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INEQUALITIES FOR EIGENVALUES OF POLYNOMIALS OF THE LAPLACIAN OPERATOR

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Abstract: Estimates for the upper bounds of eigenvalues to a kind of polynomials of the Laplace operator is obtained by using the variation method. These results are brand new and essentially different from the works done before by other mathematicians and us.

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1. Introduction

Let Ω be a bounded domain in R^m with piecewise smooth boundary and denote $2d$ as the diameter of Ω. We consider the upper bounds of the $(n+1)$-th eigenvalue λ_{n+1} of the eigenvalue problem for polynomials of the Laplacian

$$\begin{cases} P(-\Delta)u = \lambda Q(-\Delta)u, & x \in \Omega\,, \\ u = \frac{\partial u}{\partial \nu} = \cdots = \frac{\partial^{l-1} u}{\partial \nu^{l-1}} = 0, & x \in \partial\Omega\,, \end{cases} \tag{1.1}$$

where

$$P(t) = \sum_{s=r+1}^{l} a_s t^s,\ Q(t) = \sum_{s=0}^{r} b_s t^s\,,$$

where $l \geq r+1 \geq 1$, $a_l = b_r = 1$, $a_i \geq 0$ for $i = r+1, r+2, \cdots, l-1$, $b_i \geq 0$

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for $i = 0, 1, \cdots, r-1$ and $b_0 \neq 0$, $\vec{\nu}$ is the unit outward normal to $\partial\Omega$. Here we assume that $P(t)$ and $Q(t)$ are coprime.

For the similar eigenvalue problems to this paper we refer readers to the papers [3] to [7] and the references therein. Our purpose of this paper is to generalize the results in [3], [4], [6], [1], [2] to the one of a larger kind of problems. But the new result of problem (1.1) is essentially different from the old ones in [3], [4], [6], [1], [2]. We can find out that in [3], [4], [6], [1], [2] the polynomial $Q(t) = 1$ which is a degenerate polynomial and the results for it is typically proved in [2]. The natural question is: What the result is when $Q(t)$ is not degenerate? For answering this question we should typically consider the problem (1.1), where $Q(t)$ is not degenerate due to $b_r = 1$. Due to the essential difference in the inner between degenerate and non-degenerate things we can conjecture that the result for problem (1.1) will be very different from the ones in [3], [4], [6], [1], [2]. This conjecture comes true from our theorem below.

Our main result is the following theorem.

Theorem 1.1. *Denote $\tilde{r} = \#\{b_i \mid b_i \neq 0,\ i = 0, 1, \cdots, r\}$. Let $m-2 \geq 4r\tilde{r}$ and $\lambda_i (i = 1, 2, \cdots, n+1)$ be the eigenvalues of (1.1). Then the following estimates hold*

$$\lambda_{n+1} \leq \lambda_n + \frac{4(m-2)^2}{n^2 m^2 (m-2-2nr\tilde{r})^2} \sum_{i=1}^{n} A_i \times \Big\{ \sum_{i=1}^{n} \sum_{s=r+1}^{l} s a_s (2s + m - 2 + nr\lambda_1^{\frac{s-l}{l-r}}) \lambda_i^{\frac{s-r-1}{l-r}} + rd \sum_{1 \leq j < i \leq n} (\lambda_i - \lambda_j) \Big\}, \quad (1.2)$$

when $n < (m-2+dm)/(2r\tilde{r} + dm + \sum_{r+1}^{l} s a_s)$,

$$\begin{aligned}\lambda_{n+1} \leq \lambda_n &+ \frac{4(m-2)^2}{n^2 m^2 (m-2-2nr\tilde{r})^3} \sum_{i=1}^{n} A_i \Big\{ \sum_{i=1}^{n} \sum_{s=r+1}^{l} s a_s \big[(2s+m-2) \\ &\times (m-2-2nr\tilde{r}) + nr\big(n \sum_{s=r+1}^{l} s a_s + (n-1)dm\big) \lambda_1^{\frac{s-l}{l-r}}\big] \lambda_i^{\frac{s-r-1}{l-r}} \\ &+ rd\big(n \sum_{s=r+1}^{l} s a_s + (n-1)dm\big) \sum_{1 \leq j < i \leq n} (\lambda_i - \lambda_j) \Big\}, \quad (1.3)\end{aligned}$$

when $n \geq (m-2+dm)/(2r\tilde{r} + dm + \sum_{s=r+1}^{l} s a_s)$. Here

$$A_i = \sum_{s=0}^{r} b_s \lambda_i^{\frac{s+1}{l}} b_0^{\frac{s+1}{l}-1}.$$

Remark 1.1. As we said above the result here for non-degenerate polynomial $Q(t)$ depends on the diameter of the region Ω. Although it is simple but essentially different from the degenerate one in [3], [4], [6], [1], [2], where $Q(t) = 1$ and the result there does not depend on the geometry of Ω.

But if in Theorem 1.1 we take $r = 0,\ \tilde{r} = 1,\ \tilde{s} = 0,\ b_0 = 1,\ Q(t) = 1$–the degenerate case, then from (1.2) and (1.3) we obtain directly the results in [2] which is the generalization of papers [3], [4], [6], [1]. So, Theorem 1.1 is a more generalized result for the degenerate and non-degenerate $Q(t)$.

Remark 1.2. In the two-dimensional case, namely the eigenvalue problems,

$$-\Delta u = \lambda u\,.$$

Here $P(t) = t,\ Q(t) = 1$ with $l = 1 = a_1,\ r = 0,\ b_0 = 1, \tilde{r} = 1$, we conclude that $P(t)$ and $Q(t)$ are coprime, and then hypotheses of Theorem 1.1 require that $m - 2 \geq 4r\tilde{r}$, i.e $m \geq 2$.

Another example is for the biharmonic eigenvalue problem,

$$\Delta^2 u = \lambda u\,.$$

We have $P(t) = t^2$, $Q(t) = 1$ with $l = 2$, $a_1 = 0$, $a_2 = 1$, $r = 0$, $b_0 = 1$, $\tilde{r} = 1$, here in our assumption $P(t)$ and $Q(t)$ are coprime, and the hypotheses comes again $m \geq 2$.

Hence, our theorem can be applied to the most basic cases.

Remark 1.3. Our paper deals with the case $b_0 \neq 0$. However, in another problem,

$$\begin{cases} \Delta^2 u = -\lambda \Delta u, & x \in \Omega\,, \\ u = \frac{\partial u}{\partial \nu} = 0, & x \in \partial\Omega\,. \end{cases}$$

we have $P(t) = t^2,\ Q(t) = t$, due to Payne, Polya, and Weinberger,

$$\lambda_2 \leq 3\lambda_1\,.$$

In this case, $b_0 = 0$, so it is not included in our paper. But for the boundary value problem of certain kind of equations, say

$$(-\Delta)^m u = \lambda\big(\sum_{i=0}^{m-1} b_i(-\Delta)^i u\big),\ b_0 = 1,\ b_i \geq 0 \text{ for } i = 1, 2, \cdots, m-1;\ x \in \Omega$$

Theorem 1.1 gives the estimates for the eigenvalues.

2. Proof of Theorem 1.1

We take mn trial functions such that

$$\varphi_{ik} = x_k u_i - \sum_{j=1}^{n} a_{ij}^k u_j,$$
$$i = 1, 2, \cdots, n\,,\; k = 1, 2, \cdots, m\,,\; x = (x_1, x_2, \cdots, x_m) \quad (2.1)$$

with the constants given by

$$a_{ij}^k = \int x_k u_i Q(-\Delta) u_j dx,\; i, j = 1, 2, \cdots, n;\; k = 1, 2, \cdots, m.$$

Here and throughout the notation $\int$ is used for $\int_\Omega$.

Suppose that the eigenfunction $u_i (i = 1, 2, \cdots)$ of (1.1) corresponding to the eigenvalue λ_i is weighted orthogonal to u_j such that

$$\int u_i Q(-\Delta) u_j dx = \delta_{ij},\; i, j = 1, 2, \cdots.$$

Thus,

$$\int u_i Q(-\Delta) u_i dx = \sum_{t=0}^{r} b_t \int \left|\nabla^t u_i\right|^2 dx = 1\,, \quad (2.2)$$

where $\nabla^t = \Delta^{t/2}$ when t is even and $\nabla^t = \nabla(\Delta^{(t-1)/2})$ when t is odd. From (2.1) and (2.2), we know that φ_{ik} is weighted orthogonal to $u_j (j = 1, 2, \cdots, n)$, i.e.,

$$\int \varphi_{ik} Q(-\Delta) u_j dx = \int u_j Q(-\Delta) \varphi_{ik} dx = 0.$$

Moreover,

$$\varphi_{ik} = \frac{\partial \varphi_{ik}}{\partial \nu} = \cdots = \frac{\partial^{l-1} \varphi_{ik}}{\partial \nu^{l-1}} = 0,\; \text{on } \partial\Omega.$$

When $P(t)$ and $Q(t)$ are coprime, we can use the well-known Rayleigh Theorem (see [7]) to obtain that

$$\lambda_{n+1} \le \frac{\int \varphi_{ik} P(-\Delta) \varphi_{ik} dx}{\int \varphi_{ik} Q(-\Delta) \varphi_{ik} dx},\; i = 1, 2, \cdots, n. \quad (2.3)$$

To obtain a more explicit inequality between λ_{n+1} and λ_i for $i = 1, 2, \cdots, n$, we give a further calculation to the right hand side of (2.4) as follows

$$\begin{aligned} a_{ij}^k &= \int x_k u_i Q(-\Delta) u_j dx = \sum_{t=0}^{r} b_t \int x_k u_i (-\Delta)^t u_j dx \\ &= \sum_{t=0}^{r} b_t \int x_k \nabla^t u_i \cdot \nabla^t u_j dx + \sum_{t=0}^{r} b_t \int t \nabla^{t-1} u_i \cdot \nabla^{t-1} u_{j,x_k} dx. \end{aligned} \tag{2.4}$$

Here and throughout the notation u_{i,x_k} denotes $\dfrac{\partial u_i}{\partial x_k}$. A straightforward calculation yields

$$P(-\Delta)\varphi_{ik} = \lambda_i x_k Q(-\Delta) u_i - \sum_{j=1}^{n} a_{ij}^k \lambda_j Q(-\Delta) u_j - 2 \sum_{s=r+1}^{l} s a_s (-\Delta)^{s-1} u_{i,x_k}.$$

It is easy to see that

$$\begin{aligned} \sum_{t=0}^{r} b_t \int |\nabla^t \varphi_{ik}|^2 dx &= \int \varphi_{ik} Q(-\Delta) \varphi_{ik} dx \\ &= \int x_k \varphi_{ik} Q(-\Delta) u_i dx - 2 \sum_{s=0}^{r} s b_s \int \varphi_{ik} (-\Delta)^{s-1} u_{i,x_k} dx. \end{aligned}$$

By the orthogonality of φ_{ik} to u_j for $j = 1, 2, \cdots, n$, we obtain

$$\begin{aligned} \int \varphi_{ik} P(-\Delta) \varphi_{ik} dx &= \lambda_i \int \varphi_{ik} Q(-\Delta) \varphi_{ik} dx + 2 \sum_{s=0}^{r} s b_s \lambda_i \\ &\quad \times \int \varphi_{ik} (-\Delta)^{s-1} u_{i,x_k} dx - 2 \sum_{s=r+1}^{l} s a_s \int \varphi_{ik} (-\Delta)^{s-1} u_{i,x_k} dx. \end{aligned} \tag{2.5}$$

Let us define

$$\begin{aligned} I_{ik} &= 2 \sum_{s=0}^{r} s b_s \lambda_i \int \varphi_{ik} (-\Delta)^{s-1} u_{i,x_k} dx, \\ J_{ik} &= -2 \sum_{s=r+1}^{l} s a_s \int \varphi_{ik} (-\Delta)^{s-1} u_{i,x_k} dx, \\ I &= \sum_{k=1}^{m} \sum_{i=1}^{n} I_{ik}, \quad J = \sum_{k=1}^{m} \sum_{i=1}^{n} J_{ik}. \end{aligned}$$

By (2.5) we have

$$\sum_{k=1}^{m}\sum_{i=1}^{n}\int \varphi_{ik}P(-\Delta)\varphi_{ik}dx=\sum_{k=1}^{m}\sum_{i=1}^{n}\lambda_i\int \varphi_{ik}Q(-\Delta)\varphi_{ik}dx+I+J.$$

Denoting

$$S=\sum_{k=1}^{m}\sum_{i=1}^{n}\int \varphi_{ik}Q(-\Delta)\varphi_{ik}dx=\sum_{k=1}^{m}\sum_{i=1}^{n}\sum_{t=0}^{r}b_t\int |\nabla^t\varphi_{ik}|^2dx,$$

then by (2.3),

$$\lambda_{n+1}S\leq \sum_{k=1}^{m}\sum_{i=1}^{n}\lambda_i\int \varphi_{ik}Q(-\Delta)\varphi_{ik}dx+I+J. \tag{2.6}$$

Replacing λ_i in (2.6) by λ_n yields

$$(\lambda_{n+1}-\lambda_n)S\leq I+J, \tag{2.7}$$

which is the basic inequality we need. With the help of this inequality and similar to the proof in [2] we can get the following result.

Lemma 2.1. *Suppose that $u_i (i=1,2,\cdots)$ is the eigenfunction of (1.1) corresponding to the eigenvalue λ_i. Then:*

(a) $\int|\nabla^k u_i|^2dx\leq b_0^{-\frac{1}{k+1}}(\int|\nabla^{k+1}u_i|^2dx)^{\frac{k}{k+1}}$, $i=1,2,\cdots,n$, $k=0,1,\cdots,l$.

(b) $\int|\nabla^k u_i|^2dx\leq b_0^{\frac{k}{l}-1}\lambda_i^{\frac{k}{l}}$.

(c) $\sum_{s=0}^{r}b_s\int|\nabla^{s-1}u_i|^2dx\geq\big(\sum_{s=0}^{r}b_s^{\frac{s+1}{l}-1}\lambda_i^{\frac{s+1}{l}}\big)^{-1}$.

(d) $\int\left|\nabla^{r+p}u_i\right|^2dx\leq\lambda_i^{p/(l-r)}$, $p=0,1,2,\cdots,l-r$.

(e) $\sum_{s=0}^{r}b_s\int\left|\nabla^{s+1}u_i\right|^2dx\leq\big(\sum_{s=0}^{r}b_s\int\left|\nabla^{s+r}u_i\right|^2dx\big)^{1/r}$, $r>0$.

Lemma 2.2. *Let $u_i (i=1,2,\cdots)$ be an eigenfunction of (1.1) corresponding to the eigenvalue λ_i. Then:*

(a) $\sum_{k=1}^{m}\int\left|\nabla^t u_{i,x_k}\right|^2dx=\int\left|\nabla^{t+1}u_i\right|^2dx$, $t=0,1,2,\cdots,l-1$.

(b) $-\sum_{k=1}^{m}\int x_k u_i(-\Delta)^t u_{i,x_k}dx=\dfrac{1}{2}(2t+m)\displaystyle\int\left|\nabla^t u_i\right|^2dx$, $t=0,1,2,\cdots,$ $l-1$.

Lemma 2.3. *Let $\lambda_i (i=1,2,\cdots)$ be the eigenvalues of (1.1). Then, for $2\leq n<\dfrac{m-2}{2r\tilde{r}}$ we have:*

(a) $S\geq\frac{n^2(m-2-2nr\tilde{r})^2}{4(m-2)^2}m^2\big(\sum_{i=1}^{n}A_i\big)^{-1}$.

(b) $I+J \leq (\sum_{i=1}^{n}\sum_{s=r+1}^{l} sa_s\big(2s+m-2+\frac{nr}{\sigma}\big)\lambda_i^{s-l/(l-r)}\big)\lambda_i^{\frac{s-r-1}{l-r}} + \frac{rd}{\sigma}\sum_{1\leq j<i\leq r}(\lambda_i-\lambda_j)$, *where*

$$0 < \sigma < \frac{m-2-2nr\tilde{r}}{n\sum_{s=r+1}^{l} sa_s + (n-1)dm}.$$

Proof of Theorem 1.1. Applying Lemma 2.3 to (2.7) and choose $\sigma = 1$ if $n < (m-2+dm)/(2r\tilde{r}+dm+\sum_{s=r+1}^{l} sa_s)$ and $\sigma = (m-2-2nr\tilde{r})/(n\sum_{s=r+1}^{l} sa_s + (n-1)dm)$ if $n \geq (m-2+dm)/(2r\tilde{r}+dm+\sum_{s=r+1}^{l} sa_s)$, then (1.2) and (1.3) follow respectively. □

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