TAXICAB VERSIONS OF SOME EUCLIDEAN THEOREMS

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Abstract: In this paper, we give the taxicab versions of Pythagorean Theorem, Stewart’s Theorem and a median property.

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1. Introduction

The taxicab plane geometry is introduced by Menger [7] and is developed by Krause [6]. Now, there are about fifty articles published on the subject. The taxicab plane \(\mathbb{R}^2_T\) is almost the same as the Euclidean analytical plane \(\mathbb{R}^2\). The points are the same, the lines are the same, and the angles are measured in the same way. However, the distance function is different. Taxicab distance between the points \(P\) and \(Q\) is the length of a shortest path from \(P\) to \(Q\) composed of the line segments parallel to the coordinate axes. That is, if \(P = (x_1, y_1)\) and \(Q = (x_2, y_2)\) then the taxicab distance from \(P\) to \(Q\) is \(d_T(P, Q) = |x_1 - x_2| + |y_1 - y_2|\).

The taxicab plane geometry is non-Euclidean since it fails to satisfy the side-
angle-side axiom but satisfies all the remaining twelve axioms of the Euclidean plane geometry. Since the taxicab plane geometry has a different distance function it seems interesting to study the taxicab analogues of the topics that include the concept of distance in the Euclidean geometry. A few of such topics have been studied by some authors [1, 2, 3, 4, 5, 8, 9, 10, 11, 13]. The group of isometries that preserve taxicab distance is determined in [12].

Taxicab analogues of Ceva’s Theorem, Menelaus’ Theorem and Thales’ Theorems are proven in [9]. Here in this study, we give taxicab versions of Stewart’s Theorem, median property and Pythagorean Theorem.

2. A Taxicab Version of the Stewart’s Theorem

It is known that for any triangle $ABC$ in the Euclidean plane, if $X \in [BC]$ and $a = d(B, C)$, $b = d(A, C)$, $c = d(A, B)$, $p = d(B, X)$, $q = d(C, X)$, $x = d(A, X)$ then

$$x^2 = \frac{b^2p + c^2q}{p + q} - pq$$

which is known as Stewart’s Theorem. We use the following definitions given in [8] to give a taxicab version of this theorem.

Let $ABC$ be any triangle in the taxicab plane. Clearly, there exists a pair of lines passing through every vertex of the triangle, each of which is parallel to a coordinate axis. A line $l$ is called a base line of $ABC$ if and only if:

1. $l$ passes through a vertex,
2. $l$ is parallel to a coordinate axis,
3. $l$ intersects the opposite side (as a line segment) to the vertex in condition 1.

Clearly, at least one of the vertices of the triangle always has one or two base lines. Such a vertex of a triangle is called a basic vertex. A base segment is a line segment on a base line, which is bounded by a basic vertex and its opposite side.

The next theorem gives a taxicab version of the Stewart’s Theorem.

**Theorem 1.** Let the sides of a triangle $ABC$ in the taxicab plane have lengths $a = d_T(B, C)$, $b = d_T(A, C)$ and $c = d_T(A, B)$. If $X \in [BC]$ and $p = d_T(B, X)$, $q = d_T(C, X)$ and $x = d_T(A, X)$, then
x = 
\begin{align*}
\frac{bp+cq}{p+q} & \quad \text{If } ABC \text{ has no base line through the vertex } A, \\
\frac{(b-2\alpha)p+cq}{p+q} & \quad \text{If } ABC \text{ has only one base line through the vertex } A, \\
\frac{bp+(c-2\beta)q}{p+q} & \quad \text{If } ABC \text{ has two base lines through the vertex } A, \\
\frac{(b-2\alpha)p+(c-2\beta)q}{p+q} & \quad \text{and } X \text{ is between the intersection points of the base lines and the opposite side, and } D \text{ is between } X \text{ and } C, \\
\frac{(b-2\beta)p+(c-2\alpha)q}{p+q} & \quad \text{and } X \text{ is between the intersection points of the base lines and the opposite side, and } D \text{ is between } X \text{ and } B, \\
\frac{|bp-cq|}{p+q} & \quad \text{If } ABC \text{ has two base lines through the vertex } A, \text{ and } X \text{ is not between the intersection points of the base lines and the opposite side,}
\end{align*}

where \( \alpha = d_T(\text{base line}, E) \), \( \beta = d_T(A, E') \) and 

\begin{align*}
D &= \text{Intersection point of a base line and the opposite side}, \\
E &= \text{One of the vertices } B \text{ and } C \text{ such that } D \text{ lies between } X \text{ and } E, \\
E' &= \text{The point of orthogonal projection of the vertex distinct from } A \text{ and } E \text{ on the same base line.}
\end{align*}

**Proof.** Let 

\begin{align*}
B' &= \text{Orthogonal projection of } B \text{ to the line through } A \text{ and parallel to } y\text{-axis}, \\
C' &= \text{Orthogonal projection of } C \text{ to the line through } A \text{ and parallel to } x\text{-axis}, \\
X' &= \text{Orthogonal projection of } X \text{ to the line through } A \text{ and parallel to } x\text{-axis}, \\
T &= \text{Orthogonal projection of } X \text{ to the line } CC', \\
T' &= \text{Orthogonal projection of } X \text{ to the line } BB',
\end{align*}

and \( d(A, C') = b_1 \), \( d(C, C') = b_2 \), \( d(A, B') = c_2 \), \( d(B, B') = c_1 \). Thus \( b = b_1 + b_2 \), \( c = c_1 + c_2 \) and \( x = d(A, X') + d(X, X') \).
Case I. If $ABC$ is a triangle which has no base line through the vertex $A$ as in Figure 1, then one can easily obtain

$$ q \cdot (b_1 - c_1) = (p + q) \cdot d(X, T), \quad p \cdot (c_2 - b_2) = (p + q) \cdot d(X, T') $$

by [9], Theorem 5. Thus

$$ d(X, T) = q \cdot (b_1 - c_1) / (p + q), \quad d(X, T') = p \cdot (c_2 - b_2) / (p + q). $$

Since $d(A, X') = b_1 - d(X, T)$, $d(X, X') = c_2 - d(X, T')$ and $x = d(A, X') + d(X, X')$, we get

$$ x = b_1 - \frac{q(b_1 - c_1)}{p + q} + c_2 - \frac{p(c_2 - b_2)}{p + q} = \frac{b_1 p + b_2 p + c_1 q + c_2 q}{p + q} - 2b_2 p/p + q. $$

Case II. Let $ABC$ be a triangle which has only one base line through the vertex $A$. If $D$ is between $X$ and $C$ as in Figure 2, then one can easily obtain

$$ q \cdot (c_1 - b_1) = (p + q) \cdot d(X, T), \quad p \cdot (b_2 + c_2) = (p + q) \cdot d(X, T'), $$

by [9], Theorem 5. Thus

$$ d(X, T) = q \cdot (c_1 - b_1) / (p + q), \quad d(X, T') = p \cdot (b_2 + c_2) / (p + q). $$

Since $d(A, X') = b_1 + d(X, T)$, $d(X, X') = c_2 - d(X, T')$ and $x = d(A, X') + d(X, X')$ we get

$$ x = b_1 + \frac{q(c_1 - b_1)}{p + q} + c_2 - \frac{p(b_2 + c_2)}{p + q} = \frac{b_1 p + b_2 p + c_1 q + c_2 q - 2b_2 p}{p + q}. $$

Case III. Let $ABC$ be a triangle which has only one base line through the vertex $A$. If $D$ is between $X$ and $B$ as in Figure 3, then one can easily obtain

$$ q \cdot (c_1 - b_1) = (p + q) \cdot d(X, T), \quad p \cdot (b_2 + c_2) = (p + q) \cdot d(X, T'), $$

by [9], Theorem 5. Thus

$$ d(X, T) = q \cdot (c_1 - b_1) / (p + q), \quad d(X, T') = p \cdot (b_2 + c_2) / (p + q). $$

Since $d(A, X') = b_1 + d(X, T)$, $d(X, X') = c_2 - d(X, T')$ and $x = d(A, X') + d(X, X')$ we get

$$ x = b_1 + \frac{q(c_1 - b_1)}{p + q} + c_2 - \frac{p(b_2 + c_2)}{p + q} = \frac{b_1 p + b_2 p + c_1 q + c_2 q - 2b_2 p}{p + q}. $$
Figure 3:

by [9, Theorem 5]. Thus

\[ d(X, T) = q(c_1 - b_1)/(p + q), \quad d(X, T') = p(b_2 + c_2)/(p + q). \]

Since \( d(A, X') = b_1 + d(X, T) \), \( d(X, X') = d(X, T') - c_2 \) and \( x = d(A, X') + d(X, X') \) we get

\[
x = b_1 + \frac{q(c_1 - b_1) + p(b_2 + c_2)}{p + q} - c_2 = \frac{b_1 p + b_2 p + c_1 q + c_2 q - 2c_2 q}{p + q}
\]

**Case IV.** Let \( ABC \) be a triangle which has two base lines through the vertex \( A \), and \( X \) be between the intersection points of the base lines and the opposite side. If \( D \) is between \( X \) and \( C \) as in Figure 4, then one can easily obtain

\[ q(b_1 + c_1) = (p + q) \cdot d(X, T) , \quad p(b_2 + c_2) = (p + q) \cdot d(X, T') \]

by [9, Theorem 5]. Thus

\[ d(X, T) = q(c_1 - b_1)/(p + q), \quad d(X, T') = p(b_2 + c_2)/(p + q). \]

Since \( d(A, X') = d(X, T) - b_1 \), \( d(X, X') = d(X, T') - c_2 \) and \( x = d(A, X') + d(X, X') \) we get

\[
x = b_1 + \frac{q(b_1 + c_1)}{p + q} - b_1 + \frac{p(b_2 + c_2)}{p + q} - c_2 = \frac{(b - 2\alpha)p + (c - 2\beta)q}{p + q}
\]

**Case V.** Let \( ABC \) be a triangle which has two base lines through the vertex \( A \), and \( X \) be between the intersection points of the base lines and the opposite side. If \( D \) is between \( X \) and \( B \) as in Figure 5, then one can easily obtain

\[ q(b_1 + c_1) = (p + q) \cdot d(X, T) , \quad p(b_2 + c_2) = (p + q) \cdot d(X, T') \]
by [9], Theorem 5. Thus

\[ d(X, T) = q \cdot (b_1 + c_1) / (p + q), \quad d(X, T') = p \cdot (b_2 + c_2) / (p + q). \]

Since \( d(A, X') = d(X, T) - b_1 \), \( d(X, X') = d(X, T') - c_2 \) and \( x = d(A, X') + d(X, X') \) we get

\[ x = \frac{q(b_1 + c_1)}{p + q} - b_1 + \frac{p(b_2 + c_2)}{p + q} - c_2 \]
\[ = \frac{b_1 p + b_2 p + c_1 q + c_2 q - 2b_1 p - 2c_2 q}{p + q} \]
\[ = \frac{(b - 2\beta)p + (c - 2\alpha)q}{p + q}. \]

Case VI. Let \( ABC \) be a triangle which has two base lines passing through the vertex \( A \). If \( X \) is not between the intersection points of the base lines and the opposite side as in Figure 6 and Figure 7, then one can easily obtain

\[ q \cdot (b_1 + c_1) = (p + q) \cdot d(X, T), \quad p \cdot (b_2 + c_2) = (p + q) \cdot d(X, T') \]

by [9], Theorem 5. Thus

\[ d(X, T) = q \cdot (b_1 + c_1) / (p + q), \quad d(X, T') = p \cdot (b_2 + c_2) / (p + q). \]

Now, two subcases are possible. If \( D \) is between \( X \) and \( C \) as in Figure 6, then \( d(A, X') = d(X, T) - b_1 \), \( d(X, X') = c_2 - d(X, T') \) and we get

\[ x = \frac{d(A, X') + d(X, X')}{p + q} = \frac{q(b_1 + c_1)}{p + q} - b_1 + c_2 - \frac{p(b_2 + c_2)}{p + q} \]
\[ = \frac{c_1 q + c_2 q - b_1 p - b_2 p}{p + q} = \frac{cq - bp}{p + q}. \]
If $D$ is between $X$ and $B$ as in Figure 7, then $d(A, X') = b_1 - d(X, T)$, $d(X, X') = d(X, T') - c_2$ and we get

$$x = d(A, X') + d(X, X') = b_1 - \frac{q(b_1 + c_1)}{p + q} + \frac{p(b_2 + c_2)}{p + q} - c_2$$

$$= \frac{b_1p + b_2p - c_1q - c_2q}{p + q} = \frac{b_1p - c_1q}{p + q}.$$ 

Consequently, $x = \frac{|bp - cq|}{p + q}$ which completes the proof. \[ \square \]

If $X$ is the midpoint of $[BC]$ of any triangle $ABC$ in the Euclidean plane with $a = d(B, C)$, $b = d(A, C)$, $c = d(A, B)$ and $V_a = d(A, X)$ then

$$2V_a^2 = b^2 + c^2 - a^2 / 2$$

which is known as median property. The following corollary gives a taxicab version of this property, for $p = q$ in Theorem 1.
Corollary 2. Let the sides of a triangle $ABC$ in the taxicab plane have lengths $a = d_T(B, C)$, $b = d_T(A, C)$ and $c = d_T(A, B)$. If $X$ is the midpoint of $[BC]$ and $V_a = d_T(A, X)$, then

$$2V_a = \begin{cases} 
\frac{b+c}{2} & \text{If } ABC \text{ has no base line through the vertex } A, \\
\frac{b+c - 2\alpha}{2} & \text{If } ABC \text{ has only one base line through the vertex } A, \\
\frac{b+c - 2(\alpha + \beta)}{2} & \text{If } ABC \text{ has two base lines through the vertex } A \\
|b - c| & \text{If } ABC \text{ has two base lines through the vertex } A \\
& \text{and } X \text{ is not between the intersection points of the base lines and the opposite side,} \\
& \text{If } ABC \text{ has two base lines through the vertex } A \\
& \text{and } X \text{ is between the intersection points of the base lines and the opposite side,}
\end{cases}$$

where $\alpha = d_T(\text{base line}, E)$, $\beta = d_T(A, E')$ and

- $D = \text{Intersection point of a base line and the opposite side}$,
- $E = \text{One of the vertices } B \text{ and } C \text{ such that } D \text{ lies between } X \text{ and } E$,
- $E' = \text{The point of orthogonal projection of the vertex distinct from } A \text{ and } E \text{ on the same base line}$.

3. A Taxicab Version of the Pythagorean Theorem

It is well known that for any right triangle $ABC$ in the Euclidean plane, if $[BC]$ is its hypotenuse and $a = d(B, C)$, $b = d(A, C)$, $c = d(A, B)$ then

$$a^2 = b^2 + c^2,$$

which is known as the Pythagorean Theorem. A taxicab version of this theorem can be stated as follows.

**Theorem 3.** Let $a$ denote the length of the hypotenuse, $b$ and $c$ denote the lengths of the legs of a triangle $ABC$ with right angle $A$ in the taxicab plane. Then

$$a = \begin{cases} 
b + c - 2\gamma & \text{If there exists only one base line through the vertex } A, \\
b + c & \text{If there exist two base lines through the vertex } A,
\end{cases}$$

where $\gamma = d_T(A, H)$ and $H = \text{The point of orthogonal projection of } B \text{ or } C \text{ to the base segment through } A$. 
Proof. A is a basic vertex since $ABC$ is a triangle with right angle $A$. That is, this triangle always has one or two base lines passing through $A$.

**Case I.** Let $b_1$, $b_2$, $c_1$ and $c_2$ denote the parameters used in the proof of Theorem 1. If there exists only one base line through the vertex $A$ as in Figure 8, and $b = d(A, C') + d(C, C') = b_1 + b_2$, $c = d(A, B') + d(B, B') = c_2 + c_1$, then

$$a = b_1 - c_1 + b_2 + c_2 = b + c - 2c_1, \quad c_1 = \gamma = d(B, B') = d(A, H)$$

**Case II.** If there exist two base lines through the vertex $A$, then the basic lines coincide with the perpendicular sides of $ABC$ as in Figure 9. Thus, obviously, $a = b + c$ which completes the proof.

4. Another Taxicab Version of the Pythagorean Theorem

A taxicab version of the Pythagorean Theorem has been given in Section 3 using a parameter $\gamma$ which is length of a part of the base segment. In this section, we use slopes of hypotenuse and a side of the right triangle to give another version of the theorem.

**Theorem 4.** Let $a$ denote the length of the hypotenuse, $b$ and $c$ denote the lengths of the legs of a right triangle in the taxicab plane. If the slope of the hypotenuse is $m_1$ and the slope of the anyone of the legs is $m_2$, then

$$a^2 = \rho(m_1, m_2) \cdot \left(b^2 + c^2\right),$$

where

$$\rho(m_1, m_2) = \begin{cases} 
\left(\frac{1 + m_2^2}{1 + m_1^2}\right)^2 \left(\frac{1 + |m_1|}{1 + |m_2|}\right)^2, & \text{if } m_1, m_2 \in \mathbb{R}, \\
\left(\frac{1 + |m_1|^2}{1 + |m_2|^2}\right)^{\frac{1}{2}}, & \text{if } m_1 \to \infty, \\
\left(\frac{1 + |m_1|}{1 + |m_2|}\right)^2, & \text{if } m_2 \to \infty.
\end{cases}$$
Proof. We know from [4] that for any two points \( P = (x_1, y_1) \) and \( Q = (x_2, y_2) \) in the taxicab plane, if \( x_1 \neq x_2 \) then
\[
d(P, Q) = \left( (1 + m^2)^{1/2} / (1 + |m|) \right) \cdot d_T(P, Q),
\]
where \( m = (y_1 - y_2) / (x_1 - x_2) \) and if \( x_1 = x_2 \), that is, \( m \to \infty \) then
\[
d(P, Q) = d_T(P, Q)
\]
which allows us to convert a taxicab distance to the Euclidean distance.

Let \( a, b \) and \( c \) be the corresponding Euclidean lengths of the sides of the same right triangle, \( m_1 \) denote the slope of the hypotenuse, and \( m_2 \) denote the slope of the anyone of the legs. If \( m_2 \neq 0 \), then the slope of the other leg is \((1/m_2)\) and
\[
a = \left( (1 + m_1^2)^{1/2} / (1 + |m_1|) \right) \cdot a,
b = \left( (1 + m_2^2)^{1/2} / (1 + |m_2|) \right) \cdot b,
c = \left( 1 + \left( -\frac{1}{m_2} \right)^2 \right)^{1/2} / \left( 1 + \left| -\frac{1}{m_2} \right| \right) \cdot c
\]
\[
= \left( (1 + m_2^2)^{1/2} / (1 + |m_2|) \right) \cdot c.
\]

If \( m_2 = 0 \), then the slope of the other leg is \((-1/m_2) \to \infty \) or if \( m_2 \to \infty \), then the slope of the other leg is \((-1/m_2) \to 0 \) and
\[
a = \left( (1 + m_1^2)^{1/2} / (1 + |m_1|) \right) \cdot a, \quad b = b, \quad c = c.
\]

If \( m_1 \to \infty \), then \( a = a \),
\[
b = \left( (1 + m_2^2)^{1/2} / (1 + |m_2|) \right) \cdot b, \quad c = \left( (1 + m_2^2)^{1/2} / (1 + |m_2|) \right) \cdot c.
\]

Using these values of \( a, b \) and \( c \) in the Euclidean Pythagorean Theorem one obtains
\[
a^2 = \rho(m_1, m_2) \cdot (b^2 + c^2),
\]
where
\[
\rho(m_1, m_2) = \begin{cases} 
\left( \frac{1 + m_1^2}{1 + m_1} \right) \left( \frac{1 + |m_1|}{1 + |m_2|} \right)^2, & \text{if } m_1, m_2 \in \mathbb{R}, \\
\left( \frac{1 + m_2^2}{1 + m_2} \right), & \text{if } m_1 \to \infty, \\
\left( \frac{1 + |m_2|}{1 + |m_1|} \right)^2, & \text{if } m_2 \to \infty.
\end{cases}
\]

which completes the proof. \( \square \)
References


