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TWO ASPECTS OF GENERALIZED MÖBIUS FUNCTIONS: LANDAU, BRAUER-RADEMACHER IDENTITIES AND DEPENDENCE

Nittiya Pabhapote[1] §, Vichian Laohakosol[2], Pattira Ruengsinsub[3]

[1]Department of Mathematics
School of Science
The University of the Thai Chamber of Commerce
126/1 Vibhavadee – Rangsit Road, Dindaeng
Bangkok, 10400, THAILAND
e-mail: nittiyapab@utcc.ac.th

[2,3]Department of Mathematics
Faculty of Science
Kasetsart University
Bangkok, 10900, THAILAND
[2]e-mail: fscivil@ku.ac.th
[3]e-mail: fscipar@ku.ac.th

Abstract: Two investigations are made on the so-called Souriau-Hsu-Möbius function, which is a natural generalization of the usual Möbius function. The first deals with extensions of the two arithmetical identities, known as Landau and Brauer-Radamacher identities. The second deals with algebraic dependence of these generalized Möbius functions.

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1. Introduction

By an arithmetic function ([1], [13], [17], [19]) we mean a complex-valued func-

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§Correspondence author

tion whose domain is the set of positive integers. We define the addition and convolution of two arithmetic functions f and g, respectively, by

$$(f+g)(n) = f(n) + g(n), \ (f * g)(n) = \sum_{ij=n} f(i)g(j).$$

A nonzero arithmetic function f is called multiplicative if and only if

$$f(mn) = f(m)f(n) \ \text{ whenever } (m,n) = 1;$$

it is called completely multiplicative if this equality holds for all $m, n \in \mathbb{N}$.

Denote by $\mathcal{A}, \mathcal{M}, \mathcal{C}$ the sets of all arithmetic, multiplicative and completely multiplicative functions, respectively. It is easy to see that the elements of $\mathcal{M}$ are uniquely determined via their values at all prime powers, while those of $\mathcal{C}$ are uniquely determined only via their values at the primes.

The Souriau-Hsu-Möbuis function ([20], [2], [9], [12]), abbreviated here as *SHM-function*,

$$\mu_\alpha(n) = \prod_{p|n} \binom{\alpha}{\nu_p(n)} (-1)^{\nu_p(n)},$$

where α is a complex parameter and $\nu_p(n)$ is the *highest power of the prime p dividing n*, is a natural generalization of the well-known Möbius function, $\mu = \mu_1$.

Here we continue our earlier investigation on the SHM-function focusing on two aspects. In Part 1, certain arithmetical identities known as Landau and Brauer-Radamacher identities are extended, and in Part 2, the dependence among SHM-functions of various papameters α is considered.

2. Part 1

It is well-known ([3], [4], [6], [7], [10], [8]) that

$$\sum_{d|r} \frac{(\mu(d))^2}{\phi(d)} = \frac{r}{\phi(r)}, \tag{1}$$

$$\sum_{d|r} \frac{\lambda(r/d)\mu(d)}{\phi(d)} = \frac{r\lambda(r)}{\phi(r)}, \tag{2}$$

$$\phi(r) \sum_{\substack{d|r \\ (d,n)=1}} \frac{d}{\phi(d)} \mu\left(\frac{r}{d}\right) = \mu(r) \sum_{d|(n,r)} d\mu\left(\frac{r}{d}\right), \tag{3}$$

where ϕ is the *Euler totient*, λ is the *Liouville function*, and μ the *Möbuis function.* Identities (1), (2) and (3) are known as Landau, modified Landau, and Brauer-Rademacher identities, respectively. As seen from the works of Cohen ([3], [4], [5]) both Landau and Brauer-Radamacher identities arise naturally at least in two arithmetical investigations, namely, in the study of representing periodic arithmetic functions as Ramanujan sums and in the formula for the number of solutions of certain congruences. There have appeared several generalizations of these three identities, see e.g. [3], [4], [5], [6], [7], [8], [10].

We propose here to give yet another kind of generalization by replacing, in the identities above, the usual Möbuis function by the SHM-function.

3. Laudau Identities

Before stating our first theorem, recall that a positive integer n is said to be *r-th-powerful* ($r \in \mathbb{N}$) if $\nu_p(n) \geq r$ for all primes $p|n$.

Theorem 1. (SHM-Landau Identity) *Let $\alpha \in \mathbb{C}$, $f \in \mathcal{M}$ and $\beta \in \mathbb{N}$. Assume for each prime p and $j \in \mathbb{N} \cup \{0\}$ that there exists $a(p) \in \mathbb{C}\backslash\{0\}$ such that $f(p^{\beta+j}) = i_{\beta,a}(p^j) f(p^\beta)$, where $i_{\beta,a}(p^j) := a(p)^j$. Then for r being β-powerful, we have*

$$\sum_{d|r} \frac{(\mu_\alpha(d))^2}{f(d)} = \prod_{p|r} \left[\sum_{m=0}^{\beta-1} \frac{\binom{\alpha}{m}^2}{f(p^m)} + \frac{1}{f(p^\beta)} \sum_{i=\beta}^{\nu_p(r)} \frac{\binom{\alpha}{i}^2}{a(p)^{i-\beta}} \right]$$

$$= \prod_{p|r} \left[\sum_{m=0}^{\beta-1} \frac{\binom{\alpha}{m}^2}{f(p^m)} + \frac{(i_{\beta,a} * (\mu_\alpha)^2)(p^a)}{f(p^a)} - \frac{a(p)(i_{\beta,a} * (\mu_\alpha)^2)(p^{\beta-1})}{f(p^\beta)} \right].$$

Proof. Let $F(r) = \sum_{d|r} \frac{(\mu_\alpha(d))^2}{f(d)}$. It is easy to see that $F \in \mathcal{M}$. For each prime p and integer $k \geq \beta$, we have

$$F(p^k) = 1 + \frac{\binom{\alpha}{1}^2}{f(p)} + \frac{\binom{\alpha}{2}^2}{f(p^2)} + \cdots + \frac{\binom{\alpha}{\beta-1}^2}{f(p^{\beta-1})} + \frac{\binom{\alpha}{\beta}^2}{f(p^\beta)}$$

$$+\cdots+\frac{\binom{\alpha}{k}^2}{f(p^k)}$$

$$=\sum_{m=0}^{\beta-1}\frac{\binom{\alpha}{m}^2}{f(p^m)}+\left[\frac{\binom{\alpha}{\beta}^2}{f(p^\beta)}+\frac{\binom{\alpha}{\beta+1}^2}{a(p)f(p^\beta)}+\cdots+\frac{\binom{\alpha}{k}^2}{a(p)^{k-\beta}f(p^\beta)}\right]$$

$$=\sum_{m=0}^{\beta-1}\frac{\binom{\alpha}{m}^2}{f(p^m)}+\frac{(i_{\beta,a}*(\mu_\alpha)^2)(p^k)}{f(p^k)}-\frac{a(p)(i_{\beta,a}*(\mu_\alpha)^2)(p^{\beta-1})}{f(p^\beta)},$$

and the desired result follows. □

Theorem 1 is our form of generalized Landau identity. It is the crux of a large collection of well-known arithmetical identities as seen in the following corollaries.

Corollary 1. *Let $\alpha \in \mathbb{C}$ and $f \in \mathcal{M}$.*

(i) For each prime p and $j \in \mathbb{N} \cup \{0\}$, if there exists $a(p) \in \mathbb{C}\backslash\{0\}$ such that $f(p^{1+j}) = (a(p))^j f(p)$, then for any $r \in \mathbb{N}$ we have

$$\sum_{d|r}\frac{(\mu_\alpha(d))^2}{f(d)}=\prod_{p|r}\left[\frac{(i_1*(\mu_\alpha)^2)(p^{\nu_p(r)})}{f(p^{\nu_p(r)})}+\left(1-\frac{a(p)}{f(p)}\right)\right],$$

where $i_1(p^j) = a(p)^j$.

(ii) If ϕ denotes the Euler's totient, then

$$\sum_{d|r}\frac{(\mu_\alpha(d))^2}{\phi(d)}=\prod_{p|r}\left[\frac{(i_1*(\mu_\alpha)^2)(p^{\nu_p(r)})}{\phi(p^{\nu_p(r)})}-\frac{1}{\phi(p)}\right].$$

(iii) If $\mu := \mu_1$ denotes the Möbius function, we have the classical Landau identity

$$\sum_{d|r}\frac{(\mu(d))^2}{\phi(d)}=\frac{r}{\phi(r)}.$$

Proof. Taking $\beta = 1$ and noting that each positive integer is 1-powerful in Theorem 1, (i) follows. Specializing f as the Euler's totient ϕ in (i), we get (ii). Putting $\alpha = 1$, so that $a(p) = p$, in (ii), its right hand side becomes

$$\prod_{p|r}\left[\frac{(i_1*(\mu)^2)(p^{\nu_p(r)})}{\phi(p^{\nu_p(r)})}+\left(1-\frac{p}{\phi(p)}\right)\right]$$
$$=\prod_{p|r}\left[\frac{p^{\nu_p(r)}+p^{\nu_p(r)-1}}{\phi(p^{\nu_p(r)})}+\left(1-\frac{p}{p-1}\right)\right]=\prod_{p|r}\frac{p^{\nu_p(r)}}{\phi(p^{\nu_p(r)})}=\frac{r}{\phi(r)}. \quad \square$$

To prepare for the next batch of identities, let us recall from [9] and [12] that

$$h\in\mathcal{C} \implies h^{-s}=\mu_s h \quad (s\in\mathbb{N}).$$

We also need another preliminary result, which is an extension of Theorem 2.2 in [21].

Lemma 1. *Let $s\in\mathbb{N}; g,h\in\mathcal{C}$ and $f:=g*h^{-s}\,(=g*\mu_s h)$ with $f(p)\neq 0$ for each prime p. Then*

$$f(n)=g(n)\prod_{p|n}\left(1-\frac{h(p)}{g(p)}\right)^s,$$

whenever n is s-powerful.

Proof. Let $f=g*\mu_s h$. Then $f\in\mathcal{M}$ and to prove the lemma it suffices to evaluate f at prime powers which are s-powerful. For each prime p and $k\in\mathbb{N}, k\geq s$, we have

$$\begin{aligned}f(p^k)&=g(p^k)h(1)\mu_s(1)+\cdots+g(p^{k-s})h(p)^s\mu_s(p^s)+\cdots+h(p^k)\mu_s(p^k)\\&=g(p^k)h(1)\binom{s}{0}+\cdots+g(p^{k-s})h(p)^s\binom{s}{s}(-1)^s\\&=g(p^k)\left(1-\frac{h(p)}{g(p)}\right)^s. \quad \square\end{aligned}$$

Corollary 2. *(i) Let $s\in\mathbb{N},\alpha\in\mathbb{C}; g,h\in\mathcal{C}$ and $f=g*h^{-s}=g*\mu_s h$. Then*

$$\sum_{d|r}\frac{(\mu_\alpha(d))^2}{f(d)}=\prod_{p|r}\left[\sum_{m=0}^{s-1}\frac{\binom{\alpha}{m}^2}{f(p^m)}+\frac{1}{(g(p)-h(p))^s}\sum_{i=s}^{\nu_p(r)}\frac{\binom{\alpha}{i}^2}{g(p)^{\nu_p(r)-i}}\right],$$

where r is s-powerful.

(ii) Let $r, k \in \mathbb{Z}, 1-r > 1,\ \zeta_k(n) := n^k \quad (n \in \mathbb{N})$*, the* k*-th zeta function, and* $f = \zeta_k * \mu_{1-r}$*. Then for* n *being* $(1-r)$*-powerful, we have*

$$\sum_{d|n} \frac{(\mu(d))^2}{f(d)} = \prod_{p|n} \left[\frac{p^k + r}{p^k + r - 1}\right].$$

Proof. From Lemma 1, for each prime p and $j \in \mathbb{N} \cup \{0\}$, we have

$$\begin{aligned} f(p^{s+j}) = g(p^{s+j}) \left(1 - \frac{h(p)}{g(p)}\right)^s &= \left[g(p)\left(1 - \frac{h(p)}{g(p)}\right)\right]^s g(p^j) \\ &= (g(p) - h(p))^s\, g(p^j), \end{aligned}$$

and (i) follows from our SHM-Landau identity. (ii) is a direct consequence of (i) by taking $\alpha = 1, g = \zeta_k, s = 1-r$, and $h(n) = \zeta_0(n) := 1 \quad (n \in \mathbb{N})$, the unit function. □

The next lot of identities involves a generalization of the Euler's totient known as the *Jordan totient*, J_k ([3], [13], [19]).

Theorem 2. *Let* $\alpha \in \mathbb{C}$ *and* $n \in \mathbb{N}$*. Then for* $r = p_1^{a_1} \cdots p_k^{a_k}$*, we have*

$$\sum_{\substack{d|r \\ (d,n)=1}} \frac{(\mu_\alpha(d))^2}{J_k(d)} = \begin{cases} \prod_{i=1}^{k} \left[\frac{(i_k * (\mu_k)^2)}{J_k}(p_i^{a_i}) - \frac{1}{p_i^k - 1}\right] & \text{if } (r,n) = 1, \\ \prod_{j=1}^{s} \left[\frac{(i_k * (\mu_k)^2)}{J_k}(p_{i_j}^{a_{i_j}}) - \frac{1}{p_{i_j}^k - 1}\right] & \text{if } p_{i_j} \nmid n \ (1 \le j \le s), \\ & \text{but } p_{i_j} \mid n \ (s < j \le k), \\ 1 & \text{if all } p_i \mid n \quad (i = 1, \ldots, k). \end{cases}$$

Proof. Let $F(r) = \sum_{\substack{d|r \\ (d,n)=1}} \frac{(\mu_\alpha(d))^2}{J_k(d)}$, so that $F \in \mathcal{M}$. For each prime $p \nmid n$ and each $m \in \mathbb{N}$, we get

$$\begin{aligned} F(p^m) = 1 + \frac{\binom{\alpha}{1}^2}{p^k - 1} + \frac{\binom{\alpha}{2}^2}{p^{2k} - p^k} + \cdots + \frac{\binom{\alpha}{m}^2}{p^{mk} - p^{(m-1)k}} \\ = \frac{(i_k * (\mu_\alpha)^2)}{J_k}(p^m) - \frac{1}{p^k - 1}, \end{aligned}$$

while for $p \mid n$, we have $F(p^m) = \sum_{\substack{d|p^m \\ (d,n)=1}} \frac{(\mu_\alpha(d))^2}{J_k(d)} = \frac{(\mu_\alpha(1))^2}{J_k(1)} = 1.$ □

Immediate from Theorem 2 and Theorem 1 the next three identities follow, the last of which is Theorem 9 in [3].

Corollary 3. *Let $n, r \in \mathbb{N}$ and $\alpha \in \mathbb{C}$. Then*

$$\sum_{d|r} \frac{(\mu_\alpha(d))^2}{J_k(d)} = \prod_{p|r} \left(1 + \frac{1}{p^k - 1} \sum_{i=1}^{\nu_p(r)} \frac{\binom{\alpha}{i}^2}{p^{(i-1)k}}\right),$$

$$\sum_{\substack{d|r \\ (n,r)=1}} \frac{(\mu_\alpha(d))^2}{J_k(d)} = \prod_{\substack{p|r \\ (p,n)=1}} \left(1 + \frac{1}{p^k - 1} \sum_{i=1}^{\nu_p(r)} \frac{\binom{\alpha}{i}^2}{p^{(i-1)k}}\right),$$

$$\sum_{\substack{d|r \\ (d,n)=1}} \frac{(\mu(d))^2}{J_k(d)} = \frac{r^k}{(n,r)^k} \cdot \frac{J_k((n,r))}{J_k(r)}.$$

To state our generalized modified Landau identity, we need one more definition.

Definition. Let $s \in \mathbb{N} \cup \{0\}$. Define

$$\mathcal{C}_s := \{f \in \mathcal{M}; \text{there exists } a(p) \in \mathbb{C}\backslash\{0\} \text{ such that } f(p^{s+j}) = a(p)^j f(p^s)$$
$$(p \text{ prime}, j \in \mathbb{N} \cup \{0\})\}.$$

Note that the set of completely multiplicative functions, $\mathcal{C}$, is just $\mathcal{C}_0$.

Theorem 3. (SHM-Modified Landau Identity) *Let $r \in \mathbb{N}, s, t \in \mathbb{N} \cup \{0\}$ and $\alpha \in \mathbb{C}$. If $g \in \mathcal{C}_s$ and $f \in \mathcal{C}_t$, then*

$$\sum_{d|r} \frac{g(r/d)\mu_\alpha(d)}{f(d)} = \prod_{p|r} \left\{ \sum_{m=0}^{s} \binom{\alpha}{\nu_p(r) - m} (-1)^{\nu_p(r)-m} g(p^m) b(p^m) \right.$$
$$+ \frac{(-1)^{\nu_p(r)} g(p^{\nu_p(r)})}{a(p)^{\nu_p(r)} f(p^{\nu_p(r)})} \sum_{v=s}^{\nu_p(r)-t} \binom{\alpha}{\nu_p(r) - v} \left(\frac{-b(p)}{a(p)}\right)^v$$
$$\left. + g(p^{\nu_p(r)}) \sum_{u=0}^{t-1} \frac{\binom{\alpha}{u} (-1)^u}{f(p^u) a(p)^u} \right\},$$

whenever r is $(s + t)$-powerful and $a(p), b(p)$ are associated functions of g, f as defined in $\mathcal{C}_s, \mathcal{C}_t$, respectively.

Proof. Let $F(r)=\sum_{d|r}\frac{g(r/d)\mu_\alpha(d)}{f(d)}$, so that $F\in\mathcal{M}$. Since $g\in\mathcal{C}_s$ and $f\in\mathcal{C}_t$, for each prime p, there are $a(p),b(p)\in\mathbb{C}\backslash\{0\}$ such that $g(p^{s+j})=a(p)^j g(p^s)$, and $f(p^{t+i})=b(p)^i f(p^t)\quad(i,j\in\mathbb{N}\cup\{0\})$. For $k\in\mathbb{N}$ with $k\geq s+t$, we obtain

$$
\begin{aligned}
&F(p^k)=\\
&\frac{(-1)^k\binom{\alpha}{k}+b(p)g(p)\binom{\alpha}{k-1}(-1)^{k-1}+\cdots+b(p)^{s-1}g(p^{s-1})\binom{\alpha}{k-s+1}(-1)^{k-s+1}}{b(p)^{k-t}f(p^t)}\\
&\quad+\frac{g(p^s)\binom{\alpha}{k-s}(-1)^{k-s}}{b(p)^{k-s-t}f(p^t)}+\frac{a(p)g(p^s)\binom{\alpha}{k-s-1}(-1)^{k-s-1}}{b(p)^{k-s-t-1}f(p^t)}+\cdots\\
&\quad+\frac{a(p)^{k-t-s}g(p^s)\binom{\alpha}{t}(-1)^t}{f(p^t)}+\frac{a(p)^{k-t-s+1}g(p^s)\binom{\alpha}{t-1}(-1)^{t-1}}{f(p^t)}+\cdots\\
&\qquad+\frac{a(p)^{k-s}g(p^s)\binom{\alpha}{0}}{f(1)}
\end{aligned}
$$

and the desired result follows. ☐

Corollary 4. *(i) For $r\in\mathbb{N}$, we have the classical modified Landau identity*

$$\sum_{d|r}\frac{\lambda(r/d)\mu(d)}{\phi(d)}=\frac{r\lambda(r)}{\phi(r)}.$$

(ii) For $k,r\in\mathbb{N}$, we have

$$J_k(r):=\sum_{d|r}d^k\mu\left(\frac{r}{d}\right)=r^k\prod_{p|r}(1-p^{-k}).$$

Proof. Taking $f=\phi, g=\lambda$, and $\alpha=1$ in Theorem 3, we get (i). The identity (ii), which is Theorem 2 in [3], follows by taking $g=\zeta^k\in\mathcal{C}, f=\zeta_0\in\mathcal{C},\alpha=1$ in Theorem 3. ☐

4. Brauer-Rademacher Identities

Our generalized Brauer-Radamacher identity reads.

Theorem 4. (SHM-Brauer-Radamacher Identity) *Let $n,r\in\mathbb{N}, s\in\mathbb{N}\cup\{0\}$ and $\alpha\in\mathbb{C}$. If $f\in\mathcal{C}_s$ and $g\in\mathcal{C}$, then*

$$f(r)\sum_{\substack{d|r\\(d,n)=1}}\frac{g(d)}{f(d)}\mu_\alpha\left(\frac{r}{d}\right)$$

$$=\prod_{\substack{p|r\\p\nmid n}}\left\{f(p^{\nu_p(r)})\sum_{m=0}^{s-1}\frac{g(p)^m}{f(p^m)}\binom{\alpha}{\nu_p(r)-m}(-1)^{\nu_p(r)-m}+\right.$$

$$\left.\sum_{l=0}^{\nu_p(r)-s}a(p)^{\nu_p(r)-s-l}g(p)^{s+l}\binom{\alpha}{\nu_p(r)-s-l}(-1)^{\nu_p(r)-s-l}\right\}\prod_{\substack{p|r\\p|n}}(f\mu_\alpha)(p^{\nu_p(r)}),$$

where r is s-powerful and $a(p)$ is as defined in $\mathcal{C}_s$.

Proof. Let $F(r)=f(r)\sum\limits_{\substack{d|r\\(d,n)=1}}\frac{g(d)}{f(d)}\mu_\alpha\left(\frac{r}{d}\right)$, so that $F\in\mathcal{M}$. Since $f\in\mathcal{C}_s$, for each prime p, there exists $a(p)\in\mathbb{C}\cup\{0\}$ such that $f(p^{s+j})=a(p)^jf(p^s)$ ($j\in\mathbb{N}\cup\{0\}$). For each prime p and $k\in\mathbb{N}$ with $k\geq s$, for $p\nmid n$, we have

$$\begin{aligned}F(p^k)=a(p)^jf(p^s)&\left[\binom{\alpha}{k}(-1)^k+\frac{g(p)}{f(p)}\binom{\alpha}{k-1}(-1)^{k-1}+\cdots\right.\\&\left.\qquad+\frac{g(p)^{s-1}}{f(p^{s-1})}\binom{\alpha}{k-s+1}(-1)^{k-s+1}\right]\\+a(p)^jf(p^s)&\left[\frac{g(p)^s}{f(p^s)}\binom{\alpha}{k-s}(-1)^{k-s}+\cdots+\frac{g(p)^k}{f(p^k)}\binom{\alpha}{k-k}(-1)^{k-k}\right]\\&=f(p^k)\sum_{m=0}^{s-1}\frac{g(p)^m}{f(p^m)}\binom{\alpha}{k-m}(-1)^{k-m}\\&\qquad+\sum_{l=0}^{k-s}a(p)^{k-s-l}g(p)^{s+l}\binom{\alpha}{k-s-1}(-1)^{k-s-l},\end{aligned}$$

while for $p\mid n$, we have $F(p^k)=f(p^k)\mu_\alpha(p^k)$. □

Corollary 5. *Let $n,r\in\mathbb{N}$ and $\alpha\in\mathbb{C}$. Then*

$$\phi(r)\sum_{\substack{d|r\\(d,n)=1}}\frac{d}{\phi(d)}\mu_\alpha\left(\frac{r}{d}\right)$$

$$=\left(\prod_{\substack{p|r\\p\nmid n}}p^{\nu_p(r)}\sum_{i=0}^{\nu_p(r)}\binom{\alpha}{\nu_p(r)-i}(-1)^{\nu_p(r)-i}-\frac{\binom{\alpha}{\nu_p(r)}(-1)^{\nu_p(r)}}{p}\right)$$

$$\times \left(\prod_{\substack{p|r \\ p|n}} (\phi\mu_\alpha)(p^{\nu_p(r)}) \right),$$

and

$$\phi(r) \sum_{\substack{d|r \\ (d,n)=1}} \frac{d}{\phi(d)} \mu\left(\frac{r}{d}\right) = \mu(r) \sum_{d|(n,r)} d\mu\left(\frac{r}{d}\right).$$

Proof. The first identity comes from taking $f = \phi, g = \zeta$ in Theorem 4. The second identity, which is the classical Brauer-Radamacher identity, follows by further substituting $\alpha = 1$. □

5. Part 2

A (formal) Dirichlet series is an expression of the form

$$F(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}, \quad f(n) \in \mathbb{C}.$$

The set $(\mathcal{D}, +, \cdot)$ of all Dirichlet series equipped with addition and multiplication is isomorphic to $(\mathcal{A}, +, *)$, through the map $F(s) = \sum \frac{f(n)}{n^s} \leftrightarrow f$. Through this isomorphism, any algebraic relations from one setting have corresponding counterparts in the other, which allows us to refer to both interchangably.

Let $\mathcal{E}$ be a subring of $\mathcal{A}$. For $r > 1$, $f_1, f_2, \ldots, f_r \in \mathcal{A}$ are *algebraically dependent* over $\mathcal{E}$ if there exists $P \in \mathcal{E}[x_1, \ldots, x_r] \backslash \{0\}$ such that

$$P(f_1, \ldots, f_r) = \sum_{(i)} a_{(i)} * f_1^{i_1} * \cdots * f_r^{i_r} = 0,$$

and is said to be *algebraically independent* otherwise.

A *derivation* d over $\mathcal{A}$ is a mapping of $\mathcal{A}$ into itself satisfying

$$d(f * g) = df * g + f * dg\,, \quad d(c_1 f + c_2 g) = c_1 df + c_2 dg\,,$$

for all $f, g \in \mathcal{A}$ and $c_1, c_2 \in \mathbb{C}$. A typical derivation over $\mathcal{A}$ is the *p-basic derivation* d_p, where p is a prime, defined by

$$d_p f(n) = f(np)\upsilon_p(np),$$

where $v_p(n)$ denotes the exponent of highest power of p dividing n. We may also regard derivation d over $\mathcal{A}$, also as a derivation over $\mathcal{D}$ via $dF = \sum \frac{(df)(n)}{n^s}$.

Given $f_1, \ldots, f_r \in \mathcal{A}$ and derivations $d_1, \ldots, d_r$ over $\mathcal{A}$, the *Jacobian* of the f_i relative to the d_i is the determinant $J(f_1, \ldots, f_r; d_1, \ldots, d_r) = det(d_i(f_j))$. For ease of writing, when the derivations d_i $(i = 1, \ldots, n)$ are the p_i-basic derivations we denote $J(f_1, \ldots, f_r; d_{p_1}, \ldots, d_{p_r})$ by $J(f_1, \ldots, f_r; p_1, \ldots, p_r)$.

Recall an extremely useful test of algebraic independence based on Jacobian proved by Shapiro-Sparer [18].

Theorem 5. (see [18]) *Let $f_1, \ldots, f_r \in \mathcal{A}$ and $d_1, \ldots, d_r$ be distinct derivations over $\mathcal{A}$ which annihilate all elements of the subring $\mathcal{E}$. If $J(f_1, \ldots, f_r; d_1, \ldots, d_r) \neq 0$, then $f_1, \ldots, f_r$ are algebraically independent over $\mathcal{E}$.*

Note that the converse of this theorem does not hold in general as seen in the following example.

Example. Let

$$I(n) = \begin{cases} 1, & \text{if } n = 1, \\ 0, & \text{otherwise,} \end{cases}$$

$u(n) = 1$ for all $n \in \mathbb{N}$. That I and u are algebraically independent over $\mathbb{C}$ is a special case of a well-known theorem of Ostrowski [14]. But for primes $p \neq q$,

$$J(I, u; p, q)(n) = \begin{vmatrix} I(np)\nu_p(np) & u(np)\nu_p(np) \\ I(nq)\nu_q(nq) & u(nq)\nu_q(nq) \end{vmatrix} = \begin{vmatrix} 0 & \nu_p(np) \\ 0 & \nu_q(nq) \end{vmatrix} = 0.$$

The $\mathcal{T}$*-aggregate* of the arithmetic function $G_1, ..., G_r$ is the collection of sets of r distinct positive integers $\{i_1, \ldots, i_r\}$ or equivalently the collection of sets of r distinct primes $\{p_{i_1}, \ldots, p_{i_r}\}$ such that

$$J(G_1, \ldots, G_r; p_{i_1}, \ldots, p_{i_r}) \neq 0,$$

where p_j denotes the j-th prime. The functions $G_1, \ldots, G_r$ are called *strongly algebraically independent* if their $\mathcal{T}$-aggregate is nonempty, and the intersection of all the sets belonging to $\mathcal{T}$ is empty.

The *norm*, $\|f\|$, of an arithmetic function f is defined as

$$\|f\| = \begin{cases} \min\{n \in \mathbb{N} \mid f(n) \neq 0\} & \text{if } f \neq 0, \\ \infty & \text{if } f = 0. \end{cases}$$

Clearly, $\|f * g\| = \|f\| \, \|g\|$, $\|f + g\|) \geq \min\{\|f\|, \|g\|\}$, and the units of $\mathcal{A}$ are those functions whose norms are equal to 1.

For an $f \in \mathcal{A}$, define the *support* of f to be $\mathrm{supp} f = \{n \in \mathbb{N} : f(n) \neq 0\}$ and define the *prime supporting set* of f, $[\mathrm{supp} f]$, to be the smallest set of primes which generate a sub-semigroup of the positive integers containing supp f. Let $f \in \mathcal{A}$ and F be its corresponding Dirichlet series. Define $\|F\|$, supp F, and [supp F] as follows:

$$\|F\| = \|f\|\,,\ \mathrm{supp}\ F = \mathrm{supp}\ f,\ [\mathrm{supp}\ F] = [\mathrm{supp}\ f].$$

6. Products of Powers

In [18], a number of results about algebraic dependence of arithmetic functions are proved. Of particular interests to us are those showing dependence as products of powers, namely,

Theorem 6. (see [18]) *Let $G_1, \ldots, G_r \in \mathcal{A}$ be multiplicative and strongly algebraically independent. If $f \in \mathcal{A}$ is multiplicative and algebraic over $\mathbb{C}[G_1, \ldots, G_r]$, then there exist complex constants $c_1, \ldots, c_r$, such that its corresponding Dirichlet series satisfies*

$$F = G_1^{c_1} \cdots G_r^{c_r}.$$

Theorem 7. (see [18]) *Let $G_1, \ldots, G_r$ be given multiplicative Dirichlet series such that, for $G_i^* = G_i - 1 \neq 0$,*

$$\left\| \prod_{i=1}^{r} G_i^{*^{x_i}} \right\| \neq 1$$

for any sets of integers x_i not all zero. Assume also that, for every sequence $1 \le j_1 < j_2 < \cdots < j_i \le r$,

$$\bigcup_{\nu=1}^{i} [\mathrm{supp}\ G_{j_\nu}]$$

contains at least $i+1$ primes (for every i). Then if F is a multiplicative Dirichlet series such that $J(F, G_1, \ldots, G_r / p_1, \ldots, p_{r+1}) = 0$ for all sets of distinct primes $p_1, \ldots, p_{r+1}$ it follows that

$$F = G_1^{c_1} \cdots G_r^{c_r}$$

for some complex constants c_i.

Theorem 6 and Theorem 7 only guarantee the exponents $c_i's$ to be complex numbers. To deduce that they are indeed rational, Shapiro-Sparer resorted to the use of differential equation arguments, which made implicit use of the fact that the G_i's are not only algebraically independent, but also differentially independent. We now establish rationality of the $c_i' s$ by simple and direct arguments (Theorem 8) and those in Theorem 7 by imposing an extra condition.

Theorem 8. *Let $G_1, \ldots, G_r \in \mathcal{A}$ be multiplicative and strongly algebraically independent. If $f \in \mathcal{A} \setminus \{0\}$ is multiplicative and algebraic over $\mathbb{C}[G_1, \ldots, G_r]$, then there exist rational numbers $c_1, \ldots, c_r$ such that the corresponding Dirichlet series satisfies*

$$F = G_1^{c_1} \cdots G_r^{c_r}.$$

Proof. By Theorem 6, we have $F = G_1^{c_1} \cdots G_r^{c_r}$, where $c_1, \ldots, c_r \in \mathbb{C}$. It remains to show that $c_1, \ldots, c_r$ are rational. Since F is algebraic over $\mathbb{C}[G_1, \ldots, G_r]$, there exists a nontrivial polynomial $P \in \mathbb{C}[x_0, x_1, \ldots, x_r]$ such that

$$0 = \sum_{(i)} a_{(i)} F^{i_0} G_1^{i_1} \cdots G_r^{i_r} = \sum_{(i)} a_{(i)} G_1^{i_0 c_1 + i_1} \cdots G_r^{i_0 c_r + i_r}. \tag{4}$$

Without loss of generality, assume that this is the smallest degree algebraic relation. Let $\{p_1, \ldots, p_r\} \in \mathcal{T}$-aggregate of $G_1, \ldots, G_r$. Then $\det(d_{p_k} G_j) \neq 0$. For $k = 1, \ldots, r$, we have

$$\begin{aligned}
0 &= \sum_{(i)} a_{(i)} d_{p_k} (G_1^{i_0 c_1 + i_1} \cdots G_r^{i_0 c_r + i_r}) \\
&= (\sum_{(i)} a_{(i)} (i_0 c_1 + i_1) G_1^{i_0 c_1 + i_1 - 1} G_2^{i_0 c_2 + i_2} \cdots G_r^{i_0 c_r + i_r}) d_{p_k} G_1 \\
&\quad + \ldots + (\sum_{(i)} a_{(i)} (i_0 c_r + i_r) G_1^{i_0 c_1 + i_1} \cdots G_2^{i_0 c_2 + i_2} G_r^{i_0 c_r + i_r - 1}) d_{p_k} G_r \\
&= (\sum_{(i)} a_{(i)} (i_0 c_1 + i_1) F^{i_0} G_1^{i_1 - 1} G_2^{i_2} \cdots G_r^{i_r}) d_{p_k} G_1 \\
&\quad + \ldots + ((\sum_{(i)} a_{(i)} (i_0 c_r + i_r) F^{i_0} G_1^{i_1} \cdots G_{r-1}^{i_{r-1}} G_r^{i_r - 1}) d_{p_k} G_r \\
&= A_1 d_{p_k} G_1 + \cdots + A_r d_{p_k} G_r,
\end{aligned}$$

where $A_m = \sum_{(i)} a_{(i)} (i_0 c_m + i_m) F^{i_0} G_1^{i_1} \cdots G_m^{i_m - 1} \cdots G_r^{i_r}$, for $m = 1, \ldots, r$. Since $\det(d_{p_k} G_j) \neq 0$, then $A_1 = \cdots = A_r = 0$. If c_m is not rational for some $m = 1, \ldots, r$, we have a nontrivial algebraic relation of lower degree, which is a contradiction. Hence $c_1, \ldots, c_r \in \mathbb{Q}$. □

Theorem 9. *Let $G_1, \dots, G_r$ be given multiplicative Dirichlet series. Assume that:*

(i) $G_i^* = G_i - 1 \neq 0 \ \ (i = 1, \dots, r)$, and

$$\left\| \prod_{i=1}^{r} G_i^{*^{x_i}} \right\| \neq 1$$

for any sets of integers x_i not all zero.

(ii) For every sequence $1 \leq j_1 < j_2 < \cdots < j_i \leq r$,

$$\bigcup_{\nu=1}^{i} [\text{supp } G_{j_\nu}]$$

contains at least $i+1$ primes (for every i).

(iii) There exists a set of primes $p_1, \dots, p_r$ such that

$$J(G_1, \dots, G_r; p_1, \dots, p_r) := \det(d_{p_i} G_j) \neq 0.$$

If a multiplicative Dirichlet series F is algebraically dependent over $\mathbb{C}[G_1, \dots, G_r]$.

Then

$$F = G_1^{c_1} \cdots G_r^{c_r} \quad (c_1, \dots, c_r \in \mathbb{Q}).$$

Proof. Let F be a multiplicative Dirichlet series which is algebraically dependent over $\mathbb{C}[G_1, \dots, G_r]$. Then $J(F, G_1, \dots, G_r; q_1, \dots, q_{r+1}) = 0$ for all sets of distinct primes $q_1, \dots, q_{r+1}$. By Theorem 7, we have

$$F = G_1^{c_1} \cdots G_r^{c_r}$$

for some complex constants c_i. Proceeding exactly as in the proof of Theorem 8, but now taking derivations over the set of primes $p_1, \dots, p_r$ given in the condition (iii), the result follows. □

Remarks. 1) The hypothesis (iii) of Theorem 9 is compatible with $G_1, \dots, G_r$ being algebraically independent. This requirement is essential but is not made clear in Shapiro-Sparer [18]. Without this assumption, the constants c_i in Theorem 9 may not be rational as seen in the following example: let G_1 be the multiplicative Dirichlet series whose arithmetic counterpart is defined at prime powers by

$$g_1(p^\alpha) = \begin{cases} 1 & \text{if } p = 2, 3, 5; \alpha = 0, 1, \\ 0 & \text{otherwise,} \end{cases}$$

and other values being determined through multiplicativity. Let $G_2 = G_1^2$. Clearly, conditions (i) and (ii) are satisfied but not (iii), while $F := G_1^{c_1} G_2^{c_2}$ $(c_1, c_2 \in \mathbb{C})$ satisfies $J(F, G_1, G_2; p_1, p_2, p_3) = 0$ for all sets of distinct primes p_1, p_2, p_3.

2) The case $r = 1$ simplifies considerably because the assumptions (i) and (iii) hold automatically and we have the following theorem.

Theorem 10. *Let G be a multiplicative Dirichlet series such that* [supp G] *has at least two elements. If a multiplicative Dirichlet series F is algebraically dependent over $\mathbb{C}[G]$, then $F = G^c$ $(c \in \mathbb{Q})$.*

7. Dependence of SHM-Functions

In general, most SHM-functions are algebraically independent. Such independence extends even to zeta functions as confirmed in our next theorem.

Theorem 11. *Let $\alpha_1, \alpha_2, \ldots, \alpha_r, \beta_1, \ldots, \beta_k \in \mathbb{C}$ such that $\beta_i \neq 0$ and $\beta_i \neq \beta_j$ $(i \neq j)$. Then $\alpha_1, \ldots, \alpha_r$, are linearly independent over $\mathbb{Z}$ if and only if $\mu_{\alpha_1}, \ldots, \mu_{\alpha_r}, \zeta_{\beta_1}, \ldots, \zeta_{\beta_k}$ are algebraically independent over $\mathbb{C}$.*

Proof. Assume that $\alpha_1, \alpha_2, \ldots, \alpha_r$ are linearly independent over $\mathbb{Z}$. Suppose that $\mu_{\alpha_1}, \ldots, \mu_{\alpha_r}$, $\zeta_{\beta_1}, \ldots, \zeta_{\beta_k}$ are algebraically dependent over $\mathbb{C}$. Since $\mu_{\alpha_1}, \ldots, \mu_{\alpha_r}, \zeta_{\beta_1}, \ldots, \zeta_{\beta_k}$ are multiplicative, by Theorem 4 [15], there exist $j_1, \ldots, j_r, i_1, \ldots, i_k \in \mathbb{Z}$ not all zero such that

$$q = \mu_{\alpha_1}^{j_1} * \cdots * \mu_{\alpha_r}^{j_r} * \zeta_{\beta_1}^{i_1} * \cdots * \zeta_{\beta_k}^{i_k}$$

vanishes on $\mathbb{N}' = \mathbb{N}$ - semigroup generated by finitely many primes. Let $p \in \mathbb{N}'$ be prime. Then

$$\begin{aligned} 0 = q(p) &= (\mu_{\alpha_1}^{j_1} * \cdots * \mu_{\alpha_r}^{j_r} * \zeta_{\beta_1}^{i_1} * \cdots * \zeta_{\beta_k}^{i_k})(p) \\ &= \mu_{\alpha_1}^{j_1}(p) + \cdots + \mu_{\alpha_r}^{j_r}(p) + \zeta_{\beta_1}^{i_1}(p) + \cdots + \zeta_{\beta_k}^{i_k}(p) \\ &= -j_1\alpha_1 - \cdots - j_r\alpha_r + i_1 p^{\beta_1} + \cdots + i_k p^{\beta_k}. \end{aligned}$$

This relation, being true for infinitely many primes p, yields that

$$j_1\alpha_1 + \cdots + j_r\alpha_r = 0.$$

Since $\alpha_1, \ldots, \alpha_r$ are linearly independent over $\mathbb{Z}$, then $j_1 = \cdots = j_r = 0$, which is a contradiction.

Conversely, suppose that $\alpha_1, \ldots, \alpha_r$ are linearly dependent over $\mathbb{Z}$. Then there exist an α_i, say α_r, and $a_1, \ldots, a_{r-1} \in \mathbb{Z}$, not all 0, such that $\alpha_r = a_1\,\alpha_1 + \cdots + a_{r-1}\,\alpha_{r-1}$. Thus

$$\mu_{\alpha_r} = \mu_{a_1\,\alpha_1 + \cdots + a_{r-1}\,\alpha_{r-1}} = \mu_{a_1\,\alpha_1} * \cdots * \mu_{a_{r-1}\,\alpha_{r-1}} = \mu_{\alpha_1}^{a_1} * \cdots * \mu_{\alpha_{r-1}}^{a_{r-1}},$$

which indicates that even $\mu_{\alpha_1}, \ldots, \mu_{\alpha_r}$ are algebraically dependent over $\mathbb{C}$. ☐

The independence of SHM-functions can be furthered to include the Euler's totient because $\varphi = \mu_1 * \zeta_1$ and Theorem 11 clearly yields.

Corollary 6. *Let* $\alpha_1, \alpha_2, \ldots, \alpha_r, \beta_1, \ldots, \beta_k \in \mathbb{C}\backslash\{1\}$ *such that* $\beta_i \neq 0$ *and* $\beta_i \neq \beta_j \ \ (i \neq j)$. *Then* $\alpha_1, \ldots, \alpha_r$, *are linearly independent over* $\mathbb{Z}$ *if and only if* $\mu_{\alpha_1}, \ldots, \mu_{\alpha_r}, \zeta_{\beta_1}, \ldots, \zeta_{\beta_k}, \varphi$ *are algebraically independent over* $\mathbb{C}$.

Another method of establishing independence is through the use of Jacobian as expounded in the last section. However, when applied to SHM-functions, this technique works only with independence of two functions as we now elaborate.

Theorem 12. *Let* $\alpha, \beta \in \mathbb{C}$. *If* α *and* β *are linearly independent over* $\mathbb{Z}$, *then* μ_α *and* μ_β *are algebraically independent over* $\mathbb{C}$.

Proof. Let p and q be any distinct primes. Then

$$J(\mu_\alpha, \mu_\beta; p, q)(p) = \begin{vmatrix} \mu_\alpha(p^2)\upsilon_p(p^2) & \mu_\beta(p)\upsilon_p(p^2) \\ \mu_\alpha(p)\upsilon_q(pq) & \mu_\beta(p)\upsilon_q(pq) \end{vmatrix} = \alpha\,\beta\,(\alpha - \beta) \neq 0,$$

because α and β are linearly independent over $\mathbb{Z}$. By Theorem 5, μ_α, μ_β are algebraically independent over $\mathbb{C}$. ☐

We show next that using Jacobian to prove the independence of $\mu_{\alpha_1}, \ldots,$ μ_{α_k} when $k \geq 3$ fails. For ease of writing, we treat only the case $k = 3$. Let $\alpha, \beta, \gamma \in \mathbb{N} \cup \{0\}\,; p, q, r$ distinct primes and $n = p^a q^b r^c$. Then

$$\begin{aligned} J(\mu_\alpha, \mu_\beta, \mu_\gamma; p, q, r)(n) &= \upsilon_p(np)\upsilon_q(nq)\upsilon_r(nr) \begin{vmatrix} \mu_\alpha(np) & \mu_\beta(np) & \mu_\gamma(np) \\ \mu_\alpha(nq) & \mu_\beta(nq) & \mu_\gamma(nq) \\ \mu_\alpha(nr) & \mu_\beta(nr) & \mu_\gamma(nr) \end{vmatrix} \\ &= (a+1)(b+1)(c+1)(-1)^{a+b+c+1} \begin{vmatrix} \binom{\alpha}{a+1}\binom{\alpha}{b}\binom{\alpha}{c} & \binom{\beta}{a+1}\binom{\beta}{b}\binom{\beta}{c} & \binom{\gamma}{a+1}\binom{\gamma}{b}\binom{\gamma}{c} \\ \binom{\alpha}{a}\binom{\alpha}{b+1}\binom{\alpha}{c} & \binom{\beta}{a}\binom{\beta}{b+1}\binom{\beta}{c} & \binom{\gamma}{a}\binom{\gamma}{b+1}\binom{\gamma}{c} \\ \binom{\alpha}{a}\binom{\alpha}{b}\binom{\alpha}{c+1} & \binom{\beta}{a}\binom{\beta}{b}\binom{\beta}{c+1} & \binom{\gamma}{a}\binom{\gamma}{b}\binom{\gamma}{c+1} \end{vmatrix} \\ &\equiv 0. \end{aligned}$$

For n not of the form $p^a q^b r^c$, similar calculation also shows that $J(\mu_\alpha, \mu_\beta, \mu_\gamma; p, q, r)(n)$ vanishes identically, which gives no information on their independence.

However, incorporating Theorem 12 with results in the last section, we now deduce some interesting facts. Since $J(\mu_\alpha, \mu_\beta; p, q) \neq 0$ if α and β are linearly independent over $\mathbb{Z}$ and p, q are distinct primes, the $\mathcal{T}$-aggregate of μ_α, μ_β is not empty and the intersection of all the sets belonging to $\mathcal{T}$ is empty. Thus μ_α, μ_β are strongly algebraically independent. Our next theorem is interesting for it says essentially that any arithmetic function dependent on one or two SHM-functions must itself be an SHM-function whose parameter being $\mathbb{Q}$-linear combination of the former.

Theorem 13. *Let $\alpha, \beta \in \mathbb{C}$ be linearly independent over $\mathbb{Z}$.*

(i) If $f \in \mathcal{A}\backslash\{0\}$ is multiplicative and algebraic over $\mathbb{C}[\mu_\alpha]$, then there exist $r \in \mathbb{Q}$ such that

$$f = \mu_\alpha^r = \mu_{r\,\alpha}.$$

(ii) If $f \in \mathcal{A}\backslash\{0\}$ is multiplicative and algebraic over $\mathbb{C}[\mu_\alpha, \mu_\beta]$, then there are $r, t \in \mathbb{Q}$ such that

$$f = \mu_\alpha^r * \mu_\beta^t = \mu_{r\,\alpha + t\beta}.$$

Proof. The first equality, in both (i) and (ii), is immediate from Theorem 8 and strongly algebraic independence. The last equality, in both (i) and (ii), follows from the fact that $\mu_\alpha * \mu_\beta = \mu_{\alpha+\beta}$, and $\mu_\alpha^r = \mu_{r\alpha}$, where the exponential functions are as defined in Rearick [16]. $\square$

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