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ON A CLASS OF $D-$HYPERGROUPOIDS

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Abstract: In this article, besides others, a 2-partition of Hartmanis $\mathbf{P}_2(Q)$ $[|l| \geq 3, l \in \mathbf{P}_2(Q)]$ is described as a $D-$hypergroupoid $(Q; A)$. In particular, if $|l| = 3$ for every $l \in \mathbf{P}_2(Q)$, then $\mathbf{P}_2(Q)$ is a Steiner triple system, and the corresponding hypergroupoid is essentially an idempotent $TS-$quasigroup.

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1. Preliminaries

Definitions 1.1. Let Q be a non-empty set and $P(Q)$ its power set. Let A be a mapping of the set Q^2 into the set $P(Q)$. Then we say:

a) we say that the mapping A is a *hyperoperation in* Q*;* and

b) we say that the ordered pair $(Q; A)$ is a *hypergroupoid.*

Remark. A notion of a *hypergroup* was introduced by F. Marty in [5] as a generalization of the notion of a group (see, also [1] and [7]).

Definition 1.2. Let $(Q; A)$ be a hypergroupoid and α a permutation of the set $\{1, 2, 3\}$. Also let

$$A^{\alpha}(x_1, x_2) \ni x_3 \overset{def}{=} A(x_{\alpha(1)}, x_{\alpha(2)}) \ni x_{\alpha(3)}$$

for all $x_1, x_2, x_3 \in Q$. Then $(Q; A^{\alpha})$ is a hypergroupoid.

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Definition 1.3. (see [7]) Let $(Q;A)$ be a hypergroupoid and α a permutation of the set $\{1,2,3\}$. Also let

$$A^{\alpha}(x_1,x_2)\ni x_3 \overset{def}{=} A(x_{\alpha(1)},x_{\alpha(2)})\ni x_{\alpha(3)}$$

for all $x_1,x_2,x_3\in Q$. Then we say that hyperoperation is an $\alpha-$*parastrofic hyperoperation to the hyperoperation* A.

Furthermore, let:

$A^{I}\overset{def}{=}A[I\overset{def}{=}\{(x,x)|x\in\{1,2,3\}\}]$,

${}^{-1}A(x_1,x_2)\ni x_3\overset{def}{\Leftrightarrow}A(x_3,x_2)\ni x_1$,

$A^{-1}(x_1,x_2)\ni x_3\overset{def}{\Leftrightarrow}A(x_1,x_3)\ni x_2$,

$A^{*}(x_1,x_2)\ni x_3\overset{def}{\Leftrightarrow}A(x_2,x_1)\ni x_3, \quad [A^{*}(x_1,x_2)=A(x_2,x_1)]$

${}^{-1}(A^{*})(x_1,x_2)\ni x_3\overset{def}{\Leftrightarrow}A(x_3,x_1)\ni x_2$, and

$(A^{*})^{-1}(x_1,x_2)\ni x_3\overset{def}{\Leftrightarrow}A(x_2,x_3)\ni x_1$,

for all $x_1,x_2,x_3\in Q$ (see also [2] or [3]).

Proposition 1.4. (see [7]) *Let $(Q;A)$ be a hypergroupoid and α a permutation of the set $\{1,2,3\}$. Then $\{A^{\alpha},{}^{-1}(A^{\alpha}),(A^{\alpha})^{-1},(A^{\alpha})^{*},{}^{-1}((A^{\alpha})^{*}),$ $(((A^{\alpha})^{*})^{-1}\}=\{A,{}^{-1}A,A^{-1},A^{*},{}^{-1}(A^{*}),(A^{*})^{-1}\}$.*

In [4] J. Hartmanis introduced the notion of a partition of type n $(n\in N)$ for sets having at least n distinct elements, by means of the following definition.

Definition 1.5. (see [4]) Let $|Q|\geq n,\ n\in N$ and

$$Q^{(n)}\overset{def}{=}\{\{a_1^n\}|\{a_1^n\}\subseteq Q\wedge|\{a_1^n\}|=n\}.$$

Then we say that $\mathbf{P}_n(Q)$ is a *partition of Q of the type* n (briefly: $n-$partitions of Q) iff the following statements hold:

H1. For each $C\in\mathbf{P}_n(Q)$ there is at least one $\{a_1^n\}\in Q^{(n)}$ such that $\{a_1^n\}\subseteq C$; and

H2. For each $\{a_1^n\}\in Q^{(n)}$ there is exactly one $C\in\mathbf{P}_n(Q)$ such that $\{a_1^n\}\subseteq C$.

Remark. Partitions of type 1 are the ordinary partitions of a set, and partitions of type 2 are *incidence geometries.*

2. TS-Hypergroupoids and D-Hypergroupoids

Definition 2.1. Let $(Q;A)$ be a hypergroupoid. Then we say that $(Q;A)$ is a *totally symmetric hypergroupoid* (briefly: TS-hypergoupoid) iff for any permutation α on $\{1,2,3\}$ we have $A^\alpha = A$.

Definition 2.2. We say that a hypergroupoid $(Q;A)$ is a *hypergroupoid with divisions* (briefly: D-hypergoupoid) iff the following statements hold:

For all $a, b \in Q$ $A(a,b) \neq \emptyset$,
For all $a, b \in Q$ ${}^{-1}A(a,b) \neq \emptyset$ and
For all $a, b \in Q$ $A^{-1}(a,b) \neq \emptyset$.

Remark. In [6]: hyperquasigroup. See also [7].

Remarks 2.3. a) Hypergroupoid $(Q;A)$ in Table 1 is a $TS-$ and $D-$hypergroupoid; and b) $(Q;B)$ in Table 2 is a $TS-$ hypergroupoid and not a $D-$hypergroupoid.

A	1	2
1	$\{1\}$	$\{2\}$
2	$\{2\}$	$\{1,2\}$

Table 1.

B	1	2	3	4	5	6
1	$\{1\}$	$\{3\}$	$\{2\}$	$\{6\}$	$\emptyset$	$\{4\}$
2	$\{3\}$	$\{2\}$	$\{1\}$	$\{5\}$	$\{4\}$	$\emptyset$
3	$\{2\}$	$\{1\}$	$\{3\}$	$\emptyset$	$\{6\}$	$\{5\}$
4	$\{6\}$	$\{5\}$	$\emptyset$	$\{4\}$	$\{2\}$	$\{1\}$
5	$\emptyset$	$\{4\}$	$\{6\}$	$\{2\}$	$\{5\}$	$\{3\}$
6	$\{4\}$	$\emptyset$	$\{5\}$	$\{1\}$	$\{3\}$	$\{6\}$

Table 2.

Proposition 2.4. (see [7]) *Let $(Q;A)$ be a $D-$hypergroupoid. Then for all $\alpha \in \{1,2,3\}!$ the hypergroupoid $(Q;A^\alpha)$ is a $D-$hypergroupoid.*

Proposition 2.5. *Hypergroupoid $(Q;A)$ is a $TS-$hypergroupoid iff the following equalities hold: ${}^{-1}A = A^{-1} = A$.*

Sketch of a Part of the Proof.

$$\begin{aligned} A^*(x,y) \ni z \quad &\Leftrightarrow A(y,x) \ni z \\ &\Leftrightarrow A^{-1}(y,z) \ni x \\ &\Leftrightarrow {}^{-1}(A^{-1})(x,z) \ni y \\ &\Leftrightarrow ({}^{-1}(A^{-1}))^{-1}(x,y) \ni z. \end{aligned}$$

$$\begin{aligned} {}^{-1}(A^*)(x,y) \ni z \quad &\Leftrightarrow A^*(z,y) \ni x \\ &\Leftrightarrow A(y,z) \ni x \\ &\Leftrightarrow {}^{-1}A(x,z) \ni y \\ &\Leftrightarrow ({}^{-1}A)^{-1}(x,y) \ni z. \end{aligned}$$

$$\begin{aligned} (A^*)^{-1}(x,y) \ni z \quad &\Leftrightarrow A^*(x,z) \ni y \\ &\Leftrightarrow A(z,x) \ni y \\ &\Leftrightarrow A^{-1}(z,y) \ni x \\ &\Leftrightarrow {}^{-1}(A^{-1})(x,y) \ni z. \end{aligned}$$

3. $< 2, 2 >$-Invertible Hypergroupoids

Definition 3.1. Let $(Q; A)$ be a hypergroupoid. Then we say that $(Q; A)$ is $< 2, 2 > -$*invertible* iff for each $a_1, a_2, b_1, b_2 \in Q$ the following statement holds:

$$|\{a_1^2, b_1^2\}| = 4 \ \wedge \ A(a_1^2) \supseteq \{b_1^2\} \ \wedge \ A(a_2, a_1) \supseteq \{b_1^2\}$$
$$\Rightarrow A(b_1^2) \supseteq \{a_1^2\} \ \wedge \ A(b_2, b_1) \supseteq \{a_1^2\}.$$

A	1	2	3	4	5	6
1	$\{1\}$	$\{3,4,5\}$	$\{4,5,6\}$	$\{2,6\}$	$\{4\}$	$\{2\}$
2	$\{3,4,5\}$	$\{2\}$	$\{4,5\}$	$\{5\}$	$\{6\}$	$\{1,4\}$
3	$\{4,5,6\}$	$\{4,5\}$	$\{3\}$	$\{1,2\}$	$\{1,2\}$	$\{5\}$
4	$\{2,6\}$	$\{5\}$	$\{1,2\}$	$\{4\}$	$\{1,2,3\}$	$\{1,3\}$
5	$\{4\}$	$\{6\}$	$\{1,2\}$	$\{1,2,3\}$	$\{5\}$	$\{1,3\}$
6	$\{5\}$	$\{1,4\}$	$\{2\}$	$\{1,3\}$	$\{1,3\}$	$\{6\}$

Table 3

B	1	2	3	4	5
1	$\{1\}$	$\{3,4,5\}$	$\{2\}$	$\{2\}$	$\{2\}$
2	$\{3,4,5\}$	$\{2\}$	$\{1\}$	$\{1\}$	$\{1\}$
3	$\{2\}$	$\{1\}$	$\{3\}$	$\{5\}$	$\{4\}$
4	$\{2\}$	$\{1\}$	$\{5\}$	$\{4\}$	$\{3\}$
5	$\{2\}$	$\{1\}$	$\{4\}$	$\{3\}$	$\{5\}$

Table 4

Remarks 3.2. a) Hypergroupoid $(\{1, \dots, 6\}; A)$ in Table 3 and hypergroupoid $(\{1, \dots, 5\}; B)$ in Table 4 are $D-$hypergroupoids and idempotent $(C(x, x) = \{x\}$ for all $x \in Q$; $(Q; C)$ hypergroupoid).

b) Hypergroupoid $(\{1, \dots, 6\}; A)$ in Table 3 is a $< 2, 2 > -$invertible hypergroupoid and not a $TS-$hypergroupoid $A(1, 6) = \{2\}$ and $A(6, 1) = \{5\}$.

c) Hypergroupoid $(\{1, \dots, 5\}; B)$ in Table 4 is a $TS-$hypergroupoid and not a $< 2, 2 > -$invertible hypergroupoid.

4. 2-Partitions and Hypergroupoids

Theorem 4.1. *Let $\mathbf{P}_2(Q)$ be a 2-partition of Q, and let $|l| \geq 3$ for all $l \in \mathbf{P}_2(Q)$. Also, let*

$$A(x_1^2) \overset{def}{=} \begin{cases} l \setminus \{x_1^2\} \ ; \ \{x_1^2\} \subseteq l, \ x_1 \neq x_2 \, , \\ \{x_1^2\} \ ; \ x_1 = x_2 \, , \end{cases} \tag{1}$$

where $l \in \mathbf{P}_2(Q)$. Then: (a) $(Q;A)$ is a $D-$hypergroupoid;

(b) $(Q;A)$ is a $TS-$hypergroupoid;

(c) $(Q;A)$ is a $<2,2>$ $-$inverible hypergroupoid; and (d) $(Q;A)$ is an idempotent hypergroupoid.

Proof. By (1), Definition 1.6 and Proposition 2.5. □

Remarks 4.2. a) If $|l| = 3$ for every $l \in \mathbf{P}_2(Q)$, then $\mathbf{P}_2(Q)$ is a Steiner triple system (cf. [1–3]).

b) In [6]:

$$A(x_1^2) \overset{def}{=} \begin{cases} l\ ;\ \{x_1^2\} \subseteq l,\ x_1 \neq x_2\,, \\ \{x_1^2\}\ ;\ x_1 = x_2\,. \end{cases}$$

Theorem 4.3. *Let $(Q;A)$ be a hypergroupoid such that the statements $(a)-(d)$ from Theorem 4.1 hold. Also, let:*

$l_{\{x_1^2\}} \overset{def}{=} A(x_1^2) \cup \{x_1^2\}$ *for all* $\{x_1^2\} \in Q^{(2)}$ *(cf. 1.5). Moreover, let* $\mathcal{L}_A(Q) \overset{def}{=} \{l_{\{x_1^2\}} | \{x_1^2\} \in Q^{(2)}\}$.

Then:

1) $\mathcal{L}_A(Q)$ is a 2-partition of Q; and

2) $|l| \geq 3$ for all $l \in \mathcal{L}_A(Q)$.

Proof. a) By $(a), (b)$ and (d), we have 2).

b) By a), we obtain H1.

c) By (a)-(c), we have H2.

d) By b) and c), we obtain 1). □

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