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COMPLEXES OF STABLE VECTOR BUNDLES ON CURVES

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Abstract: Let X be a smooth and connected projective curve of genus $g \geq 2$. Fix integers $k \geq 3$, $n_i > 0$ and d_i, $1 \leq i \leq k$, and $r_i > 0$ and a_i, $1 \leq i \leq k-1$, such that $r_i + r_{i+1} = n_{i+1}$ for all $1 \leq i \leq k-2$, $n_1 > r_1$, $r_{k-1} < n_k$, $a_i + a_{i+1} = d_{i+1}$ for $1 \leq i \leq k-1$, $a_i/r_i < a_{i+1}/r_{i+1}$ for $1 \leq i \leq k-2$, $d_1/n_1 < a_1/r_1$. Then there is an exact sequence on X such that all vector bundles E_i, $1 \leq i \leq k$, and $F_j := \mathrm{Im}(f_j) = \mathrm{Ker}(f_{j+1})$, $1 \leq j \leq k-1$ are stable, $\mathrm{rank}(E_i) = n_i$, $\deg(E_i) = d_i$, $\mathrm{rank}(F_j) = r_j$ and $\deg(F_j) = a_j$ for all i, j. Furthermore, we may assume that each F_j, $1 \leq j \leq k-1$, is general in its moduli space of all stable vector bundles on X with that degree and that rank.

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1. Exact Sequences of Vector Bundles

Many interesting homogeneous projective varieties Y (e.g. Grassmannians or flag varieties) are equipped with a universal bundle or a universal "finite family of vector bundles and suitable maps between them", say

$$E_1 \xrightarrow{f_1} E_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{k-1}} E_k \tag{1}$$

such that any morphism $X \to Y$ is equivalent to a corresponding "finite fam-

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ily of vector bundles and suitable maps between them". Here we will study a related, but different, case in which Y is not complete. More precisely, it has a natural completion $\bar{Y}$ on which a connected reductive algebraic group acts and Y is an open G-orbit of $\bar{Y}$. Y will be the so-called "variety of complexes" ([2], [1]). More precisely, to get an irreducible Y we need to prescribe not only the rank of the vector bundles, but also the rank of certain maps. We work over an algebraically closed field $\mathbb{K}$ with $\mathrm{char}(\mathbb{K}) = 0$. Fix integers $k \geq 3$, $n_i > 0$, $1 \leq i \leq k$. Let $V(k; n_1, \dots, n_k)$ denote the set of all $(L_1, \dots, L_{k-1})$, where $L_i : \mathbb{K}^{\oplus n_i} \to \mathbb{K}^{\oplus n_{i+1}}$, $1 \leq i \leq k-1$, are linear maps such that $L_{i+1} \circ L_i = 0$ for all $1 \leq i \leq k-2$. To get irreducible components of the projective set $V(k; n_1, \dots, n_k)$ we need to fix the ranks of all linear maps L_i. To get the open orbit Y of $\bar{Y}$ we need to impose that the complex is an exact sequence of linear maps, i.e. to impose $\mathrm{Ker}(L_{i+1}) = \mathrm{Im}(L_i)$ for all $1 \leq i \leq k-1$. Fix non-negative integers r_i, $1 \leq i \leq k-1$, and set $W(k; n_1, \dots, n_k; r_1, \dots, r_{k-1}) := \{(L_1, \dots, L_{k-1}) \in V(k; n_1, \dots, n_k) : \mathrm{rank}(L_i) = r_i$ for all $1 \leq i \leq k-1$ and $\mathrm{Ker}(L_{i+1}) = \mathrm{Im}(L_i)$ for all $1 \leq i \leq k-2\}$. Of course, $W(k; n_1, \dots, n_k; r_1, \dots, r_{k-1}) \neq \emptyset$ if and only if $r_i + r_{i+1} = n_{i+1}$ for all $1 \leq i \leq k-2$, $n_1 \geq r_1$ and $r_{k-1} \leq n_k$. In our s1tuation we have an exact sequence (1) of vector bundles on X such that $\mathrm{rank}(E_i) = n_i$ for all $1 \leq i \leq k-1$, $\mathrm{Im}(f_i) = \mathrm{Ker}(f_{i+1})$ for all $1 \leq i \leq k-1$ and all locally free. Here we only consider smooth and projective curves, but we also want that all vector bundles arising from the complex (not only the k vector bundles E_i, $1 \leq i \leq k$, but also the $k-1$ vector bundles $F_i := \mathrm{Im}(f_i) = \mathrm{Ker}(f_{i+1})$, $1 \leq i \leq k-1$) are stable. This gives a very strong restriction on the degree of all vector bundles, because if A and B are stable and $h^0(X, \mathrm{Hom}(A, B)) \neq 0$, then $\mu(A) := \deg(A)/\mathrm{rank}(A) < \mu(B) := \deg(B)/\mathrm{rank}(B)$. To have many stable vector bundles on X we also need to assume that X has genus $g \geq 2$. Here we will prove the following theorem.

Theorem 1. *Let X be a smooth and connected projective curve of genus $g \geq 2$. Fix integers $k \geq 3$, $n_i > 0$ and d_i, $1 \leq i \leq k$, and $r_i > 0$ and a_i, $1 \leq i \leq k-1$, such that $r_i + r_{i+1} = n_{i+1}$ for all $1 \leq i \leq k-2$, $n_1 > r_1$, $r_{k-1} < n_k$, $a_i + a_{i+1} = d_{i+1}$ for $1 \leq i \leq k-1$, $a_i/r_i < a_{i+1}/r_{i+1}$ for $1 \leq i \leq k-2$, $d_1/n_1 < a_1/r_1$. Then there is an exact sequence (1) on X such that all vector bundles E_i, $1 \leq i \leq k$, and $F_j := \mathrm{Im}(f_j) = \mathrm{Ker}(f_{j+1})$, $1 \leq j \leq k-1$ are stable, $\mathrm{rank}(E_i) = n_i$, $\deg(E_i) = d_i$, $\mathrm{rank}(F_j) = r_j$ and $\deg(F_j) = a_j$ for all i, j. Furthermore, we may assume that each F_j, $1 \leq j \leq k-1$, is general in its moduli space of all stable vector bundles on X with that degree and that rank.*

Proof. For all integers $r > 0$ and a let $M(X; r, a)$ denote the moduli scheme of all stable vector bundles on X with rank r and degree a. The scheme $M(X;r,a)$ is a non-empty integral variety. By [3], Theorem 0.1, there are $k-1$ exact sequences

$$0 \to F_j \to E_{j+1} \to F_{j+1} \to 0\,, \tag{2}$$

$1 \le j \le k-1$, such that each F_j, E_{j+1} and F_{j+1} is stable and such that F_j is general in $M(X;r_j,a_j)$. Here we are abusing notation because we called with the same name two bundles arising in different exact sequences; we may do that by the irreducibility of each $M(X;r,a)$. We get the stable bundles E_i for all $2 \le i \le k-2$. The map f_{j+1} is the composition of the exact sequence (2) starting with the vector bundle F_j with the injective map of the exact sequence (2) starting with the vector F_{j+1}. To get E_1 we use an exact sequence

$$0 \to A \to E_1 \to F_1 \to 0\,, \tag{3}$$

where A is a general element of $M(X;n_1-r_1,d_1-a_1)$ (again with the same abuse of notation). To get E_k we use an exact sequence

$$0 \to F_{k-1} \to E_k \to B \to 0\,, \tag{4}$$

where B is a general element of $M(X;n_k-r_{k-1},d_k-a_{k-1})$ (again with the same abuse of notation). □

As the reader will have realized, this theorem does not fit exactly in the set-up introduced at the beginning of this note, because the variety $W(k;n_1,\dots,n_k;r_1,\dots,r_{k-1})$ is affine and there is no non-constant morphism $X \to W(k;n_1,\dots,n_k;r_1,\dots,r_{k-1})$. Furthermore, the "universal" vector bundles on $W(k;n_1,\dots,n_k;r_1,\dots,r_{k-1})$ are trivial. To get something projective or "almost projective" one should add a rigidification (or a framing) of at least one of the vector bundles, as in the case of the Grassmannians and the flag varieties. However, the problem studied in this note seems to be of an independent interest: it is a piece of linear algebra connected with vector bundles which is richer than the classical case of vector bundles related to quivers.

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