

REPRESENTATIONS OF FUZZY GROUPS AND ALGEBRAS

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Abstract: This paper contains some preliminary work on representations of fuzzy groups and algebras. The work is based on the notion of L -subset (or L -subgroup, L -submodule, etc.), where L is a lattice. It proves homological properties of L -modules. It defines fuzzy projective and injective submodules of a module, and shows some of the important properties of fuzzy (sub)modules. The paper also studies the Hom functor of fuzzy modules and defines short and long exact sequences of fuzzy modules and the cohomology from the derived chain complexes.

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1. Introduction

The study of *fuzzy* sets largely started from mid to late sixties last century. One of the mile stones is Zadeh's introduction [4] of systemizing the definitions of fuzzy sets. It has a wide range of applications since then in various scientific fields in both pure and applied mathematics, including the area of abstract algebra, in which one of the most important papers was by Rosenfeld [3]. A great deal of research in this area has been accomplished in recent years.

This paper contains some preliminary work on representations of fuzzy groups and algebras. Starting from a series of well known concepts in the

area, such as fuzzy sets, fuzzy groups, fuzzy rings, fuzzy fields, fuzzy modules and fuzzy group algebras, it defines fuzzy projective and injective submodules of a module and proves some basic properties of fuzzy (sub)modules. The original definition of fuzzy sets was defined as functions to the interval $[0, 1]$ in R . The work in this paper is based on the notion of L -subset (or L -subgroup, L -submodule, etc.), where L is a lattice. That is, “fuzzy” is defined as functions to a complete lattice L . The L -notion is a generalization of the original definitions of “fuzzy”. The paper also studies the Hom functor of fuzzy modules and defines short and long exact sequences of fuzzy modules and the cohomology from the derived chain complexes. We shall start our paper by some preliminary definitions in fuzzy abstract algebra in Section 2. Section 3 focuses on projective and injective modules and short exact sequences. In the last section we shall study the Hom functor of fuzzy modules and the homological/cohomological properties.

2. Fuzzy Algebra - Preliminaries

In this section, we shall briefly give some of the basic and well-known definitions for the convenience of readers who might need. Much of the details can be found in references such as [2]. Fuzzy sets were initially defined over the interval $[0, 1]$ in R , and later generalized to lattices. A complete lattice is a lattice L such that for any $A \subseteq L$, $\vee\{a \mid a \in A\}$ and $\wedge\{a \mid a \in A\}$ both defined. Unless specify we always assume in this paper that lattice is complete and 0 and 1 are always the maximal and minimal elements of a lattice, respectively. A lattice is *regular* if for any a and b , $a \wedge b = 0$ implies either a or b is 0. A complete Heyting algebra L is a complete lattice such that both distribution hold. That is, for any $A \subseteq L$ and $b \in L$,

$$\begin{aligned}\vee\{a \wedge b \mid a \in A\} &= (\vee\{a \mid a \in A\}) \wedge b \quad \text{and} \\ \wedge\{a \vee b \mid a \in A\} &= (\wedge\{a \mid a \in A\}) \vee b.\end{aligned}$$

Definition 1. For any set X , an L -subset of X is a function from X to L . In particular, if $L = [0, 1] \subseteq R$, then the L -subset is called a fuzzy subset of X .

In fact, in all of the following definitions, if $L = [0, 1] \subseteq R$, then every term L -can be replaced by *fuzzy*. The L power set of X , denoted as L^X , is the set of all L -subsets. For any L -subset $\mu \in L^X$, $\mu^* = \{x \mid \mu(x) > 0\}$ is said to be the *support* of μ . For any two L -subsets μ and ν , we say $\mu \subseteq \nu$ if $\mu(x) \leq \nu(x)$ for all $x \in X$. The union and intersection of L -subsets are defined as

$$(\mu \cup \nu)(x) = \mu(x) \vee \nu(x) \quad \text{and} \quad (\mu \cap \nu)(x) = \mu(x) \wedge \nu(x).$$

For any $Y \subseteq X$ and $a \in L$, a_Y is defined as

$$a_Y(x) = \begin{cases} a & \text{for } x \in Y, \\ 0 & \text{for } x \notin Y, \end{cases}$$

and a -cut $\mu_a = \{x \in X \mid \mu(x) \geq a\}$. For any function $f : X \rightarrow Y$ and L -subsets μ and ν in X and Y , respectively, then we can define $f(\mu)$ as

$$f(\mu)(y) = \begin{cases} \vee\{\mu(x) \mid f(x) = y\}, & \text{if } f^{-1}(y) \neq \emptyset, \\ 0 & \text{otherwise,} \end{cases}$$

and define $f^{-1}(\nu)(x) = \nu(f(x))$. Both of them are L -subsets of Y and X respectively. If μ and ν are L -subsets of X and Y , respectively, then the direct sum of μ and ν is defined as $(\mu \oplus \nu)(x, y) = \mu(x) \wedge \nu(y)$.

An L -subgroup of a group G is an L -subset μ such that $\mu(xy) \geq \mu(x) \wedge \mu(y)$ and $\mu(x^{-1}) \geq \mu(x)$ for all $x, y \in G$. An L -subgroup μ is said to be L -normal in another L -subgroup ν , denoted as $\mu \triangleleft \nu$, if $\mu \subseteq \nu$ and $\mu(xyx^{-1}) \geq \mu(y) \wedge \nu(x)$, for all $x, y \in G$. If μ is normal in the group G , then $G/\mu = \{\mu(e)_{\{x\}} \circ \mu \mid x \in G\}$ is called the *quotient* group of G to the normal L -subgroup μ , where product of two L -subgroups μ and ν is defined as $(\mu \circ \nu)(x) = \vee\{\mu(y) \wedge \nu(z) \mid y, z \in G, yz = x\}$. If G and H are two groups and μ, ν are L -subgroups of G and H respectively, a homomorphism f of G onto H is said to be a *weak homomorphism* of μ to ν if $f(\mu) \subseteq \nu$, and f is said to be a *homomorphism* of μ onto ν if $f(\mu) = \nu$. f is said to be an *isomorphism* of μ if f is an isomorphism from G to H . It is worth pointing out that if f is an L -homomorphism and e is the identity, then $\mu(e) = \mu(x \cdot x^{-1}) \geq \mu(x) \wedge \mu(x) = \mu(x)$, for any $x \in G$.

If R is a commutative ring, then an L -subset μ is called an L -subring of R if $\mu(x - y) \geq \mu(x) \wedge \mu(y)$ and $\mu(xy) \geq \mu(x) \wedge \mu(y)$, for all $x, y \in R$. An L -subring is an L -ideal if $\mu(xy) \geq \mu(x) \vee \mu(y)$ for all $x, y \in R$. Moreover, if μ is an L -ideal, then the ring $R/\mu = \{x + \mu \mid x \in R\}$ is called the *quotient ring* of R by μ . In particular, if F is a field, an L -subset of F is said to be an L -subfield if μ is an L -subgroup of the additive group of F , and an L -subgroup of the multiplicative group of F , and $\mu(1) = 1$.

In the case of M is an R -module, then an L -submodule of M is an L -subset μ such that $\mu(0) = 1$, and for all $x, y \in M$ and $r \in R$,

$$\mu(rx) \geq \mu(x), \quad \mu(x + y) \geq \mu(x) \wedge \mu(y).$$

Let M be a free R -module generated by a set B and β be an L -subset of B . An L -submodule ξ of M is said to be *free* over β if for any module Y and L -submodule η of Y with $g : B \rightarrow Y$ and $g(\beta)|_B = \eta|_B$, there exists a unique

homomorphism $h : M \rightarrow Y$ such that $h|_B = g$ and $h(\xi) \subseteq \eta$. It has been proved that for any L -subset of a set B , there exists a free L -submodule ξ with respect to β , and the free module is unique up to isomorphism.

If S is a topological space, χ is an L subset, an L -topology \mathcal{T} on χ is a collection of L -subsets such that for all $\mu \in \mathcal{T}$, $\mu \subseteq \chi$ and: (i) $\emptyset \in \mathcal{T}$; (ii) if $x_a \subseteq \chi$, where $x \in S$ and $a \in L$, then there exists $\mu \in \mathcal{T}$ such that $x_a \subseteq \mu$; (iii) if $\mathcal{R} \subseteq \mathcal{T}$, then $\bigcup_{\mu \in \mathcal{R}} \mu \in \mathcal{T}$; (iv) if $\mu, \nu \in \mathcal{T}$, then $\mu \cap \nu \in \mathcal{T}$.

Finally, let kG be the group algebra of a group G over a field k and μ an L -subset of kG . μ is said to be a L -group subalgebra of kG over $\mu|_k$ if $\mu|_k$ is an L -subfield of k , μ_G is an L -subgroup of G , and for any $\sum r_i g_i \in kG$, where $0 \neq r_i \in k$, $\mu(\sum r_i g_i) = \wedge(\mu(r_i) \wedge \mu(g_i))$.

3. Projective, Injective L -Submodules and Kernels

In this section, we shall introduce projective and injective L -submodules. The arguments can in fact be given much more nicely in the context of category theory. Moreover, the collection of all L -submodules of a module is better than a class. It is actually a set. However, for the sake of convenience, we choose to present it in a relatively concrete way.

Definition 2. M is a module. An L -submodule π is said to be *projective* if for any two L -submodules of M , μ and ν , with the given diagram

$$\begin{array}{ccc} & & \pi \\ & & \downarrow g \\ \mu & \xrightarrow{f} & \nu, \end{array}$$

where f is a homomorphism from μ onto ν and g is a weak homomorphism, there exists a homomorphism $h : \pi \rightarrow \mu$ such that the diagram commutes

$$\begin{array}{ccc} & & \pi \\ & \swarrow h & \downarrow g \\ \mu & \xrightarrow{f} & \nu \end{array}$$

Injective modules can be defined in a similar way.

Proposition 3. *Any free L -submodule is projective.*

Proof. Let π be a free L -submodule of a module P with respect to an L -subset β of a set B , consider the following diagram,

$$\begin{array}{ccc} & \pi & \\ & \downarrow g & \\ \mu & \xrightarrow{f} & \nu, \end{array}$$

where f is a homomorphism from modules M to N such that $f(\mu) = \nu$, and g is a weak homomorphsim. Define a function $h : B \rightarrow N$ as

$$h(x) = \begin{cases} \vee\{y \in M \mid f(y) = g(x)\}, & \text{if } f^{-1}(g(x)) \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

The fact that π is a free L -submodule implies that the function h can be extended to a homomorphism as desired. □

Let f be an L -homomorphism from μ to ν . We define the L -kernel of f as

$$\text{Ker}^L(f) = f^{-1}(\nu(0)) = \nu(0) \circ f.$$

Proposition 4. *L -kernel of a weak homomorphism from μ to ν is an L -submodule of M , where μ and ν are L -submodules of M and N , respectively.*

Proof. From definition, we clearly have

$$\text{Ker}^L(f)(x) \leq \nu(0), \quad \text{for all } x \in M.$$

For any $x, y \in M$, if $f(x + y) = 0$, we can imply

$$\begin{aligned} (\text{Ker}^L(f))(x + y) &= (\nu(0) \circ f)(x + y) = \nu(0)(f(x + y)) \\ &= \nu(0) \geq (\text{Ker}^L(f))(x) \geq (\text{Ker}^L(f))(x) \wedge (\text{Ker}^L(f))(y). \end{aligned}$$

On the other hand, if $f(x + y) \neq 0$, then either $f(x)$ or $f(y)$ is not zero. Hence, either $\text{Ker}^L(f)(x)$ or $\text{Ker}^L(f)(y)$ will be 0. Thus,

$$(\text{Ker}^L(f))(x + y) \geq 0 = (\text{Ker}^L(f))(x) \wedge (\text{Ker}^L(f))(y).$$

For scalar multiplication, for any $r \in R$, if $f(rx) = 0$, we have

$$(\text{Ker}^L(f))(rx) = \nu(0) \geq (\text{Ker}^L(f))(x).$$

On the other hand, if $f(rx) \neq 0$, then $f(x) \neq 0$. Hence, $\text{Ker}^L(f)(x) = 0$. Thus,

$$(\text{Ker}^L(f))(rx) \geq 0 = (\text{Ker}^L(f))(x). \quad \square$$

4. Chain Complexes, Homology and Cohomology in Fuzzy Algebra

In this section, we shall give some basic homological properties with regard to L -modules. For any two L -submodules μ and ν of modules M and N , respectively, $\text{Hom}(\mu, \nu)$ denotes the set of all weak homomorphisms from μ to ν .

Proposition 5. $\text{Hom}(\mu, \nu)$ is a Z -module. Moreover, $\text{Hom}(\mu, \nu)$ is an R -bimodule in the usual settings.

Proof. It suffices to show that for any two f and g in $\text{Hom}(\mu, \nu)$, we have $f + g$ and rf in $\text{Hom}(\mu, \nu)$, where $r \in R$, and rf is defined as $(rf)(x) = r \cdot f(x)$. For any $x \in M$, from $f(\mu), g(\mu) \subseteq \nu$ we know that $\mu(x) \leq \nu(f(x))$ and $\mu(x) \leq \nu(g(x))$. Thus, $\mu(x) \leq \nu(f(x)) \wedge \nu(g(x)) \leq \nu(f(x) + g(x))$. This implies, for any $y \in N$, if $(f + g)^{-1}(y) \neq \emptyset$,

$$\vee\{\mu(x) \mid f(x) + g(x) = y\} \leq \vee\{\nu(f(x) + g(x))\} = \nu(y).$$

Hence, $f + g \in \text{Hom}(\mu, \nu)$. Similarly, for any $r \in R$ and $x \in M$, we have $\mu(x) \leq \nu(f(x)) \leq \nu(rf(x))$, which implies if $(rf)^{-1}(y) \neq \emptyset$,

$$\vee\{\mu(x) \mid rf(x) = y\} \leq \vee\{\nu(rf(x))\} = \nu(y). \quad \square$$

Proposition 6. μ, ν and ξ are L -submodules of M, N and Q , respectively. f is a weak homomorphism from μ to ν , then f induces a module homomorphism from $\text{Hom}(\xi, \mu)$ to $\text{Hom}(\xi, \nu)$, and a module homomorphism $\text{Hom}(\nu, \xi)$ to $\text{Hom}(\mu, \xi)$.

Proof. It suffices to show that for any two f and g in $\text{Hom}(\mu, \nu)$, we have $f + g$ and rf in $\text{Hom}(\mu, \nu)$, where $r \in R$, and rf is defined as $(rf)(x) = r \cdot f(x)$. For any $x \in M$, from $f(\mu), g(\mu) \subseteq \nu$ we know that $\mu(x) \leq \nu(f(x))$ and $\mu(x) \leq \nu(g(x))$. Thus, $\mu(x) \leq \nu(f(x)) \wedge \nu(g(x)) \leq \nu(f(x) + g(x))$. This implies, for any $y \in N$, if $(f + g)^{-1}(y) \neq \emptyset$,

$$\vee\{\mu(x) \mid f(x) + g(x) = y\} \leq \vee\{\nu(f(x) + g(x))\} = \nu(y).$$

Hence, $f + g \in \text{Hom}(\mu, \nu)$. Similarly, for any $r \in R$ and $x \in M$, we have $\mu(x) \leq \nu(f(x)) \leq \nu(rf(x))$, which implies if $(rf)^{-1}(y) \neq \emptyset$,

$$\vee\{\mu(x) \mid rf(x) = y\} \leq \vee\{\nu(rf(x))\} = \nu(y). \quad \square$$

In the case of $R = kG$, a group algebra, we know that kG has the Hopf algebra structure. Namely, we can make $\text{Hom}(M, N)$ into a kG -module by defining $(gf)(x) = gf(g^{-1}x)$, where $f \in \text{Hom}(M, N)$ and $g \in G$.

Proposition 7. $\text{Hom}(\mu, \nu)$ is a kG -module.

Proof. It suffices to show that for any $g \in G$ and $f \in \text{Hom}(\mu, \nu)$, $gf \in \text{Hom}(\mu, \nu)$. Since $f(\mu) \subseteq \nu$ and μ and ν are L -submodules, we have

$$\mu(x) \leq \mu(g^{-1}x) \leq \nu(f(g^{-1}x)) \leq \nu(gf(g^{-1}x)) = \nu((gf)(x)),$$

which implies if $(gf)^{-1}(y) \neq \emptyset$, we have

$$\vee\{\mu(x) \mid (gf)(x) = y\} \leq \vee\{\nu((gf)(x))\} = \nu(y). \quad \square$$

We know that a chain complex of modules is a collection of modules $\cdots \rightarrow$

$$M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3 \rightarrow \cdots \text{ such that } f_i(M_i) \subseteq \ker f_{i+1}. \text{ A chain complex of}$$

L -submodules

$$\cdots \rightarrow \mu_1 \xrightarrow{f_1} \mu_2 \xrightarrow{f_2} \mu_3 \rightarrow \cdots$$

is induced from a chain complex of modules, where μ_i is an L -submodule of M_i and $f_i(\mu_i) \subseteq \mu_{i+1}$. The chain complex of L -submodules is said to be *exact* at i if the above module chain complex is exact at i and $f_i(\mu) = \text{Ker}^L(\mu_{i+1})$. A short exact sequence of L -submodules is a sequence

$$0 \rightarrow \mu' \xrightarrow{i} \mu \xrightarrow{p} \mu'' \rightarrow 0 \tag{1}$$

which is exact everywhere. A short exact sequence is said to be *split* if there exists a homomorphism $i' : \mu'' \rightarrow \mu$ such that $p \circ i' = 1_{\mu''}$ or there exists a homomorphism $p' : \mu \rightarrow \mu'$ such that $p' \circ i = 1_{\mu'}$. We have found that the arguments here are amazingly parallel to the ones in the study of regular (crip) module theory (see [1]). To see the two conditions in the definition of split are equivalent, note that, if we have $i' : \mu'' \rightarrow \mu$ such that $pi' = 1_{\mu''}$, then $p(1_\mu - i'p) = p - p = 0$. Thus, for any $x \in M$, $p(1_\mu - i'p)(x) = 0$. So, $(1_\mu - i'p)(x)$ is in the kernel of p . Hence, in the image of i . From exactness, we have an $x' \in \mu'$ such that $ix' = (1_\mu - i'p)(x)$ and x' is unique since i is injective. Define $p' : \mu \rightarrow \mu', x \mapsto x'$. It follows directly that $i(x') = ip'(x)$ and $ip' = 1_\mu - i'p$ or

$$i'p + ip' = 1_\mu. \tag{2}$$

Also, we have $i'pi'p = i'1_{\mu''}p = i'p$. Hence multiplying (2) on the right by $i'p$ gives $ip'i'p = 0$. Since p is surjective and i is injective, this implies $p'i' = 0$ on

μ'' . If $x' \in \mu'$, then $ip'i(x') = (1_\mu - i'p)i(x) = i(x')$ and since i is injective, we have $p'i = 1_{\mu'}$. And, vice versa.

Proposition 8. *The short exact sequence (1) splits if and only if μ is the direct sum of μ' and μ'' .*

Proof. Assume the short exact sequence splits. From the discussion above, we know that we have the following four relations,

$$pi' = 1_{\mu''}, \quad p'i' = 0, \quad p'i = 1_{\mu'}, \quad pi = 0.$$

These together with equation (2) imply that μ is isomorphic to $\mu' \oplus \mu''$. The other direction is trivial. \square

Proposition 9. *The following on an L -submodule π are equivalent:*

- (1) π is projective.
- (2) Any short exact sequence $0 \rightarrow \mu \xrightarrow{i} \nu \xrightarrow{p} \pi \rightarrow 0$ splits, where μ and ν are L -submodules.
- (3) π is a direct summand of a free L -submodule.

Proof. (1) \Rightarrow (2): Let $0 \rightarrow \mu \xrightarrow{i} \nu \xrightarrow{p} \pi \rightarrow 0$ be an exact sequence and consider the diagram

$$\begin{array}{ccc} & & \pi \\ & & \downarrow 1_\pi \\ \mu & \xrightarrow{p} & \nu. \end{array}$$

By hypothesis we can fill this in with $p' : \pi \rightarrow \nu$ to obtain a commutative diagram. Then $pp' = 1_\pi$ and the given short exact sequence splits.

(2) \Rightarrow (3): Since any L -submodule is a homomorphic image of a free L -submodule, we have a short exact sequence $0 \rightarrow \pi' \xrightarrow{i} \xi \xrightarrow{p} \pi \rightarrow 0$, where ξ

is a free L -submodule. If π satisfies (2), then $0 \rightarrow \pi' \xrightarrow{i} \xi \xrightarrow{p} \pi \rightarrow 0$ splits, and hence $\xi \cong \pi \oplus \pi'$.

(3) \Rightarrow (1): We are given that there exists a split exact sequence $0 \rightarrow$

$$\pi' \xrightarrow{i} \xi \xrightarrow{p} \pi \rightarrow 0, \text{ where } \xi \text{ is free. Now suppose we have a diagram}$$

$$\begin{array}{ccc} & & \pi \\ & & \downarrow f \\ \mu & \xrightarrow{q} & \nu. \end{array}$$

Combine the two diagrams. Since the short exact sequence splits, we obtain $i' : \pi \rightarrow \xi$ such that $pi' = 1_\pi$. Since ξ is free, hence projective from Proposition 3, we can fill in $g : \xi \rightarrow \mu$ to obtain $fp = qg$. Then $f = f \cdot 1_\pi = fpi' = qgi'$ and $gi' : \pi \rightarrow \mu$ makes the diagram

$$\begin{array}{ccc} & & \pi \\ & gi' \swarrow & \downarrow f \\ \mu & \xrightarrow{q} & \nu \end{array}$$

commutative. Hence π is projective. □

A long exact sequence of L -submodules is a sequence of L -submodules

$$0 \rightarrow \mu_0 \xrightarrow{d_0} \mu_1 \xrightarrow{d_1} \mu_2 \xrightarrow{d_2} \dots, \tag{3}$$

which is exact everywhere. For any given L -submodule μ , the cohomology of this chain complex is defined via the following derived long sequence of modules

$$0 \leftarrow \text{Hom}(\mu_0, \mu) \xleftarrow{\bar{d}_0} \text{Hom}(\mu_1, \mu) \xleftarrow{\bar{d}_1} \text{Hom}(\mu_2, \mu) \dots$$

More important case is when the underline ring is an L -subgroup algebra. To end this paper, it is well worth pointing out that although much of the results here seemed to have a direct parallelism to the traditional (*crisp* is the jargon in the community of fuzzy mathematics) homology and cohomology theory, the

actual outcome from the computing of cohomology, even some very simple finite groups, is quite different. Moreover, the definition of homology (if exists) will have to have tensor product involved, and will have a new twist.

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