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FAMILIES OF k-GONAL CURVES WITH CERTAIN SCROLLAR INVARIANTS, II

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Abstract: Fix integers $g \geq 4$, $k \geq 4$ and $m \geq 2$. Let $\mathcal{S}(g,k,m)$ denote the set of all pairs (X,R), where X is a smooth and connected curve of genus g, $R \in \mathrm{Pic}^k(X)$, R is spanned, $h^0(X,R^{\otimes(m-1)}) = m$, $h^0(X,R^{\otimes m}) \geq m+2$ and the morphism ψ induced by $H^0(X,R^{\otimes m})$ is birational. Set $\mathcal{S}(g,k,m,=) := \{(X,R) \in \mathcal{S}(g,k,m) : h^0(X,R^{\otimes m}) = m+2\}$. Here we give (for many g,k,m) of several scrollar invariants of a general $(X,R) \in \mathcal{S}(g,k,m,=)$.

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1. Scrollar Invariants

This note is a continuation of [2]. Let X be a smooth projective curve of genus $g \geq 2$ and $R \in \mathrm{Pic}^k(X)$ such that R is spanned by its global sections and $h^0(X,R) = 2$. Thus R induces a morphism $h_R : X \to \mathbf{P}^1$. From R one may produce an ordered sequence of $k-1$ integers $e_1 \geq \cdots \geq e_{k-1}$ such that $e_1 + \cdots + e_{k-1} = g-k+1$ (see [8]). The ordered sequence of $k-1$ integers $e_1,\dots,e_{k-1}$ is called the scrollar invariants of R or of f.

Fix X, R and $e_1 \geq \cdots \geq e_{k-1}$ as above. Let m be the first integer x such that $h^0(X,R^{\otimes x}) > x+1$ (i.e. set $m := e_{k-1}+2$) and $y := h^0(X,R^{\otimes m}) - 1$. Thus the complete linear system $|R^{\otimes m}|$ induces a morphism $\psi : X \to \mathbf{P}^y$. By [4], Theorem 1, if ψ is birational, then there are some strong restrictions on the scrollar invariants $e_1,\dots,e_{k-1}$ of R. Since it is very easy to construct pencils with non-birational ψ and with certain other properties (see e.g. the example at the end of [1]) we will always assume that ψ is birational. Set $Y := \psi(X)$.

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Since $h^0(X, R^{\otimes(m-1)}) = m$, it is easy to check that Y is containd in a cone $T = T_{y,m}$ with as base a smooth rational curve of $\mathbf{P}^m$ and with as vertex a $(y-m-1)$-dimensional linear subspace V of $\mathbf{P}^y$. Furthermore, $V \cap Y = \emptyset$ and the linear projection from V induces a degree k morphism $u : Y \to D$, where D is a rational normal curve of $\mathbf{P}^m$ such that the family of all degree k zero-dimensional schemes $\{u \circ \psi)^{-1}(P)\}_{P \in D}$ is the set of all divisors in $|R|$. This description of $\psi(X)$ was the key for all the results of [2].

Let $\mathcal{S}(g,k,m)$ denote the set of all pairs (X,R) where X is a smooth and connected curve of genus g, $R \in \mathrm{Pic}^k(X)$, R is spanned, $h^0(X, R^{\otimes(m-1)}) = m$, $h^0(X, R^{\otimes m}) \geq m+2$ and the morphism ψ induced by $H^0(X, R^{\otimes m})$ is birational. Set $\mathcal{S}(g,k,m,=) := \{(X,R) \in \mathcal{S}(g,k,m) : h^0(X,R^{\otimes m}) = m+2\}$. The non-emtyness of $\mathcal{S}(g,k,m,=)$ for suitable g,k and m (and much more) was proved for instance in [4]. We recall the following result (see [2], Theorem 1).

Theorem 1. *Fix integers $k \geq 4$, $m \geq 2$ and x such that $0 \leq x \leq m(k-1)(k-2)/2$. Let W be the subsets of all integral curves $B \subset F_m$ such that $B \in |kh + kmF|$ and B has exactly x nodes as only singularities. Then $W \neq \emptyset$ and W is an irreducible variety of dimension $km(k+1)/2 - 1 - x$. There is a non-empty open subset U of W such that for every $Y \in U$, calling $u : X \to Y$ the normalization map and setting $R := u^*(\mathcal{O}_Y(F))$, we have $p_a(X) = 1 + k(km-m-2)/2 - x$, R is spanned, $h^0(X,R) = 2$, $h^0(X, R^{\otimes(m-1)}) = m$, $h^0(X, R^{\otimes m}) = m+2$ and u is the morphism, ψ, associated to $H^0(X, R^{\otimes m})$; in particular ψ is birational.*

Here we will prove the following result, which is a strong improvement of [3], Theorem 1.1.

Theorem 2. *Fix integers $k \geq 4$, $m \geq 2$, x and $z \geq m$. Set $u := \lfloor z/m \rfloor$. Assume $0 \leq x \min\{\lfloor (k + mk(k-1)/2)/3 \rfloor, k-2+u+m(k-1-u)(k-2-u)/2\}$. There is $Y \in |kh + kmF|$ with exactly x nodes as only singularities which are general in F_m and such that, calling $u : X \to Y$ the normalization map and setting $R := u^*(\mathcal{O}_Y(F))$, the following properties holds:*

- *(i)* *$p_a(X) = 1 + k(km-m-2)/2 - x$, R is spanned, $h^0(X,R) = 2$, $h^0(X, R^{\otimes(m-1)}) = m$, $h^0(X, R^{\otimes m}) = m+2$, and the morphism ψ associated to $|R^{\otimes m}|$ is birational onto its image;*
- *(ii)* $h^0(X, R^{\otimes z}) = \sum_{i=0}^{u} \max\{0, z - im + 1\}$

We work over an algebraically closed field $\mathbf{K}$ with $\mathrm{char}(\mathbf{K}) = 0$.

Remark 1. Fix an integer $m \geq 3$. Let T be the cone with vertex Q and with as basis a rational normal curve $D \subset \mathbf{P}^m$. Let $\pi : F_m \to T$ be the blowing-up of Q; we use this notation because this blowing-up is isomorphic to the Hirzebruch surface with section of minimal self-intersection $-m$. Thus F_m is a smooth surface, $h := \pi^{-1}(Q) \cong \mathbf{P}^1$ and $h^2 = -m$. There is a ruling $u : F_m \to \mathbf{P}^1$. We have $\mathrm{Pic}(F_m) \cong \mathbf{Z}^{\oplus 2}$ and we take h and a fiber, F, of the ruling u as a basis of $\mathrm{Pic}(F_m)$. We have $h^0(F_m, \mathcal{O}_{F_m}(th + xF)) = 0$ for every $t < 0$, $h^0(F_m, \mathcal{O}_{F_m}(th + xF)) = \sum_{i=0}^{t} \max\{0, x - im + 1\}$ for every $t \geq 0$ and $h^1(F_m, \mathcal{O}_{F_m}(th + xF)) = \max\{0, -x - 1 - im\}$ for every $t \geq 0$. The composition $F_m \to \mathbf{P}^{m+1}$ of π with the inclusion $T \subset \mathbf{P}^{m+1}$ is given by the complete linear system $|h + mF|$. We have $\omega_{F_m} \cong \mathcal{O}_{F_m}(-2h - (m+2)F)$. Set $Y' := \pi^{-1}(Y)$. Since $Y \cap \{Q\} = \emptyset$, we have $Y' \cong Y$ and $Y' \cap h = \emptyset$. Thus $Y' \in |kh + kmF|$. We have $\omega_{F_m} \cong \mathcal{O}_{F_m}(-2h - (m+2)F)$. By the adjunction formula we have $\omega_{Y'} \cong \mathcal{O}_{Y'}((k-2)h + (km - m - 2)F)$. Hence $p_a(Y) = p_a(Y') = 1 + k(km - m - 2)/2$. Since ψ is birational, we obtain $g \leq 1 + k(km - m - 2)/2$ with equality if and only if Y is smooth, i.e. if and only if ψ is an embedding.

Remark 2. We use the notations of Remark 1. Fix an integer $x > 0$ and a general $Z \subset F_m \backslash h$ such that $\sharp(Z) = x$. Hence $\sharp(\pi(Z)) = x$ and $\pi(Z)$ is general in T. We assume the existence of an integral $E \in |kh + mkF|$ such that $\mathrm{Sing}(E) = Z$ and E has only ordinary nodes as singularities. Since E is integral and $E \in |kh + mkF|$, we have $E \cap h = \emptyset$ and hence $\pi(E) \cong E$. Let $\nu : X \to E$ be the normalization map. Since $T \backslash \{Q\}$ is a smooth surface, $u(E) \cong E$ and E has only ordinary nodes as singularities, the conductor of the normalization map $u \circ \nu : X \to u(E)$ (resp. $\nu : X \to E$) is the set $u(Z)$ (resp. Z) with its reduced structure. Since $E \in |kh + mkF|$ and $\omega_{F_m} \cong \mathcal{O}_{F_m}(-2h - (m+2)F)$, we have $\omega_E \cong \mathcal{O}_E((k-2)h + (km - m - 2)F)$. Since F_m is a smooth rational surface, we have $h^0(F_m, \omega_{F_m}) = h^1(F_m, \omega_{F_m}) = 0$. Hence $h^0(X, \omega_X) = h^0(F_m, \mathcal{I}_Z((k-2)h + (km - m - 2)F))$. Since Z is general in F_m, we have $\dim(|\mathcal{I}_Z \otimes A|) = \max\{-1, \dim(A) - x\}$ for every complete linear system $|A|$ on F_m. Hence the value of $h^1(X, R^{\otimes z})$ is computed by a suitable linear system on F_m and hence by Riemann-Roch we get $h^0(X, R^{\otimes z}) = h^0(E, \mathcal{O}_E(tF))$ if z is as in the statement of Theorem 2.

Proof of Theorem 2. Fix a general $Z \subset F_m$ with $\sharp(Z) = x$. By Remark 2 it is sufficient to show the existence of an integral curve $E \in |kh + kmF|$ such that $\mathrm{Sing}(E) = Z$ and E has only ordinary nodes as singularities. For any $P \in F_m$ let $2P$ denote the first infinitesimal neighborhood of P in F_m, i.e. the closed zero-dimensional subscheme of F_m with $\mathcal{I}_Z{}^2$ as its ideal sheaf. Set $W := \bigcup_{P \in Z} 2P$. Thus W is a closed zero-dimensional subscheme of F_m such that $W_{red} = Z$ and $\mathrm{length}(W) = 3x$. Since $k - 1 \geq 3$ and $3x \leq k + mk(k-1)/2 = h^0(F_m, \mathcal{O}_{F_m}((k-

$1)h+(k-1)mF))$, we have $h^1(F_m, \mathcal{I}_W((k-1)h+(k-1)mF))=0$ ([5]). Since $h+mF$ is ample and spanned, the existence of the nodal curve E easily follows from Castelnuovo-Mumford's Lemma (see [6], p. 100, or [7]). □

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