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EXTENSION OF HOLOMORPHIC VECTOR BUNDLES ACROSS A BOUNDARY POINT AND INDECOMPOSABILITY

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Abstract: Here we extend two previous results of mine concerning the existence of non-extendable holomorphic vector bundles across a 2-concave boundary point.

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1. Indecomposable Vector Bundles with Prescribed Rank

Here we prove two generalizations (see Theorem 1 and Theorem 2) of two results of mine (respectively [1], Theorem 1, and [1], Theorem 2) concerning the existence of holomorphic vector bundles non-extendable across a boundary point.

Theorem 1. *Let $U \subsetneq \mathbb{C}^n$, $n \geq 3$, be a domain and $P \in \partial U$ such that U is 2-concave at P. Fix an integer $r \geq n-1$. Then there exists a rank r indecomposable holomorphic vector bundle E on U which does not extend*

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across P, but it extends across every $Q \in \partial U$ with $Q \neq P$. Furthermore, $E|U \cap \Omega$ is indecomposable for every open neighborhood Ω of P in $\mathbb{C}^n$ and $f^*(E|\Omega)$ is indecomposable for every flat and finite holomorphic map $f : Y \to \Omega$.

Remark 1. Let $f : X \to Y$ be a quasi-finite and flat holomorphic map and F a reflexive coherent sheaf on Y. Then $f^*(F)$ is a reflexive coherent sheaf on X (see [2], Proposition 1.4, for the algebraic case; with only notational modifications that proof works in the analytic set-up). Now assume Y smooth. We recall that a quasi-finite map $f : X \to Y$ is flat if and only if X is locally Cohen-Macaulay, i.e. all local rings $\mathcal{O}_{X,P}$, $P \in X_{red}$, are local Cohen-macaulay rings. We recall that X is locally Cohen-Macaulay if it is smooth .

Notation 1. For any finite-dimensional Stein complex space X and any $P \in X$ let $e(X, P)$ denote the minimal integer t such that there is a holomorphic map $f : X \to \mathbb{C}^t$ which is a holomorphic embedding at P and such that $f^{-1}(f(P)) = \{P\}$. It is well-known that $\dim_P(X) \leq e(X, P) \leq \min\{\dim_P(X)+1, N\}$, where N denote the embedding dimension of X at P. Hence $\dim_P(X) \leq e(X, P) \leq \dim_P(X) + 1$ if X is smooth at P.

Theorem 2. *Let X be a Stein space, $U \nsubseteq X$ an open subset and $P \in \partial U$ such that X is smooth of dimension at least 3 at P and U is 2-concave at P. Fix an integer $r \geq e(X, P) - 1$. Then there exists a rank r indecomposable holomorphic vector bundle on U which does not extend across P, but it extends across every $Q \in \partial U$ with $Q \neq P$. Furthermore, $E|U \cap \Omega$ is indecomposable for every open neighborhood Ω of P in $\mathbb{C}^n$ and $f^*(E|\Omega)$ is indecomposable for every flat and finite holomorphic map $f : Y \to \Omega$.*

Remark 2. Let R be a Noetherian commutative local ring and N a finitely generated R-module with homological dimension at most 1. Hence there is an exact sequence

$$0 \to R^x \to R^y \to N \to 0 \tag{1}$$

for some positive integers x, y. Applying the functor $\mathrm{Hom}(- - -, N)$ to (1) we get the exact sequence

$$0 \to \mathrm{Hom}(N, N) \to N^{\oplus y} \to N^{\oplus x} \tag{2}$$

Hence if N is reflexive, then $\mathrm{Hom}(N, N)$ is reflexive ([2], Proposition 1.1).

Remark 3. Let T be a reduced complex space, $P \in T$ and $\mathcal{E}$ a reflexive sheaf on T and V an irreducible open subset of T such that $E := \mathcal{E}|V$ is locally free. The holomorphic vector bundle E on V is decomposable, say $E \cong E_1 \oplus E_2$ with neither $0 < \mathrm{rank}(E_i) < \mathrm{rank}(E)$ if and only if there is a nontrivial projector $\pi_1 \in \mathrm{Hom}(E, E)$. Now assume that $\mathcal{E}$ is reflexive and that T has depth at least

3 at each point of $T\backslash V$. By Remark 2 the coherent sheaf $\mathrm{Hom}(\mathcal{E},\mathcal{E})$ is reflexive and hence we may apply all the extension theorems (not only from $H^0(V,E)$ to $H^0(T,\mathcal{E})$, but also from $H^0(V,Hom(E,E))$ to $H^0(T,Hom(\mathcal{E},\mathcal{E}))$ quoted in [1].

Proofs of Theorem 1 and Theorem 2. Let R be an n-dimensional regular local ring ($n \geq 2$). Let $\mathbf{m}$ denote the maximal ideal of R. In the set-up of the proof of [1], Theorem 1, we started with $z_1,\dots,z_n \in \mathbf{m}$ such that their images in $\mathbf{m}/\mathbf{m}^2$ are a basis of the $R/\mathbf{m}$-vector space $\mathbf{m}/\mathbf{m}^2$. In the set-up of [1], Theorem 2, we take more elements, so that they generate $\mathbf{m}/\mathbf{m}^2$, but their are not a basis of it. In the latter case we cannot always get indecomposable non-estendable vector bundles. To prove Theorem 1 and Theorem 2 we need to fix an integer $k \gg 0$ and then starting from $r+1$ elements of $\mathbf{m}^k$ whose images in $\mathbf{m}^k/\mathbf{m}^{k+1}$ are linearly independent. Then we apply Remarks 1, 2 and 3 to complete the proofs of [1], Theorem 1 and Theorem 2. □

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