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RADON-NIKODÝM PROPERTY AND LUR POINT

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Abstract: We prove that a Banach space X has the Radon-Nikodým property if and only if for any equivalent norm $\|.\|_2$ on X, there exists an equivalent norm $\|.\|_1$ with $B_{(X,\|.\|_1)} \subseteq B_{(X,\|.\|_2)} \subseteq rB_{(X,\|.\|_1)}$ for some $r > 0$, such that $B_{(X,\|.\|_1)}$ has a LUR point. A characterization of LUR point is also given.

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1. Introduction

Recall that a Banach space X has the Radon-Nikodým property(RNP) if the Radon-Nikodým Theorem is valid for measures taking values in X, Radon-Nikodým property has attracted the attention of many researchers (see [3] for more details on the RNP). Renorming theory is that branch of functional analysis which investigates problems of the form: for which Banach spaces X does there exist a norm on X, equivalent to the given norm, with some good geometrical property of convexity? It is clear that the Radon-Nikodým property is invariant under renorming. Fabian and Godefroy [7] proved that every dual space with RNP has an equivalent locally uniformly convex norm, but for non-dual spaces, the question remains open.

For a Banach space X with dual X^*, let B_X denote the closed unit ball of X, and S_X the unit sphere. It may be useful to recall that some definitions which will be used in the paper.

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Definition 1. (see [3]) Let K be a subset of X, $x \in K$ is said to be an exposed point of K if there is a $f \in X^*$, such that $f(x) > f(y)$ for all $y \in K \setminus y$, and such a f is said to expose K at x.

Definition 2. (see [3]) Let K be a subset of X, $x \in K$ is said to be an strongly exposed point of K if there exists a $f \in X^*$, such that $f(x) > f(y)$ for all $y \in K \setminus y$, and the conditions that $x_n \in K$ and $f(x_n) \to f(x)$ imply $\lim_n x_n = y$ in norm.

Definition 3. (see [4]) Let X be a Banach space, $x \in S_X$ is a LUR point if for any $x_n \in B_X, \|x_n + x\| \to 2$, then $\|x_n - x\| \to 0$.

Definition 4. (see [4]) Let X be a Banach space, $x \in S_X$ is a wLUR point if for any $x_n \in B_X, \|x_n + x\| \to 2$, then $f(x_n) \to f(x)$, for all $f \in X^*$.

Definition 5. (see [3]) A point $x \in K$ is called a denting point of K if x is not in $\overline{co}\{K \setminus (x + \varepsilon B_X)\}$ for each $\varepsilon > 0$.

Definition 6. (see [2]) Let X be a Banach space, $x \in S_X$ is a rotund point if for any $y \in B_X, \|x + y\| = 2$, then $x = y$.

We recall that a point $x \in K$ is called an extreme point of bounded closed convex subset of X if for every $y, z \in K$ the equality $2x = y + z$ implies $y = z$. It is very easy to see that every rotund point of B_X is an extreme point, indeed an exposed point of B_X. But the converse is not generally true. For example, no extreme point of B_{l_∞} or B_{l_1} is a rotund point of B_{l_∞} or B_{l_1}. However, if every point of S_X is an extreme point of B_X, then every point of S_X is a rotund point of B_X and the space X is rotund.

From [3], it is easy to see that Banach space has RNP there are many equivalent geometric property.

Theorem 7. *Let X be a Banach space, then X has RNP if and only if one of the following assertions hold.*

(1) Every non-empty bounded subset of X has a denting point;

(2) For any equivalent norm $|||.|||$ on X, the closed unit ball of $(X, |||.|||)$ has a denting point;

(3) Every non-empty bounded subset of X has a strongly exposed point;

(4) For any equivalent norm $|||.|||$ on X, the closed unit ball of $(X, |||.|||)$ has a strongly exposed point;

(5) Every non-empty bounded subset of X has an exposed point;

(6) For any equivalent norm $|||.|||$ on X, the closed unit ball of $(X, |||.|||)$ has an exposed point.

Let us remark that evidently l_1 has Radon-Nikodým property and its unit ball has no rotund point, so it is clear that Banach space X has the Radon-Nikodým property is not equivalent to for any equivalent norm $|||.|||$ on X, the closed unit ball of $(X, |||.|||)$ has a LUR point.

Recall that a Banach space X has the Krein-Milman property (KMP) if every non-empty bounded subset of X has some extreme point. Diestel and Uhl [6] asked whether Radon-Nikodým property is equivalent to the Krein-Milman property, but it still is an open question.

In this paper, a new characterization of the Radon-Nikodým property is given in terms of LUR point and rotund point.

By the technique of [1], we have the following theorem.

Theorem 8. *Let X be a Banach space, $x \in S_X$, the following assertions are equivalent:*

(1) x is a LUR point of B_X.

(2) For every $y \in S_X$ with $\|x-y\| \geq \varepsilon$, there exists $\delta > 0$ such that

$$\lim_{t\to 0^+}\left(\frac{\|x+ty\|-\|x\|}{t}\right) < 1-\delta\,.$$

Proof. Assume that x is a LUR point, then for any $\varepsilon > 0$, there exist $\delta > 0$, such that $\|x+y\| < 2-\delta$ whenever $y \in S_X$ and $\|x-y\| \geq \varepsilon$. So

$$\frac{\|x+1y\|-\|x\|}{1} < 1-\delta\,.$$

Since the mapping

$$R \setminus 0 \longrightarrow R\,, \qquad t \longrightarrow (\|x+ty\|-\|x\|)/t$$

is increasing, we have

$$\lim_{t\to 0^+}\left(\frac{\|x+ty\|-\|x\|}{t}\right) < 1-\delta\,.$$

Assume that (2) holds. Then for any $\varepsilon > 0$, we have $\delta > 0$, such that

$$\lim_{t\to 0^+}\left(\frac{\|x+ty\|-\|x\|}{t}\right) < 1-\delta\,,$$

whenever $y \in S_X$ with $\|x-y\| \geq \varepsilon$. Thus $\|\frac{x+y}{2}\| < 1-\frac{\delta}{2}$ whenever $y \in S_X$ with $\|x-y\| \geq \varepsilon$. Otherwise, we have $\|\frac{x+y}{2}\| \geq 1-\frac{\delta}{2}$ for some $y \in S_X$ with $\|x-y\| \geq \varepsilon$. So there exist $f \in S_{X^*}$, such that

$$f\left(\frac{x+y}{2}\right) = \left\|\frac{x+y}{2}\right\|\,.$$

Since $f(x) \leq 1$, we have $f(y) > 1-\delta$. Thus

$$1-\delta < f(y) = \frac{f(x+ty)-f(x)}{t} \leq \frac{\|x+ty\|-\|x\|}{t} < 1-\delta\,,$$

which is a contradiction. □

Lemma 9. *Let X be Banach space, $\|.\|_1$ and $\|.\|_2$ are equivalent norm on X such that $B_{(X,\|.\|_1)} \subseteq B_{(X,\|.\|_2)} \subseteq rB_{(X,\|.\|_1)}$ for some $r > 0$, if $x \in S_{(X,\|.\|_1)} \cap S_{(X,\|.\|_2)}$, and x is a rotund point of $B_{(X,\|.\|_2)}$, then $x \in \exp\ B_{(X,\|.\|_1)}$.*

Proof. Since $x \in S_{(X,\|.\|_2)}$, we have $f \in S_{(X^*,\|.\|_2)}$ such that $f(x) = 1$. So $\|f\|_1 = \sup\{f(x) : \|x\|_1 \leq 1\} \leq \sup\{f(x) : \|x\|_2 \leq 1\} = 1$, by $x \in S_{(X,\|.\|_1)} \cap S_{(X,\|.\|_2)}$, we know that $\|f\|_1 = 1$, thus f is an expose functional of x in $B_{(X,\|.\|_1)}$. In fact, if $y \in B_{(X,\|.\|_1)}$ such that $f(x) = f(y)$, then $\|x + y\|_2 \geq f(x + y) = 2$, since x is a rotund point of $B_{(X,\|.\|_2)}$, we have $x = y$, hence x is an exposed point of $B_{(X,\|.\|_1)}$. □

The converse of theorem above is not true in in general. In R^2, we definite $\|x\|_1 = |x_1| + |x_2|$, and $\|x\|_2 = \max\{|x_1|, |x_2|\}$, then $\|.\|_1$ and $\|.\|_2$ are equivalent norm on R^2, and $B_{(R^2,\|.\|_1)} \subseteq B_{(R^2,\|.\|_2)} \subseteq 2B_{(R^2,\|.\|_1)}$. $S_{(R^2,\|.\|_1)} \bigcap S_{(R^2,\|.\|_2)} = \{\pm(1, 0), \pm(0, 1)\}$. $\{\pm(1, 0), \pm(0, 1)\} \in \exp\ B_{(R^2,\|.\|_1)}$, but $\{\pm(1, 0), \pm(0, 1)\}$ are not rotund points of $B_{(R^2,\|.\|_2)}$.

Lemma 10. *Let X be Banach space, $\|.\|_1$ and $\|.\|_2$ are equivalent norm on X such that $B_{(X,\|.\|_1)} \subseteq B_{(X,\|.\|_2)} \subseteq rB_{(X,\|.\|_1)}$ for some $r > 0$, if $x \in S_{(X,\|.\|_1)} \cap S_{(X,\|.\|_2)}$, and x is a LUR point of $B_{(X,\|.\|_2)}$, then $x \in s\exp\ B_{(X,\|.\|_1)}$.*

Proof. Since $x \in S_{(X,\|.\|_2)}$, we have $f \in S_{(X^*,\|.\|_2)}$ such that $f(x) = 1$. So $\|f\|_1 = \sup\{f(x) : \|x\|_1 \leq 1\} \leq \sup\{f(x) : \|x\|_2 \leq 1\} = 1$, by $x \in S_{(X,\|.\|_1)} \cap S_{(X,\|.\|_2)}$, we know that $\|f\|_1 = 1$. Next, we show that f is a strongly expose functional of x in $B_{(X,\|.\|_1)}$. Suppose that $x_n \in B_{(X,\|.\|_1)}$ such that $f(x_n) \to 1$, then $\|x + x_n\|_2 \geq f(x + x_n) \to 2$, since x is a LUR point of $B_{(X,\|.\|_2)}$, we have $x_n \to y$ in norm $\|.\|_2$, so $x_n \to y$ in norm $\|.\|_1$, hence x is a strongly exposed point of $B_{(X,\|.\|_1)}$. □

Similarly, the following lemma holds.

Lemma 11. *Let X be Banach space, $\|.\|_1$ and $\|.\|_2$ are equivalent norm on X such that $B_{(X,\|.\|_1)} \subseteq B_{(X,\|.\|_2)} \subseteq rB_{(X,\|.\|_1)}$ for some $r > 0$, if $x \in S_{(X,\|.\|_1)} \cap S_{(X,\|.\|_2)}$, and x is a weak LUR point of $B_{(X,\|.\|_2)}$, then x is a weak strongly exposed point of $B_{(X,\|.\|_1)}$.*

By above lemmas, it is easy to see that the Radon-Nikodým property can be characterized in terms of LUR point or rotund point as follows.

Theorem 12. *Let X be a Banach space, then X has RNP if and only if one of the following assertions holds.*

(1) For any equivalent norm $\|.\|_2$ on X, there exists an equivalent norm $\|.\|_1$ with $B_{(X,\|.\|_1)} \subseteq B_{(X,\|.\|_2)} \subseteq rB_{(X,\|.\|_1)}$ for some $r > 0$, such that $B_{(X,\|.\|_1)}$ has a LUR point;

(2) For any equivalent norm $\|.\|_2$ on X, there exists an equivalent norm $\|.\|_1$ with $B_{(X,\|.\|_1)} \subseteq B_{(X,\|.\|_2)} \subseteq rB_{(X,\|.\|_1)}$ for some $r > 0$, such that $B_{(X,\|.\|_1)}$ has a

wLUR point;

(3) For any equivalent norm $\|.\|_2$ *on* X*, there exists an equivalent norm* $\|.\|_1$ *with* $B_{(X,\|.\|_1)} \subseteq B_{(X,\|.\|_2)} \subseteq rB_{(X,\|.\|_1)}$ *for some* $r > 0$*, such that* $B_{(X,\|.\|_1)}$ *has a rotund point.*

Finally, it is interesting to see consider the following question.

Problem 13. Let X be Banach space, then X has Krein-Milman property if and only if for any equivalent norm $\|.\|_2$ on X, there exists an equivalent norm $\|.\|_1$ with $B_{(X,\|.\|_1)} \subseteq B_{(X,\|.\|_2)} \subseteq rB_{(X,\|.\|_1)}$ for some $r > 0$, such that $B_{(X,\|.\|_1)}$ has an extreme point holds or not?

Example 14. Banach space c_0 has not KMP, for a fixed point $x_0 \in S_X$, there exists an equivalent norm $|||.|||$ on X with $B_X \subseteq B_{(X,|||.|||)} \subseteq 3B_X$, $x_0 \in S_{(X,|||.|||)} \bigcap S_X$ and $x_0 \in ext\ B_{(X,|||.|||)}$. In fact, for any Banach space X and fixed point $x_0 \in S_X$, we have $f \in S_{X^*}$ such that $f(x_0) = 1$, set $M = ker\ f$, then $X = M + \text{span}\{x_0\}$, for any $x = (y, r) \in X$, let $|||x||| = \|y\| + |r|$, then $|||.|||$ is an equivalent norm on X with $\|x\| \leq |||x||| \leq 3\|x\|$, so $B_X \subseteq B_{(X,|||.|||)} \subseteq 3B_X$. It is easy to see $x_0 = (0, 1) \in S_{(X,|||.|||)} \bigcap S_X$ and $x_0 \in \text{ext}\ B_{(X,|||.|||)}$. Hence the answer of above problem is negative.

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