International Journal of Pure and Applied Mathematics

Volume 25 No. 3 2005, 307-310

QUIVERS, DECORATED VECTOR BUNDLES AND ÉTALE COVERINGS OF SMOOTH CURVES

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Abstract: Here we consider quiver-type stability for vector bundles on a smooth projective curve. Let $f : X \to Y$ be an unramified Galois covering between smooth and connected projective curve. We fix a quiver-type numerical datum $\tilde{\Gamma}$ on X and a geometric $\tilde{\Gamma}$-object $\underline{\underline{\tilde{E}}}$ on X. We define the quiver-type numerical datum $f_*(\tilde{\Gamma})$ on Y and the geometric $f_*(\tilde{\Gamma})$-object $f_*(\underline{\underline{\tilde{E}}})$ on Y. Then:

(i) $f_*(\underline{\underline{\tilde{E}}})$ is $f_*(\tilde{\Gamma})$-semistable if and only if $\underline{\underline{\tilde{E}}}$ is $\tilde{\Gamma}$-semistable;

(ii) $f_*(\underline{\underline{\tilde{E}}})$ is $f_*(\tilde{\Gamma})$-stable if and only if $\underline{\underline{\tilde{E}}}$ is $\tilde{\Gamma}$-stableand no two of the $\tilde{\Gamma}$-objects $\alpha^*(\underline{\underline{\tilde{E}}})$, $\beta^*(\underline{\underline{\tilde{E}}})$, $\alpha, \beta \in G$ and $\alpha \neq \beta$ are isomorphic.

AMS Subject Classification: 14H60
Key Words: holomorphic triples on curves, decorated vector bundle, vector bundles on curves, stable vector bundles, quiver

1. Quivers and Vector Bundles on Curves

Let X be a smooth and connected projective curve. Consider "decorated" vector bundles on X in the following sense. Let $\Gamma = (S, A, \underline{\underline{i}}, \underline{\underline{f}})$ a finite oriented graph, i.e. let S, A be finite sets and $\underline{\underline{i}} : A \to S$ and $\underline{\underline{f}} : A \to S$ to maps; S is the set of all vertices of Γ, A is the set of all vertices of Γ, while for any $u \in A$

Received: August 23, 2005

the vertex $\underline{\underline{i}}(u)$ (resp. $\underline{\underline{f}}(u)$) is the domain (resp. target) of the arrow u. Notice that Γ may have loops and that two vertices of Γ may be connected by more than one arrow. Let $\mathbb{N}$ denote the non-negative integers. Set $\mathbb{N}_+ := \mathbb{N}\backslash\{0\}$. Let $\mathbb{R}_{\geq 0}$ (resp. $\mathbb{R}_{>0}$) denote the set of all non-negative (resp. strincly positive) real numbers. Fix maps $\rho : A \to \mathbb{N}_+$, $\delta : A \to \mathbb{Z}$, $\gamma : S \to \mathbb{R}_{\geq 0}$ and $\gamma' : S \to \mathbb{R}_{\geq 0}$. We will say that $(\Gamma, \rho, \delta, \gamma, \gamma')$ are our data for our decorated vector bundles on X: ρ assigns the ranks, δ assigns the degrees, while γ and γ' are additional sets of weights; a common choice for them would be $\gamma'(s) = 1$ for all $s \in S$, while γ' is equivalent to the parameter α used in the theory of stable triples and in the theory of coherent pairs. A decorated vector bundle on X of type $(\Gamma, \rho, \delta, \gamma, \gamma')$ is a pair $(\underline{\underline{E}}, \underline{\underline{f}})$, where $\underline{\underline{E}} = \{E_s\}_{s\in S}$, each E_s is a rank $r_s := \rho(s)$ vector bundle on X with degree $d_s := \delta(s)$. When speacking of subobject we will allow that some rank is zero, i.e. that the corresponding sheaf is 0; we will always assume that at least one rank is non-zero and call their sum the total rank; of course, $\mu_s := d_s/r_s$ is the slope of E_s; the integer $\gamma_s r_s := \gamma(s)\cdot\rho(s)$ is called the weighted rank of E_s; the integer $\gamma'_s d_s := \gamma(s) \cdot \delta(s)$ is called the weighted degree of E_s. For any arrow $a \in A$ we fix a morphism $u_a : E_{\underline{\underline{i}}(a)} \to E_{\underline{\underline{f}}}$. Call $\tilde{\underline{\underline{E}}} := (\underline{\underline{E}}, \underline{\underline{u}})$ these geometrical objects. Set $\mathrm{rank}_\gamma((\underline{\underline{E}}, \underline{\underline{u}}) := \sum_{s\in S} \gamma_s r_s$ (the total weighted rank) and $\deg_{\gamma'}((\underline{\underline{E}}, \underline{\underline{u}}) := \sum_{s\in S} \gamma_s d_s$ (the total weighted degree) and $\tilde{\mu}(\tilde{\underline{\underline{E}}} = \mu_{\gamma,\gamma'}((\underline{\underline{E}}, \underline{\underline{u}}) := \mathrm{rank}_\gamma((\underline{\underline{E}}, \underline{\underline{u}})/\deg_{\gamma'}((\underline{\underline{E}}, \underline{\underline{u}})$ (the total weighted slope). We define subobjects, morphisms, and quotient objects as in the case of triples ([3], [5]) and using them the notions of stability with the following restriction: to check semistability or stability we only use subobjects with non-zero total rank. As in the case of triples, an obvious problem is that a morphism $i : E \to F$ between vector bundles on X which is injective as a map of sheaves may have non-locally free cokernel. This may be overcome because the saturation of its image has higher slope and this trick may be extended to the general (quiver+decoration) case. In particular we may carry over to this general set-up [5], Lemma 2.1, Lemma 2.2, Corollary 2.1 and Corollary 2.2. For general results on triples, decorated vector bundles, and moduli spaces related to bundles parametrized by a quiver, see [1], [2], [3], [4], [6], [7], [8], [9] and [10]. Let Y another smooth and connected projective curve and $f : X \to Y$ an unramified degree m Galois covering with G as its Galois group. Hence $2p_a(X) - 2 = m(2p_a(Y) - 2)$ (Riemann-Hurwitz). For any vector bundle E on X and any vector bundle F on Y we have $\deg(f^*(F) = m \cdot \deg(F)$, $\mathrm{rank}(f^*(F)) = \mathrm{rank}(F)$ and $\mathrm{rank}(f_*(E)) = m \cdot \mathrm{rank}(E)$. By Riemann-Roch it is easy to check that $\deg(f_*(E)) = \deg(E)$; here we use that f is unramified. For any set of data $\tilde{\Gamma} := (\Gamma, \rho, \delta, \gamma, \gamma')$ set

$f_*(\Gamma, \rho, \delta, \gamma, \gamma') := (\Gamma, m \cdot \rho, \delta, m \cdot \gamma, \gamma')$ (or just $f_*(\tilde{\Gamma})$) and call it the direct image of $\tilde{\Gamma}$. Since f_* is a functor from the category of coherent $\mathcal{O}_X$-sheaves to the category of coherent $\mathcal{O}_Y$-sheaves, we also get the direct image $f_*((\underline{\underline{E}}, \underline{\underline{u}}))$ of any geometric object on X with associated datum $\tilde{\Gamma}$; it is a geometric object on Y with $f_*(\Gamma)$ as its associated datum. G acts a group of automorphisms of X. Hence, as in the case of triple done in [5] for any $\alpha \in G$ and any geometric object $\underline{\underline{\tilde{E}}}$ with $\tilde{\Gamma}$ as its associated numerical datum, it is defined the pull-back decorated vector bundle $\alpha^*(\underline{\underline{\tilde{E}}})$ with the same numerical datum and the same total slope. Hence $\underline{\underline{\tilde{E}}}$ is $\tilde{\Gamma}$-stable (resp. $\tilde{\Gamma}$-semistable) if and only if $\alpha^*(\underline{\underline{\tilde{E}}})$ is $\tilde{\Gamma}$-stable (resp. $\tilde{\Gamma}$-semistable). With the just quoted analogous of [5], Lemma 2.1, Lemma 2.2, Corollary 2.1 and Corollary 2.2, it is straighforward to carry over the proof of [5], Theorem 3.1, and get the following result.

Theorem 1. *$f : X \to Y$ an unramified degree m Galois covering of smooth and connected projective curves with G as its Galois group. Fix any $\tilde{\Gamma}$ and any $\tilde{\Gamma}$-object $\underline{\underline{\tilde{E}}}$. Then:*

(i) $f_*(\underline{\underline{\tilde{E}}})$ is $f_*(\tilde{\Gamma})$-semistable if and only if $\underline{\underline{\tilde{E}}}$ is $\tilde{\Gamma}$-semistable;

(ii) $f_*(\underline{\underline{\tilde{E}}})$ is $f_*(\tilde{\Gamma})$-stable if and only if $\underline{\underline{\tilde{E}}}$ is $\tilde{\Gamma}$-stableand no two of the $\tilde{\Gamma}$-objects $\alpha^*(\underline{\underline{\tilde{E}}})$, $\beta^*(\underline{\underline{\tilde{E}}})$, $\alpha, \beta \in G$ and $\alpha \neq \beta$ are isomorphic.

We work over an algebraically closed field $\mathbb{K}$ such that $\mathrm{char}(\mathbb{K}) = 0$.

Acknowledgements

The author was partially supported by MIUR and GNSAGA of INdAM (Italy).

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