partial x} - \alpha_2 \frac{\partial u_2}{\partial x}\right) + f(x,t,u_1) - f(x,t,u_2)\right)dx \\ = -2\int_\Omega \left(\alpha_1 \frac{\partial u_1}{\partial x} - \alpha_2 \frac{\partial u_2}{\partial x}\right)\frac{\partial \overline{u}}{\partial x}dx \\ + 2\int_\Omega \overline{u}\left(f(x,t,u_1) - f(x,t,u_2)\right)dx\,. \quad (6) \end{aligned}$$

In terms of $\|(f(x,t,u_1) - f(x,t,u_2))\| \le M\|u_1 - u_2\|$, (6) can be rewritten by

$$J'(t) \le -2\int_\Omega \left(\alpha_1 \frac{\partial u_1}{\partial x} - \alpha_2 \frac{\partial u_2}{\partial x}\right)\frac{\partial \overline{u}}{\partial x}dx + 2M\int_\Omega \overline{u}^2 dx\,.$$

According to $\alpha_1 = a_1 - b_1 u$ and $\alpha_2 = a_2 - b_2 u$, then we get

$$J'(t) = -2\int_\Omega \left(a_1 \frac{\partial u_1}{\partial x} - a_2 \frac{\partial u_2}{\partial x} - \left(b_1 u_1 \frac{\partial u_1}{\partial x} - b_2 u_2 \frac{\partial u_2}{\partial x}\right)\right)\frac{\partial \overline{u}}{\partial x}dx + 2MJ(t)\,.$$

In addition,

$$\begin{aligned} &2\left(a_1 \frac{\partial u_1}{\partial x} - a_2 \frac{\partial u_2}{\partial x}\right) \\ &= (a_1 - a_2)\frac{\partial u_1}{\partial x} + a_2\left(\frac{\partial u_1}{\partial x} - \frac{\partial u_2}{\partial x}\right) + (a_1 - a_2)\frac{\partial u_2}{\partial x} + a_1\left(\frac{\partial u_1}{\partial x} - \frac{\partial u_2}{\partial x}\right), \end{aligned}$$

and $2\left(b_1 u_1 \frac{\partial u_1}{\partial x} - b_2 u_2 \frac{\partial u_2}{\partial x}\right)$ can be written in the same way. So

$$\begin{aligned} J'(t) = (a_1 + a_2)\int_\Omega \overline{u}\frac{\partial^2 \overline{u}}{\partial x^2}dx + (a_1 - a_2)\int_\Omega \overline{u}\left(\frac{\partial^2 u_1}{\partial x^2} + \frac{\partial^2 u_2}{\partial x^2}\right)dx \\ + 2MJ(t) - (b_1 - b_2)\int_\Omega \frac{\partial^2 \overline{u}}{\partial x^2}\frac{u_1^2 + u_2^2}{2}dx - \frac{(b_1 + b_2)}{2}\int_\Omega \overline{u}\frac{\partial^2 \overline{u}}{\partial x^2}(u_1 + u_2)dx \end{aligned}$$

$$\leq \delta L_2 + \eta L_3 + \int_\Omega \left(a_1 + a_2 - (b_1 + b_2)\frac{(u_1+u_2)}{2} \right) \overline{u}\frac{\partial^2 \overline{u}}{\partial x^2} dx + 2MJ(t).$$

Note that

$$-(b_1+b_2)\frac{(u_1+u_2)}{2} = (b_1-b_2)\frac{(u_1-u_2)}{2} - b_1 u_1 - b_2 u_2,$$

then

$$\begin{aligned} J'(t) &= \delta L_2 + \eta L_3 + \int_\Omega (\alpha_1+\alpha_2)\, \overline{u}\frac{\partial^2 \overline{u}}{\partial x^2} dx - \frac{b_1-b_2}{2}\int_\Omega \overline{u}^2 \frac{\partial^2 \overline{u}}{\partial x^2} dx + 2MJ(t) \\ &\leq \delta L_2 + \eta L_3 + \eta L_4/2 - \int_\Omega \left(\frac{\partial \overline{u}}{\partial x}\right)^2 (\alpha_1+\alpha_2)\, dx \\ &\quad - \int_\Omega \overline{u}\frac{\partial u}{\partial x}\frac{\partial}{\partial x}(\alpha_1+\alpha_2)\, dx + 2MJ(t) \\ &\leq \delta L_2 + \eta L_3 + \eta L_4/2 + \frac{1}{2}\int_\Omega \overline{u}^2 \frac{\partial^2}{\partial x^2}(\alpha_1+\alpha_2)\, dx + 2MJ(t) \\ &\leq \delta L_2 + \eta L_3 + \eta L_4/2 + L_1 \int_\Omega \overline{u}^2 dx + 2MJ(t). \end{aligned}$$

Let $L_1 + 2M = L$, therefore we have $J'(t) - LJ(t) \leq \delta L_2 + \eta L_3 + \eta L_4/2$.After integrating the above inequality, we get

$$\begin{aligned} J(t) &\leq (\delta L_2 + \eta L_3 + \eta L_4/2)\,(e^{Lt}-1)/L \\ &\leq (\delta L_2 + \eta L_3 + \eta L_4/2)\,(e^{LT}-1)/L, \quad t \in [0,T]. \end{aligned}$$

It follows that $\|u_1 - u_2\| < \varepsilon$. □

3. Conclusion

In summary, we have shown that the solution of the nonlinear parabolic equation with $\alpha(u)$ dependent on u is unique, stable and continuous dependent on the parameter variables.

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