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LIE-POISSON GROUPS AND THE SOLUTIONS OF THE CLASSICAL YANG-BAXTER EQUATION

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Abstract: The solutions of the classical Yang-Baxter equation over an arbitrary Lie group G can be used to assign a Lie-Poisson's structure to G. In this paper, the construction of such a structure is explained in the general case and illustrated in the particular case of the Lie group of square matrices. All solutions of the classical Yang-Baxter equation are determined for Lie groups associated with Lie algebras of dimension ≤ 3. For complex Lie algebras of higher dimensions, the existence of solutions to the classical Yang-Baxter equation is proven.

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1. Introduction

A Lie-Poisson group is a Lie group with a Poisson's structure, i.e. such that the transformation: $\pi : G \times G \to G, (g, h) \mapsto g \cdot h$, is a Poisson's morphism when $G \times G$ is provided with its product Poisson's structure (see [2]).

The goal of this article is to show how to construct a Lie-Poisson's structure over a Lie group from solutions of the classical Yang-Baxter equation (YBC) and to examine in what cases such solutions exist.

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2. Construction of Lie-Poisson Groups from Solutions of Classical Yang-Baxter Equations

Let $\mathcal{G}$ be a Lie algebra (real or complex) and $\mathcal{G}^*$ its dual. If the dimension of $\mathcal{G}$ is finite, a linear transformation $r : \mathcal{G}^* \to \mathcal{G}$ may be viewed as an element of $\mathcal{G} \otimes \mathcal{G}$ and simply denoted by r. It may also be viewed as a linear form over $\mathcal{G}^* \bigotimes \mathcal{G}$.

Let $r : \mathcal{G}^* \to \mathcal{G}$ be a skew-symmetric linear transformation, i.e., satisfying the identity $\langle \xi, r(\eta) \rangle = -\langle \eta, r(\xi) \rangle$ $\forall \xi, \eta \in \mathcal{G}^*$.

Let us consider the equation:

$$\text{(YBC)} \qquad [r, r] = 0\,, \tag{1}$$

where, $[r, r]$ denotes the trilinear form over $\mathcal{G}^*$ defined by:

$$\begin{aligned} [r,r] \quad : \quad & \mathcal{G}^* \times \mathcal{G}^* \times \mathcal{G}^* \to \mathbb{K} \quad (\mathbb{K} = \mathbb{R} \text{ or } \mathbb{C})\,, \\ & (\xi_1, \xi_2, \xi_3) \mapsto \oint \langle \xi_3, [r(\xi_1), r(\xi_2)] \rangle\,. \end{aligned}$$

The notation $[r, r]$ stands for an algebraic version of the *Shouten's* brackets, while $\oint$ denotes the sum over all cyclic permutations of indices 1, 2, 3.

The equation (1) defines what is called the classical Yang-Baxter equation, (YBC) in short.

If r is bijective, the condition (YBC) must be satisfied in order to make the reciprocal linear transformation $r^{-1} : \mathcal{G} \to \mathcal{G}^*$ a symplectic cocycle for the coadjoint representation.

Let G be a Lie group of the Lie algebra $\mathcal{G}$, and let r be a skew-symmetric linear transformation satisfying the (YBC) condition. From r we can construct a left-invariant Poisson's structure whose Poisson's bivector Λ_r^λ is defined by:

$$\Lambda_r^\lambda(g) = \left((T\lambda_g)_e \otimes (T\lambda_g)_e \right)(r) \quad \forall g \in G,$$

where, λ_g denotes the left translation of an element g of G.

The left-invariant Poisson's brackets $\{,\}_r^\lambda$ associated with r is defined by:

$$\begin{aligned} \{f_1, f_2\}_r^\lambda(g) &= \Lambda_r^\lambda(g) \Big((df_1)_g, (df_2)_g \Big) \\ &= r \Big((df_1)_g \circ (T\lambda_g)_e, (df_2)_g \circ (T\lambda_g)_e \Big) = r \left(d(f_1 \circ \lambda_g)_e, d(f_2 \circ \lambda_g)_e \right), \end{aligned}$$

for all $f_1, f_2 \in C^\infty(G)$, and all $g \in G$.

Note that because $d(f_1 \circ \lambda_g)_e$ and $d(f_1 \circ \lambda_g)_e$ belong to $\mathcal{G}^*$, $\{,\}_r^\lambda$ is well-defined. Also, because r is skew-symmetric, $\{,\}_r^\lambda$ is skew-symmetric. Finally, there is an equivalence between $\{,\}_r^\lambda$ satisfying the *Jacobi's* identity and r being a solution of the (YBC) equation (1). Therefore, $\{,\}_r^\lambda$ defines a Poisson's brackets over G.

Similarly, from r we can construct a right-invariant Poisson's structure whose Poisson's bivector Λ_r^ρ is given by:

$$\Lambda_r^\rho(g) = \left((T\rho_g)_e \otimes (T\rho_g)_e\right)(r) \qquad \forall g \in G\,,$$

where, ρ_g stands for the right translation of g.

Here, the right-invariant Poisson's brackets $\{,\}_r^\rho$ is given by:

$$\{f_1, f_2\}_r^\rho(g) = r\left(d(f_1 \circ \rho_g)_e, d(f_2 \circ \rho_g)_e\right),$$

for all $f_1, f_2 \in C^\infty(G)$, and all $g \in G$.

The left and right invariant Poisson's brackets $\{,\}_r^\lambda$ and $\{,\}_r^\rho$ are compatible in the sense that their sum (or any linear combination) is still a Poisson's brackets.

If r is a solution of the (YBC) equation, let us consider the particular bivector:

$$\Lambda_r = \Lambda_r^\lambda - \Lambda_r^\rho.$$

The bivector Λ_r is a Poisson's bivector, i.e. the brackets $\{,\}_r$ given by

$$\{f_1, f_2\}_r = \{f_1, f_2\}_r^\lambda - \{f_1, f_2\}_r^\rho\,,$$

defines a Poisson's structure over G, thus providing the group G with a Lie-Poisson's structure.

3. On the Existence of Solutions to the Classical Yang-Baxter Equation (YBC)

Let $\mathcal{G}$ be a Lie algebra over a base field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

Recall that a solution of the (YBC) equation is a skew-symmetric linear transformation $r : \mathcal{G}^* \to \mathcal{G}$ that satisfies the (YBC) equation $[r,r] = 0$, where $[r,r]$ stands for the trilinear form over $\mathcal{G}^*$ and is defined by:

$$\begin{aligned} [r,r] \quad : \quad & \mathcal{G}^* \times \mathcal{G}^* \times \mathcal{G}^* \to \mathbb{K} \qquad (\mathbb{K} = \mathbb{R} \text{ or } \mathbb{C}) \\ & (\xi_1, \xi_2, \xi_3) \mapsto \oint \langle \xi_3, [r(\xi_1), r(\xi_2)] \rangle\,. \end{aligned}$$

We have seen that such solutions provide the Lie group G of the Lie algebra $\mathcal{G}$ with a Lie-Poisson's structure.

The objective of this section is to study cases where non-trivial solutions of the (YBC) equation exist. For the study of the (YBC) equation on Lie algebras of small dimensions, we will need the classification of the Lie algebras of dimension 3 (cf. for example [1]).

If $\dim \mathcal{G} = 1$, any linear skew-symmetric transformation must be zero, therefore, the only solution of the (YBC) is the trivial solution $r = 0$.

If $\dim \mathcal{G} = 2$, by noting that the (YBC) condition is satisfied when two arguments are equal; then any linear skew-symmetric transformation r is a solution of the (YBC) equation.

Let $\dim \mathcal{G} = 3$. We will denote by $\mathcal{DG}$ the derivative ideal of $\mathcal{G}$, and by $\mathcal{Z}(\mathcal{G})$ the center of $\mathcal{G}$. Five cases are to be considered as follows:

First Case. $\mathcal{DG} = 0$.

In this case $\mathcal{G}$ is commutative and the (YBC) equation is satisfied by every skew-symmetric linear transformation.

Second Case. $\mathcal{DG} = 1$ and $\mathcal{DG} \subseteq \mathcal{Z}(\mathcal{G})$.

We know in this case that there exists a base (e_1, e_2, e_3) of $\mathcal{G}$ such that:

$$[e_2, e_3] = e_1, \qquad [e_1, e_2] = [e_1, e_3] = 0.$$

Let (e_1^*, e_2^*, e_3^*) denote the dual base of (e_1, e_2, e_3), and let us find in these bases a solution to the (YBC) equation expressed in matrix form as follows:

$$r = \begin{pmatrix} 0 & \alpha & \beta \\ -\alpha & 0 & \gamma \\ -\beta & -\gamma & 0 \end{pmatrix}, \quad \text{where } \alpha, \beta, \gamma \in \mathbb{K}.$$

The (YBC) equation writes then:

$$\langle e_1^*, [re_2^*, re_3^*]\rangle + \langle e_2^*, [re_3^*, re_1^*]\rangle + \langle e_3^*, [re_1^*, re_2^*]\rangle = 0$$

which gives $\langle e_1^*, \gamma^2 e_1\rangle = 0$, i.e. $\gamma = 0$.

The following family of solutions are obtained in this case:

$$\left\{ r = \begin{pmatrix} 0 & \alpha & \beta \\ -\alpha & 0 & 0 \\ -\beta & 0 & 0 \end{pmatrix}, \quad \text{where } \alpha, \beta \in \mathbb{K} \right\}.$$

Third Case. $\dim \mathcal{DG} = 1$ and $\mathcal{DG} \not\subset \mathcal{Z}(\mathcal{G})$.

We know in this case that there exists a base (e_1, e_2, e_3) of $\mathcal{G}$ such that:

$$[e_1, e_2] = e_1, \quad [e_1, e_3] = [e_2, e_3] = 0.$$

As a result we get: $\alpha = 0$ or $\gamma = 0$, and consequently, the two families of solutions:

$$\begin{pmatrix} 0 & 0 & \beta \\ 0 & 0 & \gamma \\ -\beta & -\gamma & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & \alpha & \beta \\ -\alpha & 0 & 0 \\ -\beta & 0 & 0 \end{pmatrix}, \quad \text{where } \alpha, \beta, \gamma \in \mathbb{K}\,.$$

Fourth Case. $\dim \mathcal{DG} = 2$.

We know in this case that there exists a base (e_1, e_2, e_3) of $\mathcal{G}$ such that:

$$[e_1, e_2] = 0, \quad [e_1, e_3] = ae_1 + be_2, \quad [e2, e3] = ce_1 + de_2\,. \tag{1}$$

The (YBC) equation reduces to: $a\beta\gamma + c\gamma^2 - b\beta^2 - d\beta\gamma = 0$. If $\mathbb{K}$ is algebraically close, the equation (1) can be split into either

$$[e_1, e_2] = 0, \quad [e_1, e_3] = e_1, \quad [e_2, e_3] = ae_1 \;\; (a \neq 0)\,,$$

or

$$[e_1, e_2] = 0, \quad [e_1, e_3] = e_1 + be_2 \;\; (b \neq 0), \quad [e_2, e_3] = e_2.$$

The (YBC) equation writes then: $(1 - a)\beta\gamma = 0$ or $\beta = 0$.

Fifth Case. $\dim \mathcal{DG} = 3$, i.e. $\mathcal{DG} = \mathcal{G}$ (in particular $\mathcal{G}$ is semi-simple).

Let us assume that $\mathcal{G}$ is splitting, i.e. there exists a $z_1 \in \mathcal{G}$ such that $ad(z_1)$ (the adjoint of z_1) has all its eigenvalues in $\mathbb{K}$, we know then that there exists a base (z_1, z_2, z_3) of $\mathcal{G}$ such that:

$$[z_1, z_2] = \lambda z_2, \quad [z_1, z_3] = \lambda z_3, \quad [z_2, z_3] = \mu z_1\,.$$

The (YBC) equations writes then: $\mu\gamma^2 - 2\lambda\alpha\beta = 0$. If $\lambda = 2$ and $\mu = 1$, we get the particular case of $sl(2, \mathbb{K})$ in which the (YBC) equation simplifies to: $\gamma^2 - 4\alpha\beta = 0$.

Remark 1. Let $\mathcal{G} = so(3)$ denote the Lie algebra of the skew-symmetric real matrices of order 3. $\mathcal{G}$ is not splitting and there exists a base (e_1, e_2, e_3) such that:

$$[e_1, e_2] = e_3, \quad [e_1, e_3] = e_2, \quad [e_2, e_3] = e_1.$$

In this case the (YBC) equation writes: $\alpha^2 + \beta^2 + \gamma^2 = 0$, i.e. $\alpha = \beta = \gamma = 0$ which shows that only the zero solution $r = 0$ fits.

Subsequently, we will assume that $\dim \mathcal{G} \geq 2$.

Lemma of Extension. *Let $\mathcal{G}$ be a Lie algebra and* a *a Lie sub-algebra of $\mathcal{G}$. If the transformation $r\,(\mathrm{a}^* \to \mathrm{a})$ is a solution of the (YBC) equation over* a*, then the transformation: $\tilde{r}\left(\mathcal{G}^* \xrightarrow{{}^t j} \mathrm{a}^* \xrightarrow{r} \mathrm{a} \xrightarrow{j} \mathcal{G}\right)$ defined by: $\tilde{r} = j \circ r \circ {}^t j$,* *where $j : \mathrm{a} \hookrightarrow \mathcal{G}$ denotes the natural injection from* a *into $\mathcal{G}$ and ${}^t j : \mathcal{G}^* \to \mathrm{a}^*$ its transpose, is a solution of the (YBC) equation over $\mathcal{G}$.*

Proof. By definition of transformation $\tilde{r} : \mathcal{G}^* \to \mathcal{G}$, we have:

$$\tilde{r}(\nu) = \left(j \circ r \circ {}^t j\right)(\nu) = (j \circ r)(\nu \circ j) = j\left(r\left(\nu \circ j\right)\right) \quad (\forall \nu \in \mathcal{G}).$$

Because r is linear and skew-symmetric so is $\tilde{r}$.

Also, if α, β, and γ are elements of $\mathcal{G}$, then,

$$\begin{aligned}\oint \langle \alpha, [\tilde{r}(\beta), \tilde{r}(\gamma)]\rangle &= \alpha\left([\tilde{r}(\beta), \tilde{r}(\gamma)]\right) + \beta\left([\tilde{r}(\gamma), \tilde{r}(\alpha)]\right) + \gamma\left([\tilde{r}(\alpha), \tilde{r}(\beta)]\right) \\ &= \alpha\left([j(r(\beta \circ j)), j(r(\gamma \circ j))]\right) + \beta\left([j(r(\gamma \circ j)), j(r(\alpha \circ j))]\right) \\ &\quad + \gamma\left([j(r(\alpha \circ j)), j(r(\beta \circ j))]\right) = (\alpha \circ j)\left([r(\beta \circ j), r(\gamma \circ j)]\right) \\ &\qquad + (\beta \circ j)\left([r(\gamma \circ j), r(\alpha \circ j)]\right) + (\gamma \circ j)\left([r(\alpha \circ j), r(\beta \circ j)]\right) = 0\end{aligned}$$

because $\alpha \circ j$, $\beta \circ j$ and $\gamma \circ j$ are in a^* and r is a solution of the (YBC) equation in a. □

Cases of Resolvable Lie Algebras

Lemma 2. *If the Lie algebra $\mathcal{G}$ contains an ideal of dimension 1, then $\mathcal{G}$ contains a Lie sub-algebra of dimension 2.*

Proof. Let $I = \mathbb{K}x$ be that ideal. Let $\mathrm{a} = \mathbb{K}x + \mathbb{K}y$, where y is an element of $\mathcal{G}$ that is not colinear to x. Then, $[x, y] \in I$ and $I \subset \mathrm{a}$, hence $[x, y] \in \mathrm{a}$ and a is a Lie sub-algebra of $\mathcal{G}$. □

Remark 2. If the Lie algebra $\mathcal{G}$ contains a Lie sub-algebra a of dimension 2, the latter contains solutions to the (YBC) equation, and according to the Lemma of Extension, $\mathcal{G}$ will also contain solutions to the (YBC) equation.

Proposition 1. *If $\mathcal{G}$ is a resolvable Lie algebra of dimension 2 then $\mathcal{G}$ contains non-trivial solutions to the (YBC) equation.*

Proof. $\mathcal{G}$ is resolvable, hence, there exists $p \in \mathbb{N}$ such that: $\mathcal{D}^{(p)}(\mathcal{G}) = 0$ and $\mathcal{D}^{(p-1)}(\mathcal{G}) \neq 0$ ($\mathcal{D}^{(p)}(\mathcal{G})$ denotes the p-th derivative ideal of $\mathcal{G}$).

If $\dim \mathcal{D}^{(p)}(\mathcal{G}) \geq 2$, we can construct a solution $r : \mathcal{G}^* \to \mathcal{G}$ to the (YBC) equation as follows: Let $(e_1, e_2, \ldots, e_k)$ denote a base of $\mathcal{D}^{(p-1)}(\mathcal{G})$. We can complet $(e_i)_{1 \leq i \leq k}$ to a base $(e_1, e_2, \ldots, e_k, e_{k+1}, \ldots, e_n)$ in $\mathcal{G}$. Let $(e_i^*)_{1 \leq i \leq n}$ denote the corresponding dual base. In these bases we shall represent r in the matrix form:

$$\begin{pmatrix} (a_{ij})_{k \times k} & 0_{k \times (n-k)} \\ 0_{(n-k) \times k} & 0_{(n-k) \times (n-k)} \end{pmatrix},$$

where the matrix $(a_{ij})_{k \times k}$ is skew-symmetric. We have:

i) r is skew-symmetric,

ii) the (YBC) equation is satisfied since $[\mathcal{D}^{(p-1)}(\mathcal{G}), \mathcal{D}^{(p-1)}(\mathcal{G})] = 0$.

Moreover, if $\dim \mathcal{D}^{(p-1)}(\mathcal{G}) = 1$, considering that $\mathcal{D}^{(p-1)}(\mathcal{G})$ is an ideal of $\mathcal{G}$, we know by Lemma 2 that we could construct a Lie sub-algebra a of $\mathcal{G}$ of dimension 2. Now using the Lemma of Extension, we conclude that $\mathcal{G}$ contains non-trivial solutions to the (YBC) equation. □

Cases of Semi-Simple Complex Lie Algebras

$\mathcal{G}$ denotes a complex Lie algebra of dimension ≥ 2. Recall the following result:

Theorem of Structure. (see [3]) *Let $\mathcal{G}$ be a semi-simple complex Lie algebra and $\mathcal{H}$ a Cartan sub-algebra of $\mathcal{G}$. Then, $\mathcal{G} = \mathcal{H} \oplus \left(\bigoplus_{\alpha\in\mathcal{R}} (g^{\alpha} \oplus g^{-\alpha})\right)$, where, $\mathcal{R}$ denotes a root system in $\mathcal{H}^*$ and g^{α} the eigen-subspace of $\mathcal{G}$ associated with the root α.*

Moreover, if $\alpha \in \mathcal{R}$, then $\dim g^{\alpha} = 1$ and $\dim h_{\alpha} = 1$, where $h_{\alpha} = [g^{\alpha}, g^{-\alpha}] \subset \mathcal{H}$; then, there exists a unique $H_{\alpha} \in h_{\alpha}$ such that $\alpha(H_{\alpha}) = 2$.

Let $\alpha \in \mathcal{R}$, for every non-zero element X_{α} of g^{α}, there exists a unique element Y_{α} of $g^{-\alpha}$ such that $[X_{\alpha}, Y_{\alpha}] = H_{\alpha}$.

We have: $[H_{\alpha}, X_{\alpha}] = 2X_{\alpha}$ and $[H_{\alpha}, Y_{\alpha}] = -2Y_{\alpha}$.

The sub-algebra $S_{\alpha} = h_{\alpha} \oplus g^{\alpha} \oplus g^{-\alpha}$ is isomorphic to sl(2).

Finally, if $\alpha, \beta \in \mathcal{R}$ and if $\alpha + \beta \neq 0$, then, $[g^{\alpha}, g^{\beta}] \subset g^{\alpha+\beta}$.

Proposition 2. *A complex semi-simple Lie algebra $\mathcal{G}$ contains non-trivial solutions of the (YBC) equation.*

Proof. By the theorem of structure, $\mathcal{G} = S_{\alpha} \oplus \mathcal{G}_1$ where $S_{\alpha} \simeq$sl(2), and using the results of the fifth case of Section 3, we know that there exists in S_{α} a non-trivial solution of the (YBC) equation. Now using the Lemma of Extension we are sure that a non-trivial solution of the (YBC) equation exists in $\mathcal{G}$. □

General Case

Theorem 2. *For every complex Lie algebra of dimension greater or equal to 2, there exist non-trivial solutions of the (YBC) equation.*

Proof. i) If $\mathcal{G}$ is semi-simple, the result is a consequence of Proposition 2.

ii) If $\mathcal{G}$ is not semi-simple, its radical $\mathcal{R}$ is not zero. There are two cases:

1. If $\dim \mathcal{R} = 1$, since $\mathcal{R}$ is an ideal, the Lemma 2 assures the existence of a sub-algebra of dimension 2 in $\mathcal{G}$. Now by the Lemma of Extension, we know that non-trivial solutions of the (YBC) equation exist in $\mathcal{G}$.

2. If $\dim \mathcal{R} = 2$, since $\mathcal{R}$ is resolvable, we know that non-trivial solutions of the (YBC) equation exist in $\mathcal{R}$. These solutions can be extended over to $\mathcal{G}$. $\square$

4. On Lie-Poisson Groups of Matrices

Let G be a Lie group whose elements are square matrices of order n with real coefficients. If $A = (A^i_j)_{n\times n} \in G$, we could consider A^i_j as a C^∞ function over G, i.e. as an element of $C^\infty(G)$.

$$A^i_j \colon G \longrightarrow \mathbb{R}, \qquad \begin{pmatrix} A^1_1 & \cdots & A^1_j & \cdots & A^1_n \\ \vdots & & & & \\ A^i_1 & \cdots & A^i_j & \cdots & A^i_n \\ \vdots & & & & \\ A^n_1 & \cdots & A^n_j & \cdots & A^n_n \end{pmatrix} \mapsto A^i_j\,.$$

Let $\mathcal{G}$ be the Lie algebra of G, $\mathcal{G} \subseteq M_n(\mathbb{R})$. We will assume that $\mathcal{G}$ contains a non-trivial solution to the (YBC) equation. Let $r \in \mathcal{G}\otimes\mathcal{G}$ denote such a solution and $\{,\}_r$ the associated Lie-Poisson's brackets over G. We have the following proposition.

Proposition 3. *$\left\{A^i_j, A^k_l\right\}_r(A) = r^{ab}_{jl}A^i_aA^k_b - r^{ik}_{ab}A^a_jA^b_l$ with $r = \left(r^{ab}_{jl}\right) \in M_n(\mathbb{R})\otimes M_n(\mathbb{R}) \simeq M_{n^2}(\mathbb{R})$.*

Proof. For all $f, g \in C^\infty(G)$ and for every $A \in G$ we have:

$$\{f,g\}_r(A) = r\left(d(f\circ\lambda_A)_e, d(g\circ\lambda_A)_e\right) - r\left(d(f\circ\rho_A)_e, d(g\circ\rho_A)_e\right),$$

where, $\lambda_A : G \to G, X \mapsto AX$ and $\rho_A : G \to G, X \mapsto XA$.

Thus, $d(f\circ\lambda_A)_e(X) = \langle d_Af, AX\rangle$ and $d(f\circ\rho_A)_e(X) = \langle d_Af, XA\rangle$, where $AX \in T_AG$ and $XA \in T_AG$ for every $X \in \mathcal{G}$, T_AG being the tangent space to G in A. If $f = A^i_j$, d_Af coincides with the function A^i_j, hence $d(A^i_j\circ\lambda_A)_e(X) = \langle A^i_j, AX\rangle = (AX)^i_j = A^i_aX^a_j$, and $d(A^i_j\circ\rho_A)_e(X) = \langle A^i_j, XA\rangle = (XA)^i_j = X^a_iA^j_a$.

Let $r = \Sigma_{\alpha,\beta}\ r^{\alpha\beta}e_\alpha\otimes e_\beta$, where, $(e_\alpha)_{1\le\alpha\le m}$ is a base of $\mathcal{G}$. Then,

$$\begin{aligned}\left\{A^i_j, A^k_l\right\}_r(A) &= r^{\alpha\beta}(Ae_\alpha)^i_j(Ae_\beta)^k_l - r^{\alpha\beta}(e_\alpha A)^i_j(e_\beta A)^k_l \\ &= r^{ab}_{jl}A^i_aA^k_b - r^{ik}_{ab}A^a_jA^b_l.\end{aligned}$$

We have also, $(A \otimes A)^{ik}_{jl} = A^i_j A^k_l$. Therefore, $\left\{A^i_j, A^k_l\right\}_r (A) = [A \otimes A, r]^{ik}_{jl}$. Here, $[A \otimes A, r]$ stands for the commutator of the two square matrices $A \otimes A$ and r in $M_{n^2}(\mathbb{R})$. $\square$

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