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ASYMPTOTIC PROPERTIES FOR MEASURES OF MULTIVARIATE KURTOSIS IN ELLIPTICAL DISTRIBUTIONS

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Abstract: In this paper, we consider the estimation of the kurtosis parameter in elliptical distributions. The consistent estimators based on measures of multivariate kurtosis independently defined by Mardia [5] and Srivastava [7] are presented. In order to investigate accuracy of estimations, asymptotic expansions of the expectation and the asymptotic variance of the estimators are derived by a perturbation method under elliptical populations. Some numerical examples obtained by Monte Carlo simulation for some selected parameters are also provided.

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1. Introduction

It is well known that the fourth order cumulant in elliptical distributions is essentially equal to the kurtosis parameter from the relation between moments and cumulants. Estimation of the kurtosis parameter is important in the study of multivariate statistical analysis for elliptical populations. The kurtosis parameter, especially with relation to the estimation problem, has been considered by many authors. It is on the basis of the measure of multivariate kurtosis in-

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troduced by Mardia [5] that used commonly as an estimator of the fourth order cumulant at present. The sample analogue of this measure is also proposed as tests of normality, and its asymptotic distribution is derived for sample from a multivariate normal population. The related discussion of Mardia's sample measure under the elliptical distribution has been given by Berkane and Bentler [2], Seo and Toyama [8], Anderson [1] and Maruyama and Seo [6]. For general distributions, the asymptotic variance of Mardia's sample measure has been discussed in Henze [3]. On the other hand, Srivastava [7] suggested another measure of multivariate kurtosis from the viewpoint of a principle component, and obtained the asymptotic distribution of its sample measure under normality, too. However there is almost no work has been done regarding estimations under non-normality compared with the case of Mardia's measure. In addition, it seems that the characteristic which the estimator based on Srivastava's measure itself has, are not completely known. Therefore, the purpose of this paper is to specify the asymptotic properties of Srivastava's sample kurtosis. In order to achieve our purpose, asymptotic expansions of the expectation and the asymptotic variance are given up to the order n^{-1} as the size n of sample tends to infinity. An overview of the present paper is as follows. In Section 2, some properties of the class of elliptical distributions are explained. Secondly we construct the consistent estimator of the kurtosis parameter based on Srivastava's measure. In Section 3, the asymptotic properties of the estimator for elliptical distributions are presented in the two cases for which the population covariance matrix is known and unknown. The theory discussed in Section 3 is applied for a moment parameter which generalized the kurtosis parameter in Section 4, and is used to find other estimation with bias correction in Section 5. Moreover, we investigate bias and MSE of the estimators by Monte Carlo simulation applied to some selected parameters, and evaluate the confidence intervals for the kurtosis parameter with the asymptotic properties.

2. Utility Notations and Definitions

In this section, we present notations, definitions and basic facts that we use in the proof of our main results. Let a p-variate random vector X be distributed as a p-variate elliptical distribution with parameters μ and Λ, i.e., $E_p(\mu, \Lambda)$, where Λ is some positive definite symmetric matrix. If the probability density function exists, it has the form $f(x) = c_p|\Lambda|^{-1/2}\, h({}^t(x-\mu)\Lambda^{-1}(x-\mu))$ for some non-negative function h, where c_p is the normalizing constant and ${}^tX$ denotes a transposition of a vector X. The characteristic function is $\phi(\theta) =$

$\exp[i\,{}^t\theta\mu]\psi({}^t\theta\Lambda\theta)$ for some function ψ, where $i=\sqrt{-1}$. Note that $E(X)=\mu$ and $\mathrm{Cov}(X)=-2\psi'(0)\Lambda=:\Sigma=(\sigma_{ij})$, respectively. For example, the multivariate normal, the multivariate t and the contaminated normal distributions belong to the class of elliptical distributions.

The following is given in Maruyama and Seo [6]. The odd order moments of $X-\mu$ are 0, the $2m$-th order moments, $\mu_{i_1i_2\cdots i_{2m}} := E[(X_{i_1}-\mu_{i_1})(X_{i_2}-\mu_{i_2})\cdots(X_{i_{2m}}-\mu_{i_{2m}})]$ are expressible in the form:

$$\mu_{i_1i_2\cdots i_{2m}} = (\mathcal{K}_{(m)}+1)\sum_{(d_m)}\sigma_{i_1i_2}\cdots\sigma_{i_{2m-1}i_{2m}}, \tag{1}$$

where $\sum_{(d_m)}$ means the sum of all $d_m := \prod_{k=1}^{m}(2k-1)$ possible combinations $(i_1,i_2,\cdots,i_{2m})$ and

$$\mathcal{K}_{(m)} := \frac{\psi^{(m)}(0)}{(\psi'(0))^m}-1.$$

We also define $\mathcal{K}_{(m)}$ as the $2m$-th order moment parameter. In case of $m=2$, $\mathcal{K}_{(2)}$ is simply denoted by κ and called a kurtosis parameter, that is a key parameter in elliptical distributions.

Next, we give careful consideration to the consistent estimators of kurtosis parameter. Suppose that $X_1,\cdots,X_n$ are independent and identically distributed random vectors according to $E_p(\mu,\Lambda)$. In Mardia [5], a multivariate coefficient of kurtosis is defined as $\beta_{2,p} := E[\{{}^t(X-\mu)\Sigma^{-1}(X-\mu)\}^2]$, and the affine invariant sample analogue of $\beta_{2,p}$ is obtained by $b_{2,p} := (1/n)\sum_{j=1}^{n}\{{}^t(X_j-\overline{X})U^{-1}(X_j-\overline{X})\}^2$, where $\overline{X}$ is the sample mean and U is the unbiased sample covariance matrix. Since $\beta_{2,p}$ can be calculated as $p(p+2)(\kappa+1)$ with (1), we have the consistent estimator of κ by $\hat{\kappa}=(1/p(p+2))b_{2,p}-1$. On the other hand, Srivastava [7] proposed a measure of multivariate kurtosis using the principle component method. Let Q be an orthogonal matrix such that ${}^tQ\Sigma Q = D_\lambda := \mathrm{diag}(\lambda_1,\cdots,\lambda_p)$, where $\lambda_1,\cdots,\lambda_p$ are the eigenvalues of Σ. Then we can write $\beta_{2,p}=E[(\mathrm{tr}Z^2)^2]$, where $Z:=\mathrm{diag}(z_1,\cdots,z_p)$ and $z := D_\lambda^{-1/2}\,{}^tQ(X-\mu) = {}^t(z_1,\cdots,z_p)$. Note that $\gamma_{2,p} := E(\mathrm{tr}Z^4)$ is another measure in [7]. Now $\gamma_{2,p}$ can be calculated as $3p(\kappa+1)$ with (1). Further suppose that $A_j := \mathrm{diag}(a_{1j},\cdots,a_{pj})$, $a_j := D_w^{-1/2}\,{}^tH(X_j-\overline{X}) = {}^t(a_{1j},\cdots,a_{pj})$, where $w_1,\cdots,w_p$ are the eigenvalues of U, and H is an orthogonal matrix such that ${}^tHUH = D_w := \mathrm{diag}(w_1,\cdots,w_p)$. Then we can write $b_{2,p}=(1/n)\sum_{j=1}^{n}(\mathrm{tr}A_j^2)^2$. Also the sample measure correspond to $\gamma_{2,p}$ is defined by $g_{2,p}=(1/n)\sum_{j=1}^{n}(\mathrm{tr}A_j^4)$. Therefore we have another consistent estimator as $\tilde{\kappa}=(1/(3p))g_{2,p}-1$.

3. The Main Results

In this section, we consider the second order asymptotic on the estimator $\tilde{\kappa}$ under elliptical populations. Let $X_1, \cdots, X_n$ and $\tilde{\kappa}$ be defined as in Section 2.

Proposition 1. *As n gets large, to the order n^{-1} with known Σ we have*

$$E(g_{2,p}) = 3p(\kappa+1) + \frac{1}{n}\big\{-12p(\kappa+1) + 6p\big\} + O(n^{-2}),$$
$$\mathrm{Var}(g_{2,p}) = \frac{1}{n}\big\{(9p^2 + 96p)(\mathcal{K}_{(4)} + 1) - 9p^2(\kappa+1)^2\big\} + O(n^{-2}).$$

Proof. It is assumed without loss of generality that $\Sigma = I_p$. Letting $Y_j := {}^tHX_j = {}^t(y_{1j}, \cdots, y_{pj})$ for $j = 1, \cdots, n$, then $g_{2,p}$ can be written

$$g_{2,p} = \frac{1}{n}\sum_{i=1}^{p}\sum_{j=1}^{n}\chi_{ij}^2,$$

where $\chi_{ij} := (y_{ij} - \overline{y}_i)^2$ and $\overline{Y} := (1/n)\sum_{j=1}^{n} Y_j = {}^t(\overline{y}_1, \cdots, \overline{y}_p)$. If we define the mean of all the observations except the j-th observation as $\overline{Y}_{(j)} := (1/(n-1))\sum_{k\neq j}^{n} Y_k$, then $Y_j - \overline{Y} = (1 - 1/n)(Y_j - \overline{Y}_{(j)})$. Note that Y_j and $\overline{Y}_{(j)}$ are independent. To obtain the expectation of $\tilde{\kappa}$ by the perturbation method, we put $\zeta := \sqrt{n-1}\,\overline{Y}_{(j)}$. Then χ_{ij}^2 can be expanded with $\zeta = {}^t(\zeta_1, \cdots, \zeta_p)$ as

$$\chi_{ij}^2 = y_{ij}^4 - \frac{1}{\sqrt{n}}4y_{ij}^3\,\zeta_i + \frac{1}{n}(6y_{ij}^2\,\zeta_i^2 - 4y_{ij}^4) + O(n^{-3/2}).$$

Therefore, calculating the expectation with respect to Y_j and ζ, we obtain the second order asymptotic mean of $g_{2,p}$. In a similar way to above, we find after some algebra the variance of $g_{2,p}$ to order n^{-1}. Thus we complete the proof. □

From Proposition 1, we have the following immediately.

Corollary 1. *Under the condition that Σ is known, the asymptotic expectation of $\tilde{\kappa}$ is*

$$E(\tilde{\kappa}) = \kappa + \frac{1}{n}\big\{-2(2\kappa+1)\big\} + O(n^{-2}), \tag{2}$$

and the asymptotic variance of $T_{\tilde{\kappa}} := \sqrt{n}(\tilde{\kappa} - \kappa)$ is

$$\sigma_{\tilde{\kappa}}^2 = \frac{35 + 3(p-1)}{3p}(\mathcal{K}_{(4)} + 1) - (\kappa+1)^2 + O(n^{-1}). \tag{3}$$

We note that in the case when Σ is known, the expectation of $\tilde{\kappa}$ is asymptotically equivalent to that of $\hat{\kappa}$ up to the order n^{-1}.

Proposition 2. *As n gets large, to the order n^{-1} with unknown Σ we have*

$$E(g_{2,p}) = 3p(\kappa+1) + \frac{1}{n}\left\{27p(\kappa+1)^2 - 15p(\kappa+1) - 30p(\mathcal{K}_{(3)}+1) + 6p\right\} + O(n^{-2}),$$

$$\mathrm{Var}(g_{2,p}) = \frac{1}{n}\Big\{(9p^2+96p)(\mathcal{K}_{(4)}+1) - 36p(p+4)(\mathcal{K}_{(3)}+1)(\kappa+1) + 36p(p+2)(\kappa+1)^3 - 9p^2(\kappa+1)^2\Big\} + O(n^{-2}).$$

Proof. Since $Y_j, \overline{Y}$ and D_w are not independent, we put $T_{ij} := w_i^{-1}(y_{ij}-\overline{y}_i)^2$ and $\overline{Y}_{(j)} := (1/(n-1))\sum_{k\neq j}^n Y_k$. Similarly, the covariance matrix Σ should be estimated without the j-th observation. That is by

$$D_{(j)} := \frac{1}{n-2}\sum_{k\neq j}^n (Y_k - \overline{Y}_{(j)})\,{}^t(Y_k - \overline{Y}_{(j)}).$$

Then, it can be shown that $D_w = (1-1/(n-1))D_{(j)} + (1/n)(Y_j - \overline{Y}_{(j)})\,{}^t(Y_j - \overline{Y}_{(j)})$ and

$$D_w^{-1} = \frac{n-1}{n-2}D_{(j)}^{-1} - \frac{(n-1)^2/(n(n-2)^2)D_{(j)}^{-1}(Y_j-\overline{Y}_{(j)})\,{}^t(Y_j-\overline{Y}_{(j)})D_{(j)}^{-1}}{1+(n-1)/(n(n-2))\,{}^t(Y_j-\overline{Y}_{(j)})D_{(j)}^{-1}(Y_j-\overline{Y}_{(j)})}.$$

Further, let $\zeta := \sqrt{n-1}\,\overline{Y}_{(j)}$ and $M := \sqrt{n-1}(D_{(j)} - I_p)$. Note that it is enough if only diagonal elements are calculated. Then, T_{ij}^2 is stochastically expanded with ζ and $M = (m_{ij})$ as

$$T_{ij}^2 = y_{ij}^4 + \frac{1}{\sqrt{n}}(-4y_{ij}^3\,\zeta_i - 2m_{ii}\,y_{ij}^4) + \frac{1}{n}(-2y_{ij}^6 - 2y_{ij}^4 + 3m_{ii}^2\,y_{ij}^4 + 6\zeta_i^2\,y_{ij}^2 + 8m_{ii}\,y_{ij}^3\,\zeta_i) + O(n^{-3/2}).$$

We can calculate the expectation for the expansion of T_{ij}^2 by using the asymptotic expanded joint probability density function (j.p.d.f) of ζ and M (see Iwashita [4] and Wakaki [9]). Then we obtain the second order asymptotic mean of $g_{2,p}$.

Using a similar idea, we may obtain the variance. In order to avoid the dependence of Y_j, Y_l, $\overline{Y}$ and D_w, we define

$$\overline{Y}_{(j,l)} := \frac{1}{n-2}\sum_{k\neq j,l}^{n} Y_k,$$
$$D_{(j,l)} := \frac{1}{n-3}\sum_{k\neq j,l}^{n}(Y_k-\overline{Y}_{(j,l)})\,{}^t(Y_k-\overline{Y}_{(j,l)}).$$

Note that $E(g_{2,p}^2) = (1/n)\mathbb{B}_1 + (1-1/n)\mathbb{B}_2$ where

$$\mathbb{B}_1 := E\left[\left(\sum_{i=1}^{p} T_{ij}^2\right)^2\right], \quad \mathbb{B}_2 := E\left[\sum_{i,\alpha}^{p} T_{ij}^2 T_{\alpha l}^2\right].$$

In order to compute $\mathbb{B}_1$ and $\mathbb{B}_2$, let $\zeta^* := \sqrt{n-2}\,\overline{Y}_{(j,l)}$ and $M^* := \sqrt{n-2}$ $(D_{(j,l)} - I_p)$. For evaluation of $\mathbb{B}_1$, we calculate the expectation with respect to Y_j, ζ and M, and then for calculating $\mathbb{B}_2$, we use an asymptotic expansion of j.p.d.f. of ζ^* and M^*. Then we have the asymptotic variance up to the order n^{-1}. Thus we complete the proof. □

An immediate consequence of Proposition 2 is given in the following results.

Corollary 2. *Under the condition that unknown* Σ*, the asymptotic expectation of* $\tilde{\kappa}$ *is*

$$E(\tilde{\kappa}) = \kappa + \frac{1}{n}\left\{9(\kappa+1)^2 - 5(\kappa+1) - 10(\mathcal{K}_{(3)}+1) + 2\right\} + O(n^{-2}), \tag{4}$$

and the asymptotic variance of $T_{\tilde{\kappa}} := \sqrt{n}(\tilde{\kappa}-\kappa)$ *is*

$$\begin{aligned}\sigma_{\tilde{\kappa}}^2 &= \frac{3p+32}{3p}(\mathcal{K}_{(4)}+1) \\ &\quad - \frac{4(p+4)}{p}(\mathcal{K}_{(3)}+1)(\kappa+1) + \frac{4(p+2)}{p}(\kappa+1)^3 - (\kappa+1)^2 + O(n^{-1}).\end{aligned} \tag{5}$$

When the underlying distribution is normal, it follows from Proposition 2 that $E(g_{2,p}) = 3p - 12p/n + O(n^{-2})$ and $\mathrm{Var}(g_{2,p}) = 24p/n + O(n^{-2})$. These expressions are essentially equivalent to those obtained by Srivastava [7].

Example 1. We shall compare the empirical distribution of $\tilde{\kappa}$ in several elliptical populations with the theoretical distribution on the basis of the expansion (4) when Σ is unknown, and investigate how rapidly the estimators

p	n	CN(0.1, 3) $\kappa = 1.78$		Normal $\kappa = 0$		MT(9) $\kappa = 0.4$		CU(1,2) $\kappa = 0.16$	
		$\hat{\kappa}$	$\tilde{\kappa}$	$\hat{\kappa}$	$\tilde{\kappa}$	$\hat{\kappa}$	$\tilde{\kappa}$	$\hat{\kappa}$	$\tilde{\kappa}$
5	100	0.965	1.124	-0.391^*	-0.390^*	0.204	0.244	0.878^*	0.951^*
		0.692	1.087	-0.400^*		0.095	0.199	0.807^*	0.958^*
	200	1.301	1.396	-0.197^*	-0.196^*	0.278	0.312	0.127	0.130
		1.235	1.432	-0.200^*		0.247	0.300	0.123	0.131
	500	1.569	1.606	-0.820^{**}	-0.801^{**}	0.337	0.359	0.150	0.152
		1.560	1.639	-0.800^{**}		0.339	0.359	0.149	0.152
	4000	1.749	1.761	-0.102^{**}	-0.100^{**}	0.380	0.385	0.164	0.165
		1.750	1.760	-0.100^{**}		0.392	0.394	0.164	0.165

$*$: value$\times 10^{-1}$, $**$: value$\times 10^{-2}$.

Table 1: The mean of estimators based on simulation study (upper) and asymptotic expansion (lower) when Σ is unknown

converge on κ. Table 1 shows the sample means for $\hat{\kappa}$ and $\tilde{\kappa}$ based on 10,000 simulations under assumption that $\Sigma = I_p$ without any loss of generality. On the other hand, the approximate values by the expansion (4) are obtained. In the case of $\hat{\kappa}$, we used the results given by Seo and Toyama [8]. The elliptical populations we considered are the contaminated normal CN(ω, τ) with $\omega = 0.1; \tau = 3$, the multivariate normal, the multivariate t with 9 degrees of freedom MT(9) and the compound normal denoted by CU(1,2) (meaning that the random vector X from CU(1,2) is the product of a standard normal vector and the inverse of a random variable according to the uniform distribution on the interval $[1, 2]$. The approximations are also illustrated with Figure 1.

Example 2. The asymptotic normality of $T_{\tilde{\kappa}}$ also enables us to easily construct the confidence intervals for κ. A confidence interval for κ with confidence coefficient $1 - \alpha$ is approximately $[\tilde{\kappa} \pm z_{\alpha/2}\, \sigma_{\tilde{\kappa}}/\sqrt{n}]$ where $z_{\alpha/2}$ is the two-tailed $100\alpha\%$ point of the standard normal distribution. In the same way, we have another interval for κ by means of $\hat{\kappa}$, that is to say $[\hat{\kappa} \pm z_{\alpha/2}\, \sigma_{\hat{\kappa}}/\sqrt{n}]$ where $T_{\hat{\kappa}} := \sqrt{n}(\hat{\kappa} - \kappa)$ and $\sigma^2_{\hat{\kappa}}$ is the asymptotic variance of $T_{\hat{\kappa}}$ (see [8]). Table 2 presents the confidence intervals for κ with nominal confidence coefficient 0.95. Those are the confidence limit with $\tilde{\kappa}$ (lower) and with $\hat{\kappa}$ (upper).

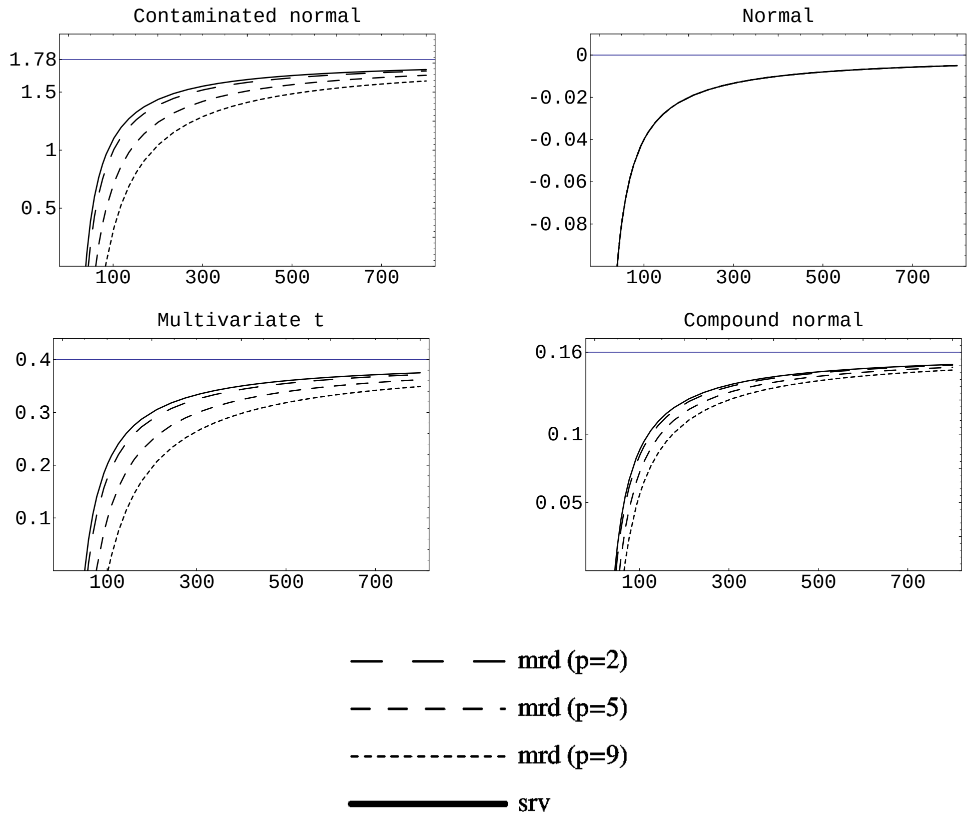

Figure 1: Approximations to the mean of estimators $\hat{\kappa}$ (mrd) and $\tilde{\kappa}$ (srv)

4. Application to the Moment Parameter

Generally it is not easy to derive the exact distribution of test statistics or the percentiles for the testing problem under the elliptical populations, and so the asymptotic expansion of the statistics is considered. In particular an asymptotic expression given up to the higher order includes not only the kurtosis parameter but the more general higher order moment parameters as well. Then we have to speculate about the estimation of the moment parameters as a practical problem.

Now we shall generalize $\beta_{2,p}$ in following so as to discuss extensions of the estimation of kurtosis parameter. We can calculate $E[\{{}^t(X-\mu)\Sigma^{-1}(X-\mu)\}^m] = 2^m(p/2)_m(\mathcal{K}_{(m)}+1)$ with (1). Then we have a consistent estimator of the $2m$-th

p	n		CN(0.1, 3) $\kappa = 1.78$		Normal $\kappa = 0$		MT(9) $\kappa = 0.4$		CU(1,2) $\kappa = 0.16$	
			lower	upper	lower	upper	lower	upper	lower	upper
2	100	$\hat{\kappa}$	−0.97	3.26	−0.23	0.16	−1.75	2.22	−0.25	0.45
		$\tilde{\kappa}$	−1.12	3.48	−0.26	0.18	−1.81	2.29	−0.29	0.49
	200	$\hat{\kappa}$	−0.07	2.93	−0.15	0.12	−1.10	1.70	−0.12	0.38
		$\tilde{\kappa}$	−0.17	3.08	−0.17	0.14	−1.14	1.76	−0.14	0.40
	500	$\hat{\kappa}$	0.67	2.57	−0.09	0.08	−0.54	1.24	−0.01	0.31
		$\tilde{\kappa}$	0.60	2.66	−0.10	0.09	−0.56	1.27	−0.02	0.32
	4000	$\hat{\kappa}$	1.41	2.10	−0.03	0.03	0.07	0.70	0.10	0.22
		$\tilde{\kappa}$	1.40	2.12	−0.03	0.03	0.06	0.70	0.10	0.22
5	100	$\hat{\kappa}$	−0.21	2.14	−0.13	0.05	−1.12	1.52	−0.10	0.28
		$\tilde{\kappa}$	−0.33	2.58	−0.18	0.10	−1.17	1.65	−0.15	0.34
	200	$\hat{\kappa}$	0.46	2.13	−0.08	0.04	−0.65	1.21	−0.01	0.26
		$\tilde{\kappa}$	0.36	2.42	−0.12	0.08	−0.68	1.31	−0.04	0.30
	500	$\hat{\kappa}$	1.04	2.09	−0.05	0.03	−0.25	0.93	0.06	0.23
		$\tilde{\kappa}$	0.96	2.25	−0.07	0.05	−0.27	0.99	0.04	0.26
	4000	$\hat{\kappa}$	1.56	1.93	−0.02	0.01	0.17	0.59	0.13	0.19
		$\tilde{\kappa}$	1.53	1.97	−0.02	0.02	0.16	0.60	0.13	0.20

Table 2: Confidence interval for κ when Σ is unknown

order moment parameter as

$$\hat{\mathcal{K}}_{(m)} = \frac{1}{n2^m (p/2)_m} \sum_{j=1}^{n} \left\{ {}^t(X_j - \overline{X}) U^{-1} (X_j - \overline{X}) \right\}^m - 1,$$

where $(a)_m := a(a+1)\cdots(a+m-1)$. The expectation of $\hat{\mathcal{K}}_{(m)}$ can be expanded as

$$E(\hat{\mathcal{K}}_{(m)}) = \mathcal{K}_{(m)} + \frac{m}{n}\mathbb{C}_1 + O(n^{-2}), \tag{6}$$

where

$$\mathbb{C}_1 = \frac{3m+2p+1}{2}(\mathcal{K}_{(m)}+1)(\kappa+1) - \frac{m+3}{2}(\mathcal{K}_{(m)}+1) - (p+2m)(\mathcal{K}_{(m+1)}+1) + \mathcal{K}_{(m-1)} + 1,$$

and the asymptotic variance of $T_1 := \sqrt{n}(\hat{\mathcal{K}}_{(m)} - \mathcal{K}_{(m)})$ is obtained by

$$\sigma_{T_1}^2 = \frac{(p/2+m)_m}{(p/2)_m}(\mathcal{K}_{(2m)}+1) + \frac{m^2(p+2)}{p}(\mathcal{K}_{(m)}+1)^2(\kappa+1) -$$

$$\frac{2m(p+2m)}{p}(\mathcal{K}_{(m+1)}+1)(\mathcal{K}_{(m)}+1)-(m-1)^2(\mathcal{K}_{(m)}+1)^2+O(n^{-1}). \quad (7)$$

Note that the expansions by (6) and (7) do not agree with the results in Maruyama and Seo [6] because it differ from the definition of this paper in a sample covariance matrix, that is $S := (1/n)\sum_{i=1}^{n}(X_i-\overline{X})\,{}^t(X_i-\overline{X})$.

In the same way, another estimator of $\mathcal{K}_{(m)}$ is also given by generalizing $\gamma_{2,p}$. Since $E(\mathrm{tr}Z^{2m}) = p\,2^m(1/2)_m(\mathcal{K}_{(m)}+1)$ under assumptions in Section 2, we have

$$\tilde{\mathcal{K}}_{(m)} = \frac{1}{np2^m(1/2)_m}\sum_{j=1}^{n}\big(\mathrm{tr}A_j^{2m}\big)-1.$$

Moreover, we have the following results for the asymptotic properties of $\tilde{\mathcal{K}}_{(m)}$ up to the order n^{-1}. The expectation of $\tilde{\mathcal{K}}_{(m)}$ is expanded as $E(\tilde{\mathcal{K}}_{(m)}) = \mathcal{K}_{(m)} + m\mathbb{C}_2/n + O(n^{-2})$, where

$$\mathbb{C}_2 = \frac{3(m+1)}{2}(\mathcal{K}_{(m)}+1)(\kappa+1)-\frac{m+3}{2}(\mathcal{K}_{(m)}+1)-(2m+1)(\mathcal{K}_{(m+1)}+1) + \mathcal{K}_{(m-1)}+1.$$

The asymptotic variance of $T_2 := \sqrt{n}(\tilde{\mathcal{K}}_{(m)}-\mathcal{K}_{(m)})$ with unknown Σ is given by

$$\sigma_{T_2}^2 = \frac{(m+1/2)_m+(p-1)(1/2)_m}{p(1/2)_m}(\mathcal{K}_{(2m)}+1)+\frac{m^2(p+2)}{p}(\mathcal{K}_{(m)}+1)^2(\kappa+1) - \frac{2m(p+2m)}{p}(\mathcal{K}_{(m+1)}+1)(\mathcal{K}_{(m)}+1)-(m-1)^2(\mathcal{K}_{(m)}+1)^2+O(n^{-1}).$$

5. Bias Correction

In this section, we shall correct the bias of $\tilde{\kappa}$. It follows from (2) that the bias in $\tilde{\kappa}$ is of order n^{-1} when Σ is known. Then an estimator with the bias correction is given by

$$\tilde{\kappa}^{\star} := \tilde{\kappa} + \frac{1}{n}\big\{2(2\tilde{\kappa}+1)\big\}. \quad (8)$$

In the case of unknown Σ, we may obtain another estimator as

$$\tilde{\kappa}^{\star} := \tilde{\kappa} - \frac{1}{n}(9\tilde{\kappa}^2+13\tilde{\kappa}-10\tilde{\mathcal{K}}_{(3)}-4). \quad (9)$$

Remark that the bias in $\tilde{\kappa}^{\star}$, both (8) and (9), are of order n^{-2} with the same variance up to the order n^{-1} as that of $\tilde{\kappa}$. Further, estimators with bias correction for $\mathcal{K}_{(m)}$ can be obtained in a similar way.

Example 3. We make investigations into the bias and the MSE for estimators by Monte Carlo simulation for some selected values of parameters. Consider the elliptical distributions of Example 1 as reproduced, the contaminated normal CN(0.1, 3), the multivariate normal, the multivariate t MT(9) and the compound normal CU(1,2). Computations are made for each case where the population covariance matrix Σ is unknown. If Σ is known, we use only (8). Table 3 gives the bias for two estimators $\hat{\kappa}$ and $\tilde{\kappa}$ and the modified them $\hat{\kappa}^{\star}$ and $\tilde{\kappa}^{\star}$ in (9) (see also [8]) based on 10,000 simulations when Σ is unknown. The MSE is presented in Table 4.

6. Conclusion

From Table 1, we can see that the simulation results nearly coincide with the approximate values for normal and CU(1,2). But in the other cases CN(0.1, 3) and MT(9), it seems that the convergence is quite slow and the approximate expression (4) is not always precise. Moreover it may be found from Figure 1 that the approximations for $\hat{\kappa}$ become inaccurate as the value of p is large. But $\tilde{\kappa}$ is not so because the approximation (4) is independent of p. In all cases, both values agree for a sufficiently large n.

It can be seen from Table 2 that the large n has the small range of the confidence interval. The confidence interval by means of $\hat{\kappa}$ has a slightly small range by contrast with that of $\tilde{\kappa}$. Because $T_{\tilde{\kappa}}$ has a little bigger asymptotic variance than $\sigma^2_{\hat{\kappa}}$. It may be found that the intervals are small for fixed n with large p. In addition, note that both of the asymptotic variances $\sigma^2_{\hat{\kappa}}$ and $\sigma^2_{\tilde{\kappa}}$ decrease monotonously when p is large. In a practical situation, we will need to use the estimators $\tilde{\sigma}^2_{\tilde{\kappa}}$ (or $\hat{\sigma}^2_{\hat{\kappa}}$) instead of $\sigma^2_{\tilde{\kappa}}$ (or $\sigma^2_{\hat{\kappa}}$) where the $\tilde{\sigma}^2_{\tilde{\kappa}}$ (or $\hat{\sigma}^2_{\hat{\kappa}}$) is defined from $\tilde{\sigma}^2_{\tilde{\kappa}}$ (or $\hat{\sigma}^2_{\hat{\kappa}}$) by replacing unknown parameters $\mathcal{K}_{(m)}$ by the consistent estimators $\tilde{\mathcal{K}}_{(m)}$ (or $\hat{\mathcal{K}}_{(m)}$), respectively. In this case we have seen that the proposed approximations do not very bad as in the case when the moment parameters $\mathcal{K}_{(m)}$ are known, through a numerical study.

It may be seen from simulation results in Table 3 that the expectation of the estimators converges to the kurtosis parameter when the sample size is large. Especially in normal population, it is noted that $\tilde{\kappa}$ rapidly approaches κ and $\tilde{\kappa}^{\star}$ is more than. The values obtained for the estimators were acceptable for a large n and improved as ω increased for the contaminated normal distributions, and

p	Model	n	$\hat{\kappa}$	$\hat{\kappa}^{\star}$	$\tilde{\kappa}$	$\tilde{\kappa}^{\star}$
2	CN(0.1, 3)	100	-0.633	-0.282	-0.594	-0.210
	$\kappa = 1.78$	200	-0.348	-0.911^{*}	-0.325	-0.783^{*}
		500	-0.153	-0.216^{*}	-0.139	-0.210^{*}
		4000	-0.220^{*}	-0.289^{**}	-0.114^{*}	-0.258^{**}
	Normal	100	-0.405^{*}	-0.520^{**}	-0.403^{*}	-0.173^{**}
	$\kappa = 0$	200	-0.197^{*}	-0.899^{***}	-0.196^{*}	-0.145^{***}
		500	-0.707^{**}	-0.770^{***}	-0.687^{**}	-0.720^{***}
		4000	-0.106^{**}	$-0.677^{\dagger}$	-0.104^{**}	$-0.346^{\dagger}$
	MT(9)	100	-0.165	-0.680^{*}	-0.156	-0.564^{*}
	$\kappa = 0.4$	200	-0.101	-0.371^{*}	-0.929^{*}	-0.324^{*}
		500	-0.510^{*}	-0.169^{*}	-0.456^{*}	-0.150^{*}
		4000	-0.195^{*}	-0.144^{*}	-0.180^{*}	-0.135^{*}
	CU(1,2)	100	-0.691^{*}	-0.106^{*}	-0.662^{*}	-0.510^{**}
	$\kappa = 0.16$	200	-0.380^{*}	-0.547^{**}	-0.372^{*}	-0.462^{**}
		500	-0.137^{*}	-0.388^{***}	-0.129^{*}	-0.105^{***}
		4000	-0.213^{**}	-0.288^{***}	-0.195^{**}	-0.191^{***}
5	CN(0.1, 3)	100	-0.812	-0.375	-0.652	-0.636^{*}
	$\kappa = 1.78$	200	-0.476	-0.141	-0.381	-0.403^{*}
		500	-0.208	-0.314^{*}	-0.171	-0.299^{*}
		4000	-0.279^{*}	-0.154^{**}	-0.175^{*}	-0.135^{**}
	Normal	100	-0.391^{*}	-0.374^{**}	-0.390^{*}	-0.300^{**}
	$\kappa = 0$	200	-0.197^{*}	-0.997^{***}	-0.196^{*}	-0.973^{***}
		500	-0.820^{**}	-0.411^{***}	-0.801^{**}	-0.405^{***}
		4000	-0.102^{**}	$-0.308^{\dagger}$	-0.100^{**}	$-0.303^{\dagger}$
	MT(9)	100	-0.195	-0.836^{*}	-0.155	-0.150^{*}
	$\kappa = 0.4$	200	-0.121	-0.424^{*}	-0.871^{*}	-0.779^{**}
		500	-0.625^{*}	-0.207^{*}	-0.408^{*}	-0.976^{**}
		4000	-0.199^{*}	-0.131^{*}	-0.147^{*}	-0.101^{*}
	CU(1,2)	100	-0.788^{*}	-0.145^{*}	-0.715^{*}	-0.134^{*}
	$\kappa = 0.16$	200	-0.396^{*}	-0.242^{**}	-0.363^{*}	-0.237^{**}
		500	-0.165^{*}	-0.409^{***}	-0.142^{*}	-0.266^{***}
		4000	-0.197^{**}	-0.161^{***}	-0.180^{**}	$-0.289^{\dagger}$

$*$: value$\times 10^{-1}$, $**$: value$\times 10^{-2}$, $***$: value$\times 10^{-3}$, $\dagger$: value$\times 10^{-4}$

Table 3: Simulation results for the bias of estimators when Σ is unknown

when n increased for the compound normal CU(1,2). Also from Table 3, we note that the estimator $\tilde{\kappa}$ is underestimated as well as $\hat{\kappa}$ for elliptical populations. The bias of $\tilde{\kappa}$ is actually smaller in magnitude than that of $\hat{\kappa}$. It may be noted that the size of p does not have much effect on the bias of estimators. These results are true for small sample sizes, not more than $n = 50$, when Σ is known.

p	Model	n	$\hat{\kappa}$	$\hat{\kappa}^\star$	$\tilde{\kappa}$	$\tilde{\kappa}^\star$
2	CN(0.1, 3)	100	0.805	0.830	0.851	1.307
	$\kappa = 1.78$	200	0.459	0.557	0.517	0.611
		500	0.209	0.243	0.244	0.272
		4000	0.287*	0.294*	0.335*	0.340*
	Normal	100	0.968**	0.109*	0.127*	0.134*
	$\kappa = 0$	200	0.494**	0.538**	0.637**	0.656**
		500	0.199**	0.209**	0.261**	0.267**
		4000	0.257***	0.258***	0.336***	0.336***
	MT(9)	100	0.108	0.167	0.135	0.197
	$\kappa = 0.4$	200	0.835*	0.130	0.107	0.149
		500	0.680*	0.106	0.926*	0.121
		4000	0.116*	0.125*	0.135*	0.142*
	CU(1,2)	100	0.263*	0.332*	0.323*	0.381*
	$\kappa = 0.16$	200	0.142*	0.165*	0.175*	0.192*
		500	0.607**	0.661**	0.750**	0.788**
		4000	0.812***	0.820***	0.999***	0.100**
5	CN(0.1, 3)	100	0.767	0.368	0.632	0.331
	$\kappa = 1.78$	200	0.320	0.187	0.307	0.185
		500	0.985*	0.770*	0.115	0.101
		4000	0.952**	0.921**	0.139*	0.136*
	Normal	100	0.325**	0.257**	0.599**	0.530**
	$\kappa = 0$	200	0.139**	0.125**	0.281**	0.250**
		500	0.501***	0.477***	0.107**	0.101**
		4000	0.578†	0.575†	0.136***	0.134***
	MT(9)	100	0.577*	0.550*	0.805*	0.760*
	$\kappa = 0.4$	200	0.377*	0.347*	0.724*	0.602*
		500	0.216*	0.202*	0.417*	0.397*
		4000	0.520**	0.500**	0.856**	0.802**
	CU(1,2)	100	0.115*	0.935**	0.161*	0.155*
	$\kappa = 0.16$	200	0.507**	0.492**	0.792**	0.746**
		500	0.188**	0.187**	0.312**	0.302**
		4000	0.233***	0.231***	0.401***	0.399***

* : value$\times 10^{-1}$, ** : value$\times 10^{-2}$, *** : value$\times 10^{-3}$, †: value$\times 10^{-4}$

Table 4: Simulation results for the MSE of estimators when Σ is unknown

As for the MSE, from Table 4, when Σ is unknown with large p, the MSE of the modified estimators $\hat{\kappa}^\star$ and $\tilde{\kappa}^\star$ are smaller than that of each original one, that is, $\hat{\kappa}$ and $\tilde{\kappa}$. But, on the contrary, the MSE of each estimator gets larger in inverse proportion to the bias of them for fixed n with small p. It can be seen form Table 3 and Table 4 that the bias as well as the MSE for estimators is reduced when the sample size is large and the covariance matrix is unknown. As far as we can judge these results, $\tilde{\kappa}$ is better than $\hat{\kappa}$, and $\tilde{\kappa}^{\star}$ is good more than in the point of bias.

The facts mentioned above may be applied to the case when Σ is known, but details are abbreviated in the present paper. Here, we cite only one or two instances. For the mean of estimators, it was seen that the simulation results almost agree with the approximate values (2) in each elliptical population. Also we found that the MSE of estimators are not as small as those in the case of an unknown Σ. As for the asymptotic variance of $T_{\tilde{\kappa}}$, the approximated value given in (3) with known Σ is large in comparison with that in (5) when Σ is unknown.

As a result, we conclude to recommend $\tilde{\kappa}^{\star}$ as a better estimator of κ judging from bias only. Taking MSE into consideration, we can suggest $\tilde{\kappa}$ rather than $\hat{\kappa}$ on condition that the value of p is large and the sample size is also moderately large.

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References

[1] T.W. Anderson, *An Introduction to Multivariate Statistical Analysis*, Third Edition, John Wiley and Sons, New York (2003).

[2] M. Berkane, P.M. Bentler, Mardia's coefficient of kurtosis in elliptical populations, *Acta Math. Appl. Sinica*, English Ser., **6** (1990), 289-294.

[3] N. Henze, On Mardia's kurtosis test for multivariate normality, *Comm. Statist. Theory Methods*, **23** (1994), 1031-1045.

[4] T. Iwashita, Asymptotic null and nonnull distribution of Hotelling's T^2-statistic under the elliptical distribution, *J. Statist. Plann. Inference*, **61** (1997), 85-104.

[5] K.V. Mardia, Measures of multivariate skewness and kurtosis with applications, *Biometrika*, **57** (1970), 519-530.

[6] Y. Maruyama, T. Seo, Estimation of moment parameter in elliptical distributions, *J. Japan Statist. Soc.*, **33** (2003), 215-229.

[7] M.S. Srivastava, A measure of skewness and kurtosis and a graphical method for assessing multivariate normality, *Statist. Probab. Lett.*, **2** (1984), 263-267.

[8] T. Seo, T. Toyama, On the estimation of kurtosis parameter in elliptical distributions, *J. Japan Statist. Soc.*, **26** (1996), 59-68.

[9] H. Wakaki, Asymptotic expansion of the joint distribution of sample mean vector and sample covariance matrix from an elliptical population, *Hiroshima Math. J.*, **27** (1997), 295-305.