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QED HOPF ALGEBRAS ON PLANAR BINARY BITREES

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Abstract: In this paper the Hopf algebras on planar binary bitrees used to renormalize propagators of quantum electrodynamics are described and the coactions which describes renormalization procedure are constructed.

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1. Introduction

In [2] and [6], it has been demonstrated that the renormalization of the electron propagator in quantum electrodynamics can be described in terms of the semidirect coproduct of Hopf algebras $\mathcal{H}^\alpha \ltimes \mathcal{H}^{\mathrm{e}}$ on the set of rooted planar binary trees. There, $\mathcal{H}^\alpha$ was a commutative Hopf algebra which represented the renormalization of the electric charge, while $\mathcal{H}^{\mathrm{e}}$ was a Hopf algebra which represented the electron propagators, and which was neither commutative nor cocommutative. The renormalization was then a coaction of $\mathcal{H}^\alpha \ltimes \mathcal{H}^{\mathrm{e}}$ on $\mathcal{H}^{\mathrm{e}}$, obtained as restriction of the coproduct.

The main objective of this paper is, in the spirit of wave-particle duality, to replace planar binary trees and belonging algebraic structures used in renormalization of QED, with planar binary bitrees and their corresponding structures. These structures will generalize Hopf algebras structures of the electron and photon propagators and QED Hopf algebra structure on trees introduced in [2].

Concerning the use of planar binary bitrees, we recall that they have appeared in [3] where addition and multiplication on bitrees closely related to

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addition and multiplication on the integers was constructed. These operations were generalizations of addition and multiplication on trees which were closely related to addition and multiplication on natural numbers.

We would like to remark that although it seems that planar binary bitrees are ordered pairs of planar binary trees, they could be much more, as integers are much more than pairs of natural numbers.

The paper is organized as follows. In Section 2 we recall basic facts about planar binary bitrees. In Section 3, Section 4, and Section 5 we define the non-commutative Hopf algebra which corresponds to the electron and photon propagator; the charge Hopf algebra, which corresponds to the renormalization of the coupling constant of QED; and finally the renormalization Hopf algebra and renormalization coactions for the electron and the photon propagators.

Notations. In the following sections it will be supposed that all vector spaces and algebras are defined over the field of complex numbers $\mathbb{C}$, but this choice is not necessary. We will denote by $\mathbb{C}X$, the vector space spanned by arbitrary set X, by $\mathbb{C}\langle X\rangle$ the tensor algebra on X (non-commutative polynomials), and by $\mathbb{C}[X]$ the symmetric algebra on X (commutative polynomials).

2. About Planar Binary Bitrees

2.1. Introduction

We already mentioned that in [3] addition and multiplication on the set of planar binary bitrees are constructed, closely related to addition and multiplication on integers. This gives rise to a new kind of non-commutative arithmetic theory. Also in [4] planar binary bitrees are used to construct prefix and Huffman prefix code and coding English alphabet in such a way that characters have codewords different from already established ones.

2.2. Planar Binary Bitrees

A *planar binary bitree* (p.b. bitree) is an oriented planar graph which contains the upper and lower binary tree whose roots are connected by the edge. This edge is called the root of the planar binary bitree. In every p.b. tree, each internal vertex has two leaves and one root.

The number of internal vertices of the planar binary bitree is called the *order* of the p.b. bitree.

By $\mathbb{Y}_n$ the set of p.b. bitrees of the order n is denoted. The set $\mathbb{Y}_{n-1}$ has c_n

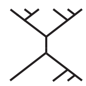


Figure 1: Planar binary bitree

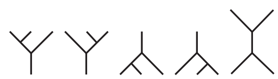
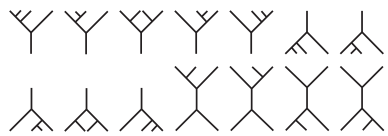
order	planar binary bitrees
0	
1	
2	
3	

Table 1: Planar binary bitrees

elements, where $c_n = \frac{(2n)!}{n!(n+1)!}$ is the so-called Catalan number (the set of p.b. trees of the order n has c_n elements).

Let Y_n be the set of trees of the order n as it appears in [5], [8] and [9]. We will call it upper-trees in this article. Observe that $Y_n \subset \mathbb{Y}_n$.

By $\mathbb{Y} = \bigoplus\limits_{n \geq 0} \mathbb{Y}_n$ the set of all p.b. bitrees is denoted.

We can divide the set of bitrees into three subsets: the set of upper-trees $\mathbb{Y}^{(u)}$, the set of lower-trees $\mathbb{Y}_{(d)}$ and the set of "real" bitrees $\mathbb{Y}^{(u)}_{(d)}$.

2.2.1. Operations on Planar Binary Bitrees

On p.b. bitrees the operation of *cutting* is defined, so that by cutting a p.b. bitree we get the upper-tree and the lower-tree (the inverse operation of cutting we will call it *welding*).

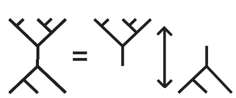


Figure 2: Cutting of p.b. bitree

For any bitree x we have $x = x_u \updownarrow x_d$.

For upper- and lower-trees the operation of *grafting* is defined identically as in [8]. The grafting of p.b. trees x and y (upper- or lower-) is a new p.b. tree, denoted by $x \vee y$, obtained by joining the two roots to a new vertex. Observe that if we graft upper- or lower-trees of orders k and l, we get the upper- or lower-tree of order $k + l + 1$. Grafting is a noncommutative operation.

Also, for "real" bitrees x and y, we define $x \vee y = (x_u \vee y_u) \updownarrow (x_d \vee y_d)$.

2.2.2. Involution on Bitrees

For p.b. bitrees a symmetry around the axis perpendicular to the root is defined. The symmetry τ is an involution on $\mathbb{Y}$.

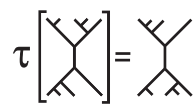


Figure 3: Involution τ

It is obvious that $\tau(x_u \updownarrow x_d) = \tau(x_u) \updownarrow \tau(x_d)$.

3. The Photon-Electron Propagator Hopf Algebra on Planar Binary Bitrees

In this section we introduce the non-commutative algebra on planar binary bitrees dual to the sets of electron and photon propagators. This algebra represents generalization of the electron and photon propagator Hopf algebras introduced in [2]. The only non-trivial parts of this algebra are coproduct and antipode.

3.1. The Product of Bitrees

In this subsection new operation on p.b. bitrees is introduced. This operation is generalization of the corresponding operation on planar binary trees.

Definition 1. For $x \in \mathbb{Y}_k$ and $y \in \mathbb{Y}_l$, the p.b. bitree $x * y$ in $\mathbb{Y}_{k+l}$ is obtained by identifying the root of x_u with the leftmost leaf of y_u, and by identifying the root of y_d with the leftmost leaf of x_d :

$$x * y := (x_u / y_u) \updownarrow (x_d \backslash y_d),$$

where "/" and "\" are operations "over" and "under" defined on planar binary trees (cf. [9]) and $x_d \backslash y_d := \tau(\tau(x_d) \backslash \tau(y_d))$.

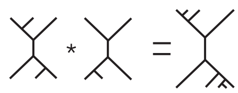


Figure 4: Product of two bitrees

It is obvious that defined graded product is associative (since operations over and under on planar binary trees are associative), non-commutative (for a similar reason) and the p.b. bitree | is the neutral element on both sides.

Remark 2. It is easy to see that operation of welding is essentially restriction of the defined product to pairs of upper- and lower-trees.

Also we can describe the product by recurrence relation by the following way:

$$x * y = ((x_u / y_u^l) \vee y_u^r) \updownarrow (x_d^l \vee (x_d^r \backslash y_d)),$$

and $x * | = | * x = x$.

3.2. The Prunning Coalgebra

We define the coproduct

$$\Delta^p : \mathbb{CY} \to \mathbb{CY} \otimes \mathbb{CY}$$

dual of the product $*$ in the following way:

$$\Delta^p(x) = \sum_{x = x' * x''} x' \otimes x'', \quad \text{and} \quad \Delta^p(|) = | \otimes |,$$

where the standard Sweedler's notation is used.

Naturally, the coproduct is a graded coassociative operation, and together with the counit ε (dual to the unit $|$), defined as $\varepsilon(|) = 1$ and $\varepsilon(x) = 0$ if $x \neq |$, define on $\mathbb{CY}$ structure of a graded coalgebra.

The coproduct breaks all the branches of bitrees which are on the left of the root and places them on the same side.

Observe that the coproduct is neither commutative nor co-commutative and that restriction of the coproduct to the pairs of upper- and lower-trees represents operation of welding on bitrees.

3.3. The Photon-Electron Propagator Hopf Algebra

If we extend the prunning coproduct by multiplicity on tensor product of bitrees, we obtain Hopf algebra $\mathcal{H}^{\mathrm{p}}$, which is neither commutative nor co-commutative. Therefore we set $\mathcal{H}^{\mathrm{p}} := \mathbb{C}\langle\mathbb{Y}\rangle/(1-|)$ as a free associative algebra on the set of bitrees where we identify the formal unit 1 with the bitree $|$.

We can define a total order on a tensor product of bitrees as the sum of the orders of the bitrees,

$$|x_1 \cdots x_k| = |x_1| \cdots |x_k|.$$

Then, the algebra $\mathcal{H}^{\mathrm{p}}$ is graded connected Hopf algebra, with homogeneous components

$$\mathcal{H}^{\mathrm{p}}{}_n = \bigoplus_{i_1+\ldots+i_k=n} \mathbb{C}\mathbb{Y}_{i_1} \otimes \cdots \otimes \mathbb{C}\mathbb{Y}_{i_k}.$$

In particular, the pruning antipode S^p is a graded algebra antimorphism automatically defined on bitrees by the recursive formula $S^p(|) = |$ and

$$S^p(x) = -x - \sum_{P(x)} S^p(x')x'' = -x - \sum_{P(x)} x' S^p(x''),$$

where $P(x) = \Delta^p(x) - x \otimes | - | \otimes x$.

Since S^p plays an explicit role in the renormalization of the electron propagator, we give a few examples in Table 2.

4. Charge Hopf Algebra on Bitrees

In this section the Hopf algebra on bitrees, as a generalization of the corresponding Hopf algebra on trees, is introduced. The Hopf algebra on trees was dual to the renormalization group acting on the electric charge.

The novelty here is a fact that each element of the non-commutative Hopf algebra on bitrees has a substructure determined by the shuffle relation.

Definition 3. For $x \in \mathbb{Y}_k$ and $y \in \mathbb{Y}_l$, the p.b. bitree $\mathbf{x}/\!\!/\mathbf{y}$ in $\mathbb{Y}_{k+l}$ is obtained by identifying the root of x_u with the leftmost leaf of y_u, and by identifying the root of x_d with the leftmost leaf of y_d

Let $\mathbb{C}\langle V(x_u), V(x_d)\rangle$ be the algebra of noncommutative polynomials generated by the upper-trees of the form $V(x_u) = | \vee x_u$ and the lower-trees of the form $V(x_d) = | \vee x_d$. We introduce an equivalence relation on this algebra on a following way: we define $[x] := u \sqcup d$ as the set of all elements

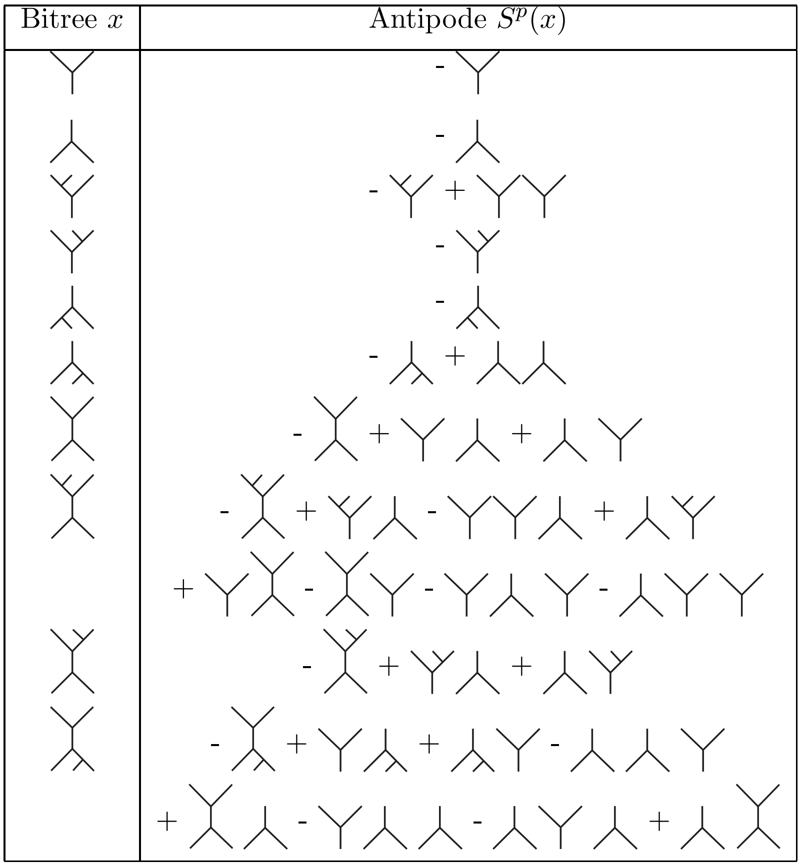


Table 2: Pruning antipodes

$x = x_1 \cdots x_{p+q} \in \mathbb{C}\langle V(x_u), V(x_d)\rangle$ such that there exists two complementary subsequences $i_1 < i_2 < \cdots < i_p$ and $j_1 < j_2 < \cdots < j_q$ of $[1, p+q]$, with

$$u = x_{i_1} \cdots x_{i_p} \in \mathbb{C}\langle V(x_u)\rangle \quad \text{and} \quad d = x_{j_1} \cdots x_{j_q} \in \mathbb{C}\langle V(x_d)\rangle.$$

Remark. In other words, $[x]$ is a shuffle product of u and d.

Furthermore, let $\mathcal{H}^{\mathrm{ch}} := \mathbb{C}[V(x_u), V(x_d)]$ be the polynomial algebra generated by the upper-trees of the form $V(x_u)$ and the lower-trees of the form $V(x_d)$.

Since each bitree (except upper- or lower-trees) $x \in \mathbb{Y}$ can be uniquely decomposed (up to the shuffle relation) as $x = (x_u^l /\!\!/ (| \vee x_u^r)) /\!\!/ (x_d^l /\!\!/ (| \vee x_d^r))$, the map $V(t) \mapsto V(t)$ (where $t = x_u$ or $t = x_d$) and $1 \mapsto |$ is an algebra isomorphism from $\mathcal{H}^{\mathrm{ch}}$ to the abelianization of $(\mathbb{C}\mathbb{Y}, /\!\!/)$.

Under the inverse of this isomorphism, the natural homogenous component $\mathbb{C}\mathbb{Y}_n$ of degree n corresponds to the subspace

$$\mathcal{H}^{\mathrm{ch}}{}_n = \bigoplus_{n_1 \leq \ldots \leq n_k} \mathbb{C}V(\mathbb{Y}_{n_1}) \otimes \cdots \otimes \mathbb{C}V(\mathbb{Y}_{n_k})$$

of total degree $n = n_1 + \ldots + n_k + k$ in $\mathcal{H}^{\mathrm{ch}}$.

From now on, we identify $\mathcal{H}^{\mathrm{ch}}$ with $(\mathbb{C}\mathbb{Y}, /\!\!/)_{ab}$, and represent the unit 1 as the root bitree $|$.

Furthermore, we define a coproduct $\Delta^{\mathrm{ch}} : \mathcal{H}^{\mathrm{ch}} \to \mathcal{H}^{\mathrm{ch}} \otimes \mathcal{H}^{\mathrm{ch}}$ and a coaction $\delta^{\mathrm{ch}} : \mathcal{H}^{\mathrm{ch}} \to \mathcal{H}^{\mathrm{ch}} \otimes \mathcal{H}^{\mathrm{ch}}$ as two linear operations satisfying the following recursive relations:

$$\Delta^{\mathrm{ch}}| = | \otimes |,$$
$$\Delta^{\mathrm{ch}} V(x_u) = | \otimes V(x_u) + \delta^{\mathrm{ch}} V(x_u), \quad \Delta^{\mathrm{ch}} V(x_d) = | \otimes V(x_d) + \delta^{\mathrm{ch}} V(x_d),$$
$$\Delta^{\mathrm{ch}}(x_u \vee y_u) = \Delta^{\mathrm{ch}} x_u /\!\!/ \Delta^{\mathrm{ch}} V(y_u), \quad \Delta^{\mathrm{ch}}(x_d \vee y_d) = \Delta^{\mathrm{ch}} x_d /\!\!/ \Delta^{\mathrm{ch}} V(y_d),$$
$$\Delta^{\mathrm{ch}}(x \vee y) = (\Delta^{\mathrm{ch}}(x_u \vee y_u)) \updownarrow (\Delta^{\mathrm{ch}}(x_d \vee y_d))$$

and

$$\delta^{\mathrm{ch}}| = | \otimes |,$$
$$\delta^{\mathrm{ch}} V(x_u) = (V \otimes \mathrm{Id})\delta^{\mathrm{ch}}(x_u), \quad \delta^{\mathrm{ch}} V(x_d) = (V \otimes \mathrm{Id})\delta^{\mathrm{ch}}(x_d),$$
$$\delta^{\mathrm{ch}}(x_u \vee y_u) = \Delta^{\mathrm{ch}} x_u /\!\!/ \delta^{\mathrm{ch}}(V(y_u)), \quad \delta^{\mathrm{ch}}(x_d \vee y_d) = \Delta^{\mathrm{ch}} x_d /\!\!/ \delta^{\mathrm{ch}}(V(y_d)),$$
$$\delta^{\mathrm{ch}}(x \vee y) = (\delta^{\mathrm{ch}}(x_u \vee y_u)) \updownarrow (\delta^{\mathrm{ch}}(x_d \vee y_d)).$$

The coproduct and the coaction on small generator bitrees are given in Table 3.

Let $\varepsilon : \mathcal{H}^{\mathrm{ch}} \to \mathbb{C}$ be the linear map which sends all the bitrees to 0 except the bitree $|$ which is sent to 1.

Theorem 4. *The algebra $\mathcal{H}^{\mathrm{ch}}$ is a graded connected commutative Hopf algebra. Moreover, δ^{ch} is a right Δ^{ch}-coaction, that is*

$$(\delta^{\mathrm{ch}} \otimes \mathrm{Id})\delta^{\mathrm{ch}} = (\mathrm{Id} \otimes \Delta^{\mathrm{ch}})\delta^{\mathrm{ch}}.$$

Proof. First of all, we observe that the coproduct preserves the grading of $\mathcal{H}^{\mathrm{ch}}$:

$$\Delta^{\mathrm{ch}}(\mathcal{H}^{\mathrm{ch}}{}_n) \subset \bigoplus_{p+q=n} \mathcal{H}^{\mathrm{ch}}{}_p \otimes \mathcal{H}^{\mathrm{ch}}{}_q.$$

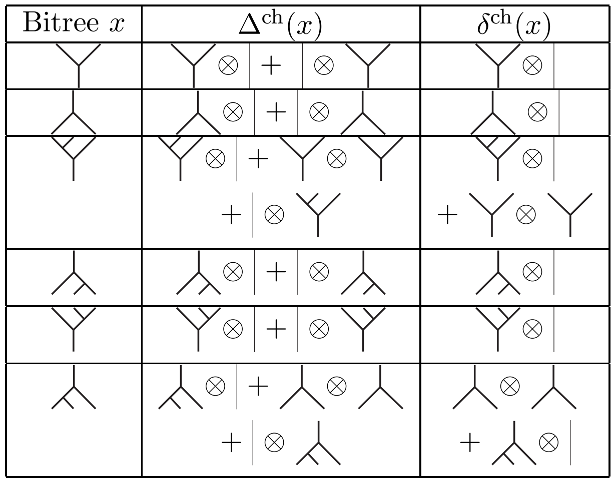


Table 3: Charge coproduct and coaction

Since ${\mathcal{H}^{\mathrm{ch}}}_0$ is spanned by a single bitree | the graded algebra $\mathcal{H}^{\mathrm{ch}}$ is connected.

The map ε is counit for Δ^{ch}, and the antipode $S^{\mathrm{ch}} : \mathcal{H}^{\mathrm{ch}} \to \mathcal{H}^{\mathrm{ch}}$ is a graded algebra isomorphism automatically defined on the generators by the recursive formula $S^{\mathrm{ch}}(|) = |$ and

$$S^{\mathrm{ch}}(x) = -x - \sum_{\bar{\Delta}^{\mathrm{ch}}(x)} S^{\mathrm{ch}}(x') /\!\!/ x''.$$

We prove by induction that the operator δ^{ch} defines a left Δ_{ch}-coaction of $\mathcal{H}^{\mathrm{ch}}$ on itself. The proof is quite analogical to the proof which was demonstrated in [2] for planar binary trees.

It is true on x=|. Suppose that is true for all upper- or lower-trees up to order n, and let $V(t)$ (where $t = x_u$ or $t = x_d$) has order $n+1$. Then

$$\begin{aligned}(\delta^{\mathrm{ch}} \otimes \mathrm{Id})\delta^{\mathrm{ch}}(V(t)) &= (\delta^{\mathrm{ch}} \circ V \otimes \mathrm{Id})\delta^{\mathrm{ch}}(t) = (V \otimes \mathrm{Id} \otimes \mathrm{Id})(\delta^{\mathrm{ch}} \otimes \mathrm{Id})\delta^{\mathrm{ch}}(t) \\ &= (V \otimes \mathrm{Id} \otimes \mathrm{Id})(\mathrm{Id} \otimes \Delta^{\mathrm{ch}})\delta^{\mathrm{ch}}(t) = (V \otimes \Delta^{\mathrm{ch}})\delta^{\mathrm{ch}}(t) = (\mathrm{Id} \otimes \Delta^{\mathrm{ch}})\delta^{\mathrm{ch}}(V(t)),\end{aligned}$$

Now let $s \vee t = s /\!\!/ V(t)$, (s and t are upper- or lower-trees) has order $n+1$, with $s \neq |$. Then both s and $V(t)$ have order smaller or equal to n. Using the Sweedler's notations

$$\Delta^{\mathrm{ch}}(s) = \sum s^{(1)} \otimes s^{(2)}, \qquad \delta^{\mathrm{ch}}(t) = \sum t^{l} \otimes t^{r},$$

we have

$$\begin{aligned}(\delta^{\mathrm{ch}}\otimes\mathrm{Id})\delta^{\mathrm{ch}}(s)/\!\!/V(t) &= (\delta^{\mathrm{ch}}\otimes\mathrm{Id})[\Delta^{\mathrm{ch}}(s)/\!\!/\delta^{\mathrm{ch}}V(t)]\\ &= (\delta^{\mathrm{ch}}\otimes\mathrm{Id})\sum_{\delta^{\mathrm{ch}}(t),\Delta^{\mathrm{ch}}(s)} s^{(1)}/\!\!/V(t^l)\otimes s^{(2)}/\!\!/t^r\\ &= \sum_{\delta^{\mathrm{ch}}(t),\Delta^{\mathrm{ch}}(s)} \Delta^{\mathrm{ch}}(s^{(1)})/\!\!/\delta^{\mathrm{ch}}(V(t^l))\otimes s^{(2)}/\!\!/t^r\\ &= [(\Delta^{\mathrm{ch}}\otimes\mathrm{Id})\Delta^{\mathrm{ch}}(s)]/\!\!/[(\delta^{\mathrm{ch}}\otimes\mathrm{Id})\delta^{\mathrm{ch}}V(t)],\end{aligned}$$

But, on the other hand:

$$(\mathrm{Id}\otimes\Delta^{\mathrm{ch}})\delta^{\mathrm{ch}}(s/\!\!/V(t)) = [(\mathrm{Id}\otimes\Delta^{\mathrm{ch}})\Delta^{\mathrm{ch}}(s)]/\!\!/[(\mathrm{Id}\otimes\Delta^{\mathrm{ch}})\delta^{\mathrm{ch}}V(t)],$$

so the equality holds by inductive hypothesis.

Now we prove by induction that the operator Δ^{ch} is coassociative, that is

$$(\mathrm{Id}\otimes\Delta^{\mathrm{ch}})\Delta^{\mathrm{ch}} = \Delta^{\mathrm{ch}}(\mathrm{Id}\otimes\Delta^{\mathrm{ch}}),$$

using the fact that δ^{ch} is a coaction. Since Δ^{ch} is multiplicative, we only need to prove it on the generators $V(t)$. It is true for x=|. Suppose that Δ^{ch} is coassociative on all upper- (and lower-) trees with order up to n, and $V(t)$ be a generator with order $n+1$. Then we have

$$\begin{aligned}(\mathrm{Id}\otimes\Delta^{\mathrm{ch}})\Delta^{\mathrm{ch}}(V(t)) &= |\otimes\Delta^{\mathrm{ch}}V(t) + (\mathrm{Id}\otimes\Delta^{\mathrm{ch}})\delta^{\mathrm{ch}}(V(t))\\ &= |\otimes|\otimes V(t) + |\otimes\delta^{\mathrm{ch}}(V(t)) + (\mathrm{Id}\otimes\Delta^{\mathrm{ch}})\delta^{\mathrm{ch}}(V(t)).\end{aligned}$$

On the other side

$$\begin{aligned}(\Delta^{\mathrm{ch}} &\otimes \mathrm{Id})\Delta^{\mathrm{ch}}(V(t)) = \Delta^{\mathrm{ch}}(|)\otimes V(t) + (\Delta^{\mathrm{ch}}\otimes\mathrm{Id})\delta^{\mathrm{ch}}(V(t))\\ &= |\otimes|\otimes V(t) + (\Delta^{\mathrm{ch}}\circ V\otimes\mathrm{Id})\delta^{\mathrm{ch}}(t)\\ &= |\otimes|\otimes V(t) + (\mathrm{Id}\otimes V\otimes\mathrm{Id})(|\otimes\delta^{\mathrm{ch}}(t)) + (\delta^{\mathrm{ch}}\circ V\otimes\mathrm{Id})\delta^{\mathrm{ch}}(t)\\ &= |\otimes|\otimes V(t) + |\otimes(V\otimes\mathrm{Id})\delta^{\mathrm{ch}}(t) + (\delta^{\mathrm{ch}}\otimes\mathrm{Id})(V\otimes\mathrm{Id})\delta^{\mathrm{ch}}(t)\\ &= |\otimes|\otimes V(t) + |\otimes\delta^{\mathrm{ch}}(V(t)) + (\delta^{\mathrm{ch}}\otimes\mathrm{Id})\delta^{\mathrm{ch}}(V(t))\,.\end{aligned}$$

Then, the two sides are equal because

$$(\mathrm{Id}\otimes\Delta^{\mathrm{ch}})\delta^{\mathrm{ch}}(V(t)) = (\delta^{\mathrm{ch}}\otimes\mathrm{Id})\delta^{\mathrm{ch}}(V(t)). \qquad \square$$

4.1. The Non-Commutative Charge Hopf Algebra

Let $\tilde{\mathcal{H}}^{\mathrm{ch}} := \mathbb{C}\langle V(x_u), V(x_d)\rangle/_{\mathrm{shuff}}$ be the quotient algebra of non-commutative polynomials generated by the upper-trees of the form $V(x_u) = | \vee x_u$ and the lower-trees of the form $V(x_d) = | \vee x_d$, with respect of "shuffle" relation. The charge algebra $\mathcal{H}^{\mathrm{ch}}$ is then the Abelian quotient of $\tilde{\mathcal{H}}^{\mathrm{ch}}$.

Moreover, the isomorphism $\mathcal{H}^{\mathrm{ch}} \tilde{\to} (\mathbb{C}\mathbb{Y}, /\!/)_{ab}$ can be lifted to an isomorphism $\tilde{\mathcal{H}}^{\mathrm{ch}} \tilde{\to} (\mathbb{C}\mathbb{Y}, /\!/)$. Therefore, the formulas which define a coproduct Δ^{ch} and a coaction δ^{ch} on $\mathcal{H}^{\mathrm{ch}}$ can be adopted to define some lifted maps $\tilde{\Delta}^{\mathrm{ch}}$ and a coaction $\tilde{\delta}^{\mathrm{ch}}$ from $\tilde{\mathcal{H}}^{\mathrm{ch}}$ to $\tilde{\mathcal{H}}^{\mathrm{ch}} \otimes \tilde{\mathcal{H}}^{\mathrm{ch}}$.

Theorem 5. *The algebra $\tilde{\mathcal{H}}^{\mathrm{ch}}$ is a graded connected Hopf algebra, which is neither commutative nor co-commutative.*

Proof. Since we never used the commutativity of the product in $\mathcal{H}^{\mathrm{ch}}$, we can repeat the proof of previous theorem. □

5. QED Hopf Algebra and Coaction on Bitrees

In this section we assemble the structures defined on bitrees in Section 3 and Section 4. The resulting maps describe the renormalization coactions on bitrees, for the photon-electron propagator.

5.1. The Photon-Electron Coaction

Since $\tilde{\mathcal{H}}^{\mathrm{ch}} \cong \mathbb{C}\mathbb{Y}$ as a vector space, the coaction $\tilde{\delta}$ on $\tilde{\mathcal{H}}^{\mathrm{ch}}$ can be seen as a linear map $\tilde{\delta} : \mathbb{C}\mathbb{Y} \to \mathbb{C}\mathbb{Y} \otimes \mathbb{C}\mathbb{Y}$. Since $\mathbb{Y}$ is the set of generators of the algebra $\mathcal{H}^{\mathrm{p}}$ and $\tilde{\delta}(|) = | \otimes |$, we can extend $\tilde{\delta}$ to the map $\delta^p : \mathcal{H}^{\mathrm{p}} \to \mathcal{H}^{\mathrm{p}} \otimes \mathcal{H}^{\mathrm{ch}}$, defined as $\tilde{\delta}$ on the generators, extended by multiplicatively on tensor products,

$$\delta^p(x_1 \cdots x_n) := \tilde{\delta}(x_1) \cdots \tilde{\delta}(x_n),$$

and finally passed to the quotient $\tilde{\mathcal{H}}^{\mathrm{ch}} \to \mathcal{H}^{\mathrm{ch}}$.

Lemma 6. *The map δ^p is a right Δ^{ch}-coaction, i.e. it satisfies*

$$(\delta^p \otimes Id)\delta^p = (Id \otimes \Delta^{\mathrm{ch}})\delta^p,$$

and it commutes with Δ^p, i.e. it satisfies

$$(\Delta^p \otimes Id)\delta^p = m^3_{24}(\delta^p \otimes \delta^p)\Delta^p.$$

Proof. The map δ^p is a right Δ^{ch}-coaction, because we proved already that identity

$$(\delta^p \otimes \mathrm{Id})\delta^p = (\mathrm{Id} \otimes \Delta^{\mathrm{ch}})\delta^p$$

holds on single bitrees, and on a product $x_1 \cdots x_n$, it follows from the fact that

$$(\delta^p \otimes \mathrm{Id})\delta^p(x_1 \cdots x_n) = [(\delta^p \otimes \mathrm{Id})\delta^p(x_1)] \cdots [(\delta^p \otimes \mathrm{Id})\delta^p(x_n)]$$

and similarly

$$(\mathrm{Id} \otimes \Delta^{\mathrm{ch}})\delta^p(x_1 \cdots x_n) = [(\mathrm{Id} \otimes \Delta^{\mathrm{ch}})\delta^p(x_1)] \cdots [(\mathrm{Id} \otimes \Delta^{\mathrm{ch}})\delta^p(x_n)].$$

Furthermore, we prove that δ^p commutes with Δ^p. It holds

$$(\Delta^p \otimes \mathrm{Id})\delta^p = m_{24}^3(\delta^p \otimes \delta^p)\Delta^p,$$

where m_{24}^3 multiplies what is on the position 2 by what is on the position 4 and puts it on the position 3. We prove it by induction on single bitrees. It is true for the bitree $|$, so let us suppose that equality holds for all bitrees up to order n, and let $x \vee y$ has order $n+1$. Then on the left side we have

$$\begin{aligned}(\Delta^p \otimes \mathrm{Id})\delta^p(x \vee y) &= \sum_{\Delta^{\mathrm{ch}}x, \delta^p y} \Delta^p(x^{(1)}) \vee y^{(p)} \otimes x^{(2)} /\!\!/ y^{(ch)} \\ &= \sum_{\Delta^{\mathrm{ch}}x, \delta^p y} | \otimes x^{(1)} \vee y^{(p)} \otimes x^{(2)} /\!\!/ y^{(ch)} \\ &+ \sum_{\Delta^{\mathrm{ch}}x, \delta^p y, \Delta^p y^{(p)}} x^{(1)} \vee y^{(p1)} \otimes y^{(p2)} \otimes x^{(2)} /\!\!/ y^{(ch)}\end{aligned}$$

and on the right side we have

$$m_{24}^3(\delta^p \otimes \delta^p)\Delta^p(x \vee y) = m_{24}^3(\delta^p \otimes \delta^p)[| \otimes x \vee y + \sum_{\Delta^p y} x \vee y^{(1)} \otimes y^{(2)}]$$

$$= m_{24}^3(\sum_{\Delta^{\mathrm{ch}}x, \delta^p y} | \otimes | \otimes x^{(1)} \vee y^{(1)} \otimes x^{(2)} /\!\!/ y^{(2)} + \sum_{\Delta^p y} \delta^p(x \vee y^{(1)}) \otimes \delta^p(y^{(2)}))$$

$$= \sum_{\Delta^{\mathrm{ch}}x, \delta^p y} | \otimes x^{(1)} \vee y^{(p)} \otimes x^{(2)} /\!\!/ y^{(ch)}$$

$$+ \sum_{\Delta^p y, \Delta^{\mathrm{ch}}x, \delta^p y^{(1)}, \delta^p y^{(2)}} x^{(1)} \vee y^{(1p)} \otimes y^{(2p)} \otimes x^{(2)} /\!\!/ y^{(1ch)} /\!\!/ y^{(2ch)}.$$

The two sides coincide, because for the bitree y holds

$$\sum_{\delta^p y, \Delta^p y^{(p)}} y^{(p1)} \otimes y^{(p2)} \otimes y^{(ch)} = \sum_{\Delta^p y, \delta^p y^{(1)}, \delta^p y^{(2)}} y^{(p1)} \otimes y^{(p2)} \otimes y^{(1ch)} /\!\!/ y^{(2ch)}.$$

The proof of assertion that equality holds on tensor product is quite analogous to the proof of similar assertion for planar binary trees, so we omit them here. □

5.2. QED Hopf Algebra

In [2] it has been explained that if we have two Hopf algebras $\mathcal{H}^{(1)}$ and $\mathcal{H}^{(2)}$ with multiplications $m^{(1)}$, $m^{(2)}$ and coproducts $\Delta^{(1)}$, $\Delta^{(2)}$, and if $\mathcal{H}^{(1)}$ coacts on $\mathcal{H}^{(2)}$ from the right and the coaction satisfies some conditions, then the semidirect product $\mathcal{H}^{(1)} \ltimes \mathcal{H}^{(2)}$ is a tensor algebra, and in the same time a coalgebra.

Moreover, if $\mathcal{H}^{(1)}$ is commutative, then $\mathcal{H}^{(1)} \ltimes \mathcal{H}^{(2)}$ is a Hopf algebra.

So, we conclude that in our case semidirect product $\mathcal{H}^{\text{qed}} := \mathcal{H}^{\text{ch}} \ltimes \mathcal{H}^{\text{p}}$ is graduated connected Hopf algebra, which is neither commutative nor cocommutative. The grading is given by the sum of the orders of all bitrees appearing in a monomial.

The coproduct $\Delta^{\text{qed}} : \mathcal{H}^{\text{qed}} \to \mathcal{H}^{\text{qed}} \otimes \mathcal{H}^{\text{qed}}$ is explicitly given by:

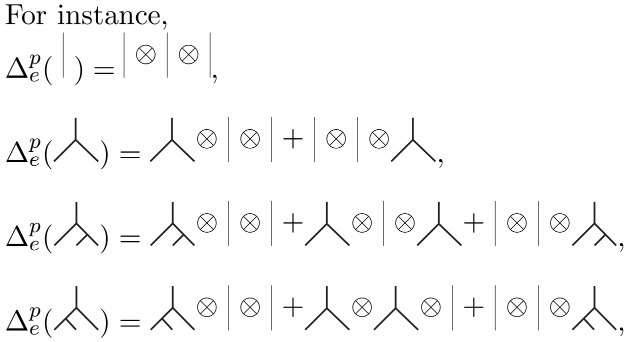
$$\Delta^{\text{qed}}(x \otimes y_1 \cdots y_n) := \Delta^{\text{ch}}(x)[(\delta^p \otimes \text{Id})\Delta^{\text{p}}(y_1 \cdots y_n)].$$

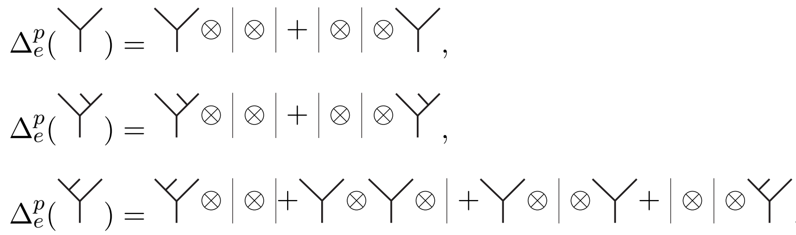
5.3. The Electron Renormalization Coaction

We define a coaction of $\mathcal{H}^{\text{qed}}$ on $\mathcal{H}^{\text{p}}$, as the map $\Delta_e^p : \mathcal{H}^{\text{p}} \to \mathcal{H}^{\text{p}} \otimes \mathcal{H}^{\text{qed}}$ given by

$$\Delta_e^p(x_1 \cdots x_n) := (\delta^p \otimes \text{Id})\Delta^p(x_1 \cdots x_n).$$

For instance,





Observe that electron renormalization coaction of bitrees restricted to the subspace of lower-trees coincides with the electron renormalization coaction of trees.

5.4. The Photon Renormalization Coaction

As we said, the semidirect coproduct $\mathcal{H}^{\mathrm{ch}} \ltimes \mathcal{H}^{\mathrm{p}}$ is a graded connected Hopf algebra, with twisted coproduct

$$x \otimes y_1 \cdots y_n \mapsto \Delta^{\mathrm{ch}}(x)[(\delta \otimes \mathrm{Id})\Delta^{\mathrm{p}}(y_1 \cdots y_n)],$$

which coacts on $\mathcal{H}^{\mathrm{p}}$ from the right, with coaction given by the restriction of the coproduct to the subspace $\mathcal{H}^{\mathrm{p}}$. However it is not the semidirect coproduct $\mathcal{H}^{\mathrm{ch}} \ltimes \mathcal{H}^{\mathrm{p}}$ which describes the renormalization of the photon propagators, than the photon renormalization Hopf algebra is the charge algebra $\mathcal{H}^{\mathrm{ch}}$.

Let $\sigma : \mathcal{H}^{\mathrm{p}} \to \mathcal{H}^{\mathrm{ch}}$ be the algebra morphism defined by

$$\sigma(x_1 \cdots x_n) := x_1 /\!\!/ \cdots /\!\!/ x_n.$$

Then we define $\Delta_\gamma^p := \mathcal{H}^{\mathrm{p}} \to \mathcal{H}^{\mathrm{p}} \otimes \mathcal{H}^{\mathrm{ch}}$ as the map

$$\Delta_\gamma^p := m_{23}^3(\delta^p \otimes \sigma)\Delta^p.$$

Since σ is an algebra morphism, Δ_γ^p is also an algebra morphism.

Lemma 7. *The map Δ_γ^p is a right coaction of $\mathcal{H}^{\mathrm{ch}}$ on $\mathcal{H}^{\mathrm{p}}$, i.e., it is coassociative with respect to Δ^{ch}.*

Proof. It is sufficient to show that the map σ interwines Δ^{ch} and Δ_γ^p, i.e. for any $x_1 \cdots x_n \in \mathcal{H}^{\mathrm{p}}$ we have

$$\Delta^{\mathrm{ch}}\sigma(x_1 \cdots x_n) = (\sigma \otimes \mathrm{Id})\Delta_\gamma^p(x_1 \cdots x_n).$$

If $n > 1$, the result follows from the fact that all the maps are algebra morphisms. So we have to check it on a single bitree. We prove it by induction on the order of the bitrees. The equality holds for $|$, suppose that it holds for a bitree x, that

$$\sum_{\Delta^{\mathrm{ch}} x} x^{(1)} \otimes x^{(2)} = \sum_{\Delta^p x, \delta^p x^{(1)}} \sigma(x^{(1p)}) \otimes x^{(2)} /\!\!/ x^{(1ch)} = \sum_{\Delta^p x, \delta^p x^{(1)}} x^{(1p)} \otimes x^{(2)} /\!\!/ x^{(1ch)}.$$

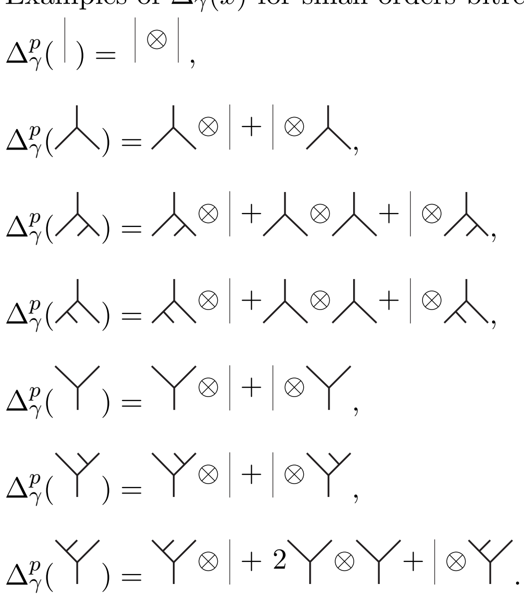
For a bitree $x \vee y$ of a larger order we have

$$\begin{aligned}(\sigma \otimes \mathrm{Id})\Delta^p_\gamma(x \vee y) &= (\sigma \otimes \mathrm{Id}) m^3_{23} (\delta^p \otimes \mathrm{Id})\Delta^p(x \vee y) \\ &= (\sigma \otimes \mathrm{Id}) m^3_{23} [\delta^p(x \vee y) \otimes | + \sum_{\Delta^p x} \delta^p(x^{(1)}) \otimes x^{(2)} \vee y] \\ &= \delta^p(x \vee y) + \sum_{\Delta^p x, \delta^p x^{(1)}} \sigma(x^{(1p)}) \otimes x^{(1ch)} /\!\!/ (x^{(2)} \vee y) \\ &= \delta^p(x \vee y) + \sum_{\Delta^{\mathrm{ch}} x} x^{(1)} \otimes x^{(2)} /\!\!/ V(y) = \Delta^{\mathrm{ch}}(x \vee y)\,. \quad \square\end{aligned}$$

Corollary 8. *The photon renormalization coaction Δ^p_γ restricted to the subspace of single upper-trees coincides with the noncommutative charge coproduct,*

$$\Delta^p_\gamma(x) = \tilde{\Delta}^{\mathrm{ch}}(x) \qquad \forall x \in \mathbb{Y}^{(u)}.$$

Examples of $\Delta^p_\gamma(x)$ for small orders bitrees can be then easy constructed:



References

[1] C. Brouder, On the trees of quantum fields, *Eur. Phys. J.C.*, **12** (2000), 535-549.

[2] C. Brouder, A. Frabetti, QED Hopf algebras on planar binary trees, *Journal of Algebra*, **267** (2003), 298-322.

[3] Z. Erjavec, *Planarna Binarna Bistabla i Primjene*, Ph.D. Thesis, University of Zagreb (2005).

[4] Z. Erjavec, An application of planar binary bitrees to prefix and huffman prefix code, *JIOS*, **28**, No. 1-2 (2004), 33-39.

[5] A. Frabetti, Simplicial properties of the set of planar binary trees, *J. Alg. Combin.*, **13**, No. 1 (2001), 41-65.

[6] A. Frabetti, C. Brouder, Renormalization of QED with planar binary trees, *Eur. Phys. J.C.*, **19** (2001), 715-741.

[7] C. Kassel, *Quantum Groups*, Springer-Verlag New York (1995).

[8] J.-L. Loday, Arithmetree, *Journal of Algebra*, **258**, No. 1 (2002), 275-309; math.CO/0112034.

[9] J.-L. Loday, M.O. Ronco, Hopf algebra of the planar binary trees, *Adv. in Maths.*, **139** (1998), 293-309.