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FIXED POINTS FOR WEAK COMPATIBLE TYPE AND PARAMETRICALLY $\phi(\epsilon, \delta; a)$-CONTRACTION MAPPINGS

H.K. Pathak[1], R.K. Verma[2], S.M. Kang[3 §], M.S. Khan[4]

[1]Department of Mathematics
Kalyan Mahavidyalaya
Bhilai Nagar-490006, C.G., INDIA
[1]e-mail: hkpathak@sify.com

[2]Department of Mathematics
Govt. C.L.C. College
Patan, Distt.-Durg -491111, C.G., INDIA

[3]Department of Mathematics
Research Institute of Natural Science
Gyengsang National University
#900 Gazwa-Dong, Chinju 660-701, KOREA
e-mail: smkang@nongae.gsnu.ac.kr

[4]Department of Mathematics and Computing
Sultan Qaboos University
P.O.Box 36, Postal Code 123
Alkhod, Muscat, SULTANATE OF OMAN
e-mail: mohammad@squ.edu.om

Abstract: In this paper, we introduce the notion of parametrically $\phi(\epsilon, \delta; a)$-contraction mappings. In the sequel, we obtain common fixed point and coincidence results for family of four self-mappings of a metric space satisfying weak compatibility and parametrically $\phi(\epsilon, \delta; a)$-contractions. Another coinci-

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§Correspondence author

dence and common fixed point result is obtained for parametrically $\phi(\epsilon, \delta; a)$-contraction on star-shaped compact subsets of a normed linear space X satisfying weak compatible conditions on X. Our results generalize the result of Jungck [4], Pathak et al [15], etc.

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1. Introduction and Preliminaries

Let (X, d) be be a metric space. A mapping $T : X \to X$ is said to be *nonexpansive* if for all $x, y \in X$,

$$d(Tx, Ty) \leq d(x, y).$$

The first fixed point theorems for nonlinear nonexpansive mappings in noncompact settings were proved by Browder [1] and Göhde [2] independently. They proved that a nonexpansive self-mappings of a bounded closed convex subset C of a uniformly Banach space X has a fixed point. Kirk [7], [8] proved the same result under reflexive Banach space X with a closed bounded convex subset K having normal structure in X. Since then, a number of generalizations of nonexpansive mappings and their fixed point theorems have been discussed by many authors.

In 1969, Meir and Keeler [9] obtained a remarkable generalization of Banach contraction principle. This is known as (ϵ, δ)-contraction mapping. This was further generalized by many authors for one pair of mappings and for three mappings (see [11], [13] and [16]).

The Meir-Keeler type (ϵ, δ)-contraction mapping was further generalized by giving the notion of (ϵ, δ, a)-contraction for two pairs of self-mappings by Pathak [15].

We give the following definitions.

Definition 1.1. Two self-maps A and B of a metric space (X, d) are said to be $(\epsilon, \delta; a)$-*contraction* with respect to mappings $S, T : X \to X$ if for $AX \subseteq TX$, $BX \subseteq SX$, there is a function $\delta : (0, \infty) \to (0, \infty)$ such that $\delta(\epsilon) > \epsilon$ for all $\epsilon > 0$, $\frac{1}{2} < a \leq 1$ and for all $x, y \in X$,

$$\epsilon \leq ad(Sx, Ty) + (1 - a)d(Sx, Ax) < \delta(\epsilon)$$

$$\text{implies} \quad ad(Ax, By) + (1-a)d(By, Ty) < \epsilon, \tag{1.1}$$

where $d(Sx, Ax) \leq d(By, Ty)$,

$$Ax = By \quad \text{whenever} \quad Sx = Ty. \tag{1.2}$$

Remark 1.1. Of course, if $a = 1$ we shall say that A and B are (ϵ, δ)-contractions with respect to S and T (see [3]).

Remark 1.2. If $a = 1$, $A = B$, $S = T$ and $\delta(\epsilon) = \epsilon + h(\epsilon)$, where $h(\epsilon) > 0$ for given $\epsilon > 0$, then Definition 1.1 reduces to (ϵ, δ)-contraction of Meir-Keeler type for a pair of self-mappings (see [9]).

Definition 1.2. (see [15]) Let A, B, S and T be self-mappings of a metric space (X, d). The pair (A, B) is said to be *parametrically nonexpansive* with respect to the pair (S, T) if

$$ad(Ax, By) + (1-a)d(By, Ty) \leq ad(Sx, Ty) + (1-a)d(Sx, Ax) \tag{1.3}$$

for all $x, y \in X$, where $\frac{1}{2} < a \leq 1$.

Remark 1.3. Of course if $a = 1$, $A = B$ and $S = T$, we shall say that A is *S-nonexpansive* if S is continuous (see Park [12]). In general case, we shall say that A is parametrically nonexpansive with respect to S.

Remark 1.4. It is to be noted that the ordering of mapping-pair in this definition is crucial. Thus the pair (A, B) may be parametrically nonexpansive relative to the pair (S, T), whereas the pair (B, A) may not be parametrically nonexpansive relative to the pair (S, T) (see Remark 3.1 of Jungck [3]).

In 1998, Jungck and Rhoades [6] introduced the notion of weak compatible pair of mappings.

Definition 1.3. (see [6]) A pair of mappings (A, S) is called *weakly compatible* if they commute at their coincidence points.

From above definition it is clear that $STt = TSt$ whenever $St = Tt$. The following Example 1.1 shows that the *continuity* is not a necessary condition for a pair of mappings to be weakly-compatible.

Example 1.1. Let $X = [0, 4]$ be equipped with the usual metric $d(x, y) = |\, x - y \,|$. Define $S, T : [0, 4] \to [0, 4]$ by

$$Sx = \begin{cases} 2 & \text{if } x = 2, \\ 3 & \text{if } x \neq 2, \end{cases} \quad \text{and} \quad Tx = \begin{cases} 2 & \text{if } x = 2, \\ 6 - x & \text{if } x \neq 2. \end{cases}$$

Then pair (S,T) is weak compatible at $x=2$ and $x=3$ as $TS2=ST2$ for $T2=S2$ and $TS3=ST3$ for $S3=T3$. Both the mappings are discontinuous at $x=2$ but continuous at $x=3$.

The following example shows that, in a weak-compatible pair:

(1) one mapping may be continuous and the other may be discontinuous at the point of weak-compatibility, or

(2) both the mappings may be discontinuous at the point of weak compatibility as shown in the following Example 1.2.

Example 1.2. Let $X=R$ and d be the usual metric on X. Define mappings $S,T:X\to X$ by

$$Sx = \begin{cases} 2x^2 & \text{if } x\in(-\infty,1), \\ x^2-1 & \text{if } x\in[1,\infty), \end{cases}$$

$$Tx = \begin{cases} 0 & \text{if } x \text{ is a rational number}, \\ 1 & \text{if } x \text{ is an irrational number}. \end{cases}$$

Here $x=\sqrt{2}$ is a point of weak-compatibility, since $S\sqrt{2}=1=T\sqrt{2}$ and $ST\sqrt{2}=0=TS\sqrt{2}$. But S is continuous and T is discontinuous at the point of weak-compatibility.

Similarly, $x=1$ is a point of weak-compatibility, since S and T commutes at their coincidence point $(x=1)$ as, $S1=T1=0$ and $ST1=TS1=0$. Here S and T are discontinuous at $x=1$ since $S(1+0)=0\neq S(1-0)=2$.

Therefore it may be say that in a weak compatible pair, it is not necessary whether one or both the mappings taken are continuous or discontinuous. This indicates that weak-compatibility of pair of mappings is independent from the continuity of mappings chosen.

Further, it may be observed from Example 1.1 that pair (S,T) is compatible. Suppose $\{x_n\}=\{3+\frac{1}{2n}\}$ in X, then $STx_n, TSx_n\to t=3$ whenever $Sx_n, Tx_n\to t\in X$.

Similarly in the following Example 1.3, the pair (S,T) is noncompatible but it is weak compatible at the coincidence point $x=2$.

Example 1.3. Let $X=[2,20]$ and d be the usual metric on X. Define mappings $S,T:X\to X$ by

$$Sx = \begin{cases} 2 & \text{if } x=2 \text{ or } >5, \\ 6 & \text{if } 2<x\leq 5, \end{cases} \quad \text{and} \quad Tx = \begin{cases} 2 & \text{if } x=2, \\ 12 & \text{if } 2<x\leq 5, \\ x-3 & \text{if } x>5. \end{cases}$$

Define a sequence $\{x_n\}$ by $\{x_n\} = \{5+\frac{1}{n}\}$, $n \geq 1$. Then $Tx_n \to 2$, $Sx_n = 2$, $TSx_n = 2$ and $STx_n = 6$.

Thus the pair (S, T) is noncompatible. But they are weak compatible since they commute at their coincidence point $x = 2$.

Therefore weak-compatibility does not necessarily imply the compatibility (or the noncompatibility). But only weak-compatibility does not assuers the fixed point of the mapping pair.

Definition 1.4. (see [15]) A subset C of a linear space X is said to be *star-shaped* with respect to q (or shortly *star-shaped*) if there exists $q \in C$ such that

$$kx + (1-k)q \in C \tag{1.4}$$

for any $k \in [0, 1]$ and $x \in C$. Of course, if C is convex, then it is star-shaped with respect to $q \in C$. Here q is called the *star-center* of C.

Now we introduce the notion of parametrically $\phi(\epsilon, \delta; a)$-contraction mappings.

Definition 1.5. Let A, B, S and T be self-maps of a metric space (X, d) such that $AX \subseteq TX$ and $BX \subseteq SX$. Let us define a function $\delta{:}(0,\infty){\to}(0,\infty)$ such that $\delta(\epsilon) > \epsilon$ for all $\epsilon > 0$. The pair (A, B) is said to be *parametrically $\phi(\epsilon, \delta; a)$-contraction* with respect to the pair (S, T) if for some $a \in (\frac{1}{2}, 1]$ and for all $x, y \in X$, the following conditions satisfy

$$ad(Ax, By) + (1-a)d(By, Ty) \leq \phi(ad(Sx, Ty) + (1-a)d(Ax, Sx)), \tag{1.5}$$

where $\phi : R_+ \to R_+$ such that

(a) ϕ is continuous,

(b) $\phi(t) < t$ for all $t > 0$,

(c) $\epsilon \leq d(By, Ty) < \delta(\epsilon) \Rightarrow \phi(d(Ax, Sx)) < \epsilon$,

(d) $\phi(0) = 0$.

Remark 1.5. Here we observe that the following cases are the special cases of our notion viz. parametrically $\phi(\epsilon, \delta; a)$-contraction. Suppose ϕ is a continuous mapping as stated in (1.5)(a) above.

(1) If $\phi(t) = t$ for all $t > 0$, then the pair (A, B) is parametrically nonexpansive with respect to the pair (S, T) (see Pathak, Khan and Kang [15]).

(2) If $\phi(t) = t$ for all $t > 0$ and $\phi(0) = 0$ whenever $a = 1$, then the pair (A, B) is (ϵ, δ)-contraction with respect to the pair (S, T) (see Jungck [3]).

(3) If $\phi(0) = 0$ whenever $a = 1$ and $\delta(\epsilon) = \epsilon + h(\epsilon)$, where $h(\epsilon) > 0$ and if $A = B$ and $S = T$, our notion reduces to well known (ϵ, δ)-contraction of Meir-Keeler type mapping pair (see Meir-Keeler [9]).

(4) If $\phi(t) = t$ for all $t > 0$ and $\phi(0) = 0$ whenever $a = 1$ and $A = B$, $S = T$ and S is continuous, we shall say that A is S-nonexpansive if S is continuous (see Park [12]).

Remark 1.6. Suppose A and B are parametrically $\phi(\epsilon, \delta, a)$-contraction with respect to S and T. Then since $AX \subseteq TX$ and $BX \subseteq SX$ for any $x_0 \in X$ and $n \in N$, we can define inductively a sequence $\{x_n\}$ in X as follows:

$$y_{2n-1} = Tx_{2n-1} = Ax_{2n-2} \quad \text{and} \quad y_{2n} = Sx_{2n} = Bx_{2n-1}. \tag{1.6}$$

We shall call this as (S,T)-*iteration* on X.

2. Main Results

Before giving our main results we need the following lemmas.

Lemma 2.1. *Let (X, d) be a metric space and the mappings A, B be parametrically $\phi(\epsilon, \delta, a)$-contration with respect to S,T. If $x_0 \in X$ and $\{y_n\}$ be an (S,T)-iteration on x_0 under A and B, then for each $\epsilon > 0$ there exist $\delta(\epsilon) >0$ such that*

$$\begin{aligned} & ad(y_{p+1}, y_{q+1}) + (1-a)d(y_{q+1}, y_q) \\ \leq \; & \phi(ad(y_p, y_q) + (1-a)d(y_p, y_{p+1})) \end{aligned} \tag{2.1}$$

provided p and q are of opposite parties.

Proof. Since $AX \subseteq TX$ and $BX \subseteq SX$, we can define the (S,T)-iteration on x_0 under A and B as in (1.6).

So we have

$$\begin{aligned} & ad(Ax_{2n}, Bx_{2m-1}) + (1-a)d(Bx_{2m-1}, Tx_{2m-1}) \\ = \; & ad(y_{2n+1}, y_{2m}) + (1-a)d(y_{2m}, y_{2m-1}) \\ = \; & ad(y_{p+1}, y_{q+1}) + (1-a)d(y_{q+1}, y_q) \end{aligned}$$

and

$$\begin{aligned} & ad(Sx_{2n}, Tx_{2m-1}) + (1-a)d(Sx_{2n}, Ax_{2n}) \\ = \; & ad(y_{2n}, y_{2m-1}) + (1-a)d(y_{2n}, y_{2n+1}) \\ = \; & ad(y_p, y_q) + (1-a)d(y_p, y_{q+1}), \end{aligned}$$

where $p = 2n$ and $q = 2m - 1$. Hence, from (1.5) it follows that

$$ad(y_{p+1}, y_{q+1}) + (1-a)d(y_{q+1}, y_q) \leq \phi(ad(y_p, y_q) + (1-a)d(y_p, y_{p+1})),$$

where p and q are of opposite parties. This completes the proof. □

Lemma 2.2. *Let A, B, S and T be self-maps of a metric space (X, d) satisfying the conditions of Lemma 2.1. Then*

$$\lim_{n\to\infty} d(y_n, y_{n+1}) = 0.$$

Proof. Let x_0 be an arbitrary point of X. For $p = 2n$ and $q = 2n-1$, the conditions (2.1) and (1.5)(b) yields

$$\begin{aligned} & ad(y_{2n+1}, y_{2n}) + (1-a)d(y_{2n}, y_{2n-1}) \\ \leq\ & \phi(ad(y_{2n}, y_{2n-1}) + (1-a)d(y_{2n}, y_{2n+1})) \\ \leq\ & ad(y_{2n}, y_{2n-1}) + (1-a)d(y_{2n}, y_{2n+1}) \end{aligned}$$

or, equivalently,

$$(2a-1)d(y_{2n+1}, y_{2n}) < (2a-1)d(y_{2n}, y_{2n-1}),$$

that is,

$$d(y_{2n+1}, y_{2n}) < d(y_{2n}, y_{2n-1}).$$

Similarly for $p = 2n+1$ and $q = 2n$, from (2.1) we have

$$d(y_{2n+2}, y_{2n+1}) < d(y_{2n+1}, y_{2n}).$$

Thus we obtain a sequence $\{y_n\}$ of monotone decreasing, which converges to the greatest lower bound of its range $t \geq 0$.

We claim that $t = 0$. Otherwise, from condition (1.5)(c), for $x = x_{2n}$ and $y = x_{2n-1}$ we have

$$\epsilon \leq d(Bx_{2n-1}, Tx_{2n-1}) < \delta(\epsilon) \quad \Rightarrow \quad \phi(d(Ax_{2n}, Sx_{2n})) < \epsilon,$$

that is,

$$\epsilon \leq d(y_{2n}, y_{2n-1}) < \delta(\epsilon) \quad \Rightarrow \quad \phi(d(y_{2n+1}, y_{2n})) < \epsilon.$$

Making $n \to \infty$ and letting $\lim_{n\to\infty} d(y_n, y_{n+1}) = L$ we have

$$\epsilon \leq L < \delta(\epsilon) \quad \Rightarrow \quad \phi(L) < \epsilon,$$

i.e.,

$$\phi(L) < \epsilon \leq L < \delta(\epsilon).$$

Using (1.5) and since ϕ is continuous, letting $n \to \infty$ we have

$$L \leq \phi(L)\,.$$

Hence

$$L \leq \phi(L) < \epsilon \leq L < \delta(\epsilon) .$$

Thus we obtain a contradiction. This shows that

$$\lim_{n\to\infty} d(y_n, y_{n+1}) = L = 0.$$

This completes the proof. □

Lemma 2.3. *Let the mappings A, B, S and T be as in Lemma 2.2. Then the sequence $\{y_n\}$ or the (S,T)-iteration (1.6) defined on X is a Cauchy sequence.*

Proof. Let us assume that the sequence $\{y_n\}$ or the (S,T)-iteration as defined by (1.6) on X is not a Cauchy sequence. Then there exists an $\epsilon' > 0$ and a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ such that $d(y_{n_i}, y_{n_{i+1}}) \geq 2\epsilon'$.

Since $\lim_{n\to\infty} d(y_n, y_{n+1}) = 0$, there exists an integer m_i satisfying $n_i < m_i < n_{i+1}$ such that $d(y_{n_i}, y_{m_i}) \geq \epsilon'$.

If not, then

$$\begin{aligned} d(y_{n_i}, y_{n_{i+1}}) &\leq d(y_{n_i}, y_{n_{i+1}-1}) + d(y_{n_{i+1}-1}, y_{n_{i+1}}) \\ &< \epsilon' + d(y_{n_{i+1}-1}, y_{n_{i+1}}) < 2\epsilon', \end{aligned}$$

which is a contradiction. If m_i be the smallest integer such that $d(y_{n_i}, y_{m_i}) > \epsilon'$, then

$$\begin{aligned} \epsilon' &\leq d(y_{n_i}, y_{m_i}) \\ &\leq d(y_{n_i}, y_{m_i-2}) + d(y_{m_i-2}, y_{m_i-1}) + d(y_{m_i-1}, y_{m_i}) \\ &< \epsilon' + d(y_{m_i-2}, y_{m_i-1}) + d(y_{m_i-1}, y_{m_i}), \end{aligned}$$

that is, there exist integer m_i satisfying $n_i < m_i < n_{i+1}$ such that $d(y_{n_i}, y_{m_i}) \geq \epsilon'$ and

$$\lim_{n_i\to\infty} d(y_{n_i}, y_{m_i}) = \epsilon'. \tag{2.2}$$

Without loss of generality we can assume that n_i is odd and m_i is even. Now by virtue of (2.1), we have

$$\begin{aligned} & ad(y_{n_i+1}, y_{m_i+1}) + (1-a)d(y_{m_i+1}, y_{m_i}) \\ \leq\ & \phi(ad(y_{n_i}, y_{m_i}) + (1-a)d(y_{n_i}, y_{n_i+1})). \end{aligned}$$

Now on letting $n_i \to \infty$ and in view of (2.2) and $\phi(t) < t$ for all $t > 0$, the above relation yields

$$a.\epsilon' + (1-a)0 \leq \phi(a\epsilon' + (1-a)0) < a\epsilon',$$

which is a contradiction.

Thus the sequence $\{y_n\}$ or the (S,T)-iteration as defined by (1.6) on X is a Cauchy sequence. This completes the proof. □

Now, we give the main theorems.

Theorem 2.1. *Let S and T be self-maps of a metric space (X,d) and the pair (A,B) is parameyrically $\phi(\epsilon,\delta;a)$-contration with respect to the mappings (S,T). Let TX is complete,then there exist $u,v,w \in X$ such that*

$$Au = Su = w = Bv = Tv. \tag{2.3}$$

Furthermore, if the pair (A,S) (resp. (B,T)) is weakly compatible, then w may be chosen so that

$$Aw = Sw = w \quad (\text{resp. } Bw = Tw = w). \tag{2.4}$$

If both the pairs $(A,S)$and (B,T) are weakly compatible, then w is the unique common fixed point of the mappings A, B, S and T.

Proof. Let us make a sequence $\{y_n\}$ of (S,T)-iteration as in (1.6). Then by Lemma 2.3, the sequence $\{y_n\}$ is Cauchy. Therefore the subsequence $\{y_{2n-1}\}$ of $\{y_n\}$, which is in TX is also Cauchy. Since TX is complete, $\{y_{2n-1}\}$ converges to a point $w = Tv$ for some $w \in X$. Thus $y_n \to w$.

We now prove that $w = Bv$. For, $x = x_{2n}$ and $y = v$ the condition (1.5) is

$$ad(Ax_{2n}, Bv) + (1-a)d(Bv,Tv) \leq \phi(ad(Sx_{2n},Tv) + (1-a)d(Ax_{2n},Sx_{2n}))$$

or,

$$ad(y_{2n+1}, Bv) + (1-a)d(Bv,w) \leq \phi(ad(y_{2n},w) + (1-a)d(y_{2n+1},y_{2n})).$$

Letting $n\to\infty$ we have

$$ad(w,Bv) + (1-a)d(Bv,w) \leq \phi(ad(w,w) + (1-a)d(w,w)),$$

i.e.,

$$d(Bv,w) \leq \phi(0) = 0.$$

Thus $Bv = w$,Therefore $w = Bv = Tv$. But, since $BX \subseteq SX$, there exist $u \in X$ such that $Su = Bv = Tv$. Therefore $w = Bv = Tv = Su$.

We claim $Au = w$. Otherwise from condition (1.5), for $x = u$ and $y = v$ we get

$$ad(Au,w) \leq \phi((1-a)d(Au,w)) < (1-a)d(Au,w),$$

that is, $(2a-1)d(Au,w)<0$, which is a contradiction. Therefore $Au=w$. Thus we get $Au=Su=w=Bv=Tv$.

Further, suppose the pair (A,S) is weak compatible. Then by definition the pair is commuting at their coincidence point u. So that $ASu=SAu$ or $Aw=Sw$ as $Au=w=Su$.

We claim that $Aw=w$. Otherwise, for $x=w$ and $y=v$ in condition (1.5), we get

$$ad(Aw,Bv)+(1-a)d(Bv,Tv)\leq\phi(ad(Sw,Tv)+(1-a)d(Aw,Sw)),$$

that is,

$$\begin{aligned} ad(Aw,w)+(1-a)0 &\leq \phi(ad(Aw,w)+(1-a)0) \\ &= \phi(ad(Aw,w))<ad(Aw,w), \end{aligned}$$

which is a contradiction. Thus $Aw=w=Sw$.

Similarly, $w=Bw=Tw$ provided that B and T are weak compatible. Therefore we get $Aw=w=Sw$ (resp. $Bw=w=Tw$).

However if both the pairs (A,S) and (B,T) are weak compatible, then both the pairs are commuting at their coincidence point.

Suppose $Aw=Sw=w$ and $Bw'=Tw'=w'$, and if $w\neq w'$, then using (1.5), we get

$$\begin{aligned} & ad(Aw,Bw')+(1-a)d(Bw',Tw') \\ \leq\ & \phi(ad(Sw,Tw')+(1-a)d(Aw,Sw)) \\ =\ & \phi(ad(Aw,Bw')) \\ <\ & ad(Aw,Bw'), \end{aligned}$$

which is a contradiction. Therefore w is a common fixed point of A, B, S and T. Thus we get $Aw=Sw=w=Tw=Bw$.

Uniqueness of fixed point follows easily. For, if $w''(\neq w)$ is another common fixed point, then we have

$$\begin{aligned} ad(w,w'') &= ad(w,w'')+(1-a)d(w'',w'') \\ &= ad(Aw,Bw'')+(1-a)d(Bw'',Tw'') \\ &\leq \phi(ad(Sw,Tw'')+(1-a)d(Aw,Sw)) \\ &= \phi(ad(w,w'')+(1-a)d(w,w)) \\ &= \phi(ad(w,w'')) \\ &< ad(w,w''), \end{aligned}$$

which is a contradiction. Thus w is the unique common fixed point of A, B, S and T. This completes the proof. □

We now slightly modify the Theorem 2.1 to obtain the following result.

Corollary 2.1. *Let A, B, S and T be self-maps of a complete metric space (X, d) and the pair (A, B) is parametrically $\phi(\epsilon, \delta, a)$-contraction with respect to the pair (S, T) as defined by (1.6). If S and T are surjective then there exist $u, v, w \in X$ such that $Au = Su = w = Bv = Tv$. If, moreover, the pairs (A, S) and (B, T) are each weak compatible, then A, B, S and T have a unique common fixed point.*

The following Theorem 2.2 is proved for a star-shaped compact subset C of a normed linear space X. The mapping pair (S, T) is surjective. The parametrically $\phi(\epsilon, \delta, a)$-contraction and weak-compatibility is used to prove the theorem.

Theorem 2.2. *Let A, B, S and T be self-maps of a compact subset C of a normed linear space $(X, \|.\|)$. Suppose that C is star-shaped with respect to $q \in C$, and S and T are surjective and there exist $y \in X$ such that $Ty = q$. If (A, B) is a parametrically $\phi(\epsilon, \delta; a)$-contraction with respect to (S, T), where $\frac{1}{2} < a \leq 1$, then there exist $u, v, w \in C$ such that $Au = Su = w = Bv = Tv$.*

If (A, S) (resp. (B, T)) is weak compatible pair of mappings, then $Aw = Sw$ (resp. $Bw = Tw$). If both the pairs are weak compatible then w is the unique common fixed point of C.

Proof. Let $\alpha_n \in (0, 1)$ for $n \in N$ such that $\alpha_n \to 1$. For any $x \in C$, we define

$$A_n x = \alpha_n Ax + (1 - \alpha_n)q, \quad B_n x = \alpha_n Bx + (1 - \alpha_n)q. \tag{2.5}$$

Since A and B are self-maps of C and C is star-shaped with respect to q, (2.5) assures that $A_n, B_n : C \to C$ are mappings. Further parametrically $\phi(\epsilon, \delta; a)$-contraction with (2.5) imply that, for all $x \in C$, $y \in X$ and $n \in N$,

$$\begin{aligned} & a \parallel A_n x - B_n y \parallel + (1 - a) \parallel B_n y - Ty \parallel \\ = \; & \alpha_n (a \parallel Ax - By \parallel + (1 - a) \parallel By - Ty \parallel) \\ \leq \; & \alpha_n \phi (a \parallel Sx - Ty \parallel + (1 - a) \parallel Sx - Ax \parallel). \end{aligned}$$

But $C = S(C) = T(C)$ is compact and so complete. Corollary 2.1 implies that, for each $n \in N$ there exist $x_n, y_n \in C$ such that

$$A_n x_n = Sx_n = w_n = B_n y_n = Ty_n. \tag{2.6}$$

Since C is compact, we have a subsequence $\{n_i\}$ such that $A_{n_i}x_{n_i} \to w \in C$. Moreover it follows that

$$\begin{aligned}\|Ax_{n_i} - Sx_{n_i}\| &= \|Ax_{n_i} - A_{n_i}x_{n_i}\| \\ &= \|Ax_{n_i} - (\alpha_{n_i}Ax_{n_i} + (1-\alpha_{n_i})q)\| \\ &\leq (1-\alpha_{n_i})(\|Ax_{n_i}\| + \|q\|) \\ &\leq (1-\alpha_{n_i})M\end{aligned}$$

for some $M > 0$ by (2.5) and (2.6) since C is bounded.

Therefore we have

$$\|Ax_{n_i} - Sx_{n_i}\| \to 0 \quad \text{as } \alpha_{n_i} \to 1$$

and hence

$$Ax_{n_i} \quad \text{and} \quad Sx_{n_i} \to w \quad \text{as } i \to \infty. \tag{2.7}$$

Similarly, we have

$$By_{n_j} \quad \text{and} \quad Ty_{n_j} \to w \quad \text{as } j \to \infty. \tag{2.8}$$

Again, since $S(C) = C$, $Su = w$ for some $u \in C$ and so by (1.5), we have

$$\begin{aligned}& a\|Au - w\| + (1-a)\|Bu - w\| \\ \leq\ & a[\|Au - By_{n_j}\| + \|By_{n_j} - w\|] \\ & +(1-a)[\|Bu - By_{n_j}\| + \|By_{n_j} - Ty_{n_j}\|] \\ =\ & [a\|Au - By_{n_j}\| + (1-a)\|By_{n_j} - Ty_{n_j}\|] + (1-a)\|Bu - By_{n_j}\| \\ & +(1-a)\|Ty_{n_j} - w\| + a\|By_{n_j} - w\| \\ \leq\ & \phi(a\|Ty_{n_j} - Su\| + (1-a)\|Su - Au\|) + (1-a)\|Bu - By_{n_j}\| \\ & +(1-a)\|Ty_{n_j} - w\| + a\|By_{n_j} - w\| \\ <\ & a\|Ty_{n_j} - Su\| + (1-a)\|Su - Au\| + (1-a)\|Bu - By_{n_j}\| \\ & +(1-a)\|Ty_{n_j} - w\| + a\|By_{n_j} - w\| \\ =\ & a\|Ty_{n_j} - w\| + (1-a)\|w - Au\| + (1-a)\|Bu - By_{n_j}\| \\ & +(1-a)\|Ty_{n_j} - w\| + a\|By_{n_j} - w\|.\end{aligned}$$

Now $i, j \to \infty$ we have

$$a\|Au - w\| + (1-a)\|Bu - w\| < (1-a)\|w - Au\| + (1-a)\|Bu - w\|.$$

As $2a > 1$ this implies $Au = w$. Thus $Au = w = Su$. The weak-compatibility of pair (A, S) implies $ASu = SAu$, i.e., $Aw = Sw$.

Similarly, if B and T are weak compatible, $Bw = Tw$. If both the pairs (A, S) and (B, T) are weak-compatible, then $Aw = Sw$ and $Bw' = Tw'$ for coincidence points w and w'. Using the condition (1.5) for $w \neq w'$ we get a contradiction as in Theorem 2.1. Thus w is a common fixed point of A, B, S and T. Uniqueness of fixed point follows easily by using (1.5). Thus w is the unique common fixed point of A, B, S and T. This completes the proof. □

Remark 2.1. Theorem 2.1 and Theorem 2.2 are true if the mapping pairs (A, S) and (B, T) are compatible (Jungck [3]), compatible of type (A) (Jungck-Murthy-Cho [5]) and the mappings S, T must be continuous as mentioned below in Lemma 2.1.

Lemma 2.4. *Let S and T be self-maps of a metric space (X, d). Suppose that S and T are compatible of type (A) (resp. compatible (see Remark 1.1 of Pathak-Khan-Kang [15])), and Sx_n and $Tx_n \to t$ for some $t \in X$. Then:*
(1) $\lim_{n\to\infty} TSx_n = St$ *if S is continuous,*
(2) $STt = TSt$ *if both S and T are continuous at t.*

Remark 2.2. In 1994, Pant [10] introduced the notion of R-weakly commuting mappings for a pair of mappings.

Two self-maps S and T of a metric space (X, d) are said to be *pointwise R-weakly commuting* 'at the point' $x \in X$ if there exist a real number $R > 0$ such that

$$d(STx, TSx) \leq Rd(Sx, Tx). \tag{2.9}$$

From above definition it is clear that pointwise R-weakly commuting 'fails' when the pair has such coincidence point (say u) where it does not commute, that is, $Su = Tu$ but $STu \neq TSu$ at $u \in X$.

Since the definition of weak-compatibility assures the commutativity at coincidence point, it shows that Theorem 2.1 and Theorem 2.2 will be true if the mapping pairs (A, S) and (B, T) are R-weakly commuting instead of weak-compatibility.

Remark 2.3. In the sequel, we now give the definition of R-weakly commuting of type (P).

Definition 2.1. (see [17]) Two self-maps S and T of a metric space (X, d) are said to be *pointwise R-weakly commuting of type* (P) at a point $x \in X$ if there exist a real number $R > 0$ such that

$$d(SSx, TTx) \leq Rd(Sx, Tx). \tag{2.10}$$

It may be noted that R-weakly commuting of type (P) does not guarantee the *weak-compatibility* or *compatibility* or *R-weakly commuting* since 'commuting at coincidence point' is not necessary for (2.10) above. In Example 1.2, pair (S,T) commutes at their coincidence points $x_1 = 1$ and $x_2 = \sqrt{2}$; but pointwise R-weakly commuting of type (P) does not follows at $x_1 = 1$. Here the mappings involved are discontinuous at $x_1 = 1$.

Thus we conclude that pointwise R-weakly commuting of type (P) is helpful to us where the mapping pair is *not necessarily continuous* as well as 'commuting at coincidence point'.

Our Theorem 2.1 and Theorem 2.2 are true for pointwise R-weakly commuting of type (P) also.

Example 2.1. Let $X = [2, 20]$ and d be the usual metric on X. Define mappings A, B, S and T on X by

$$Ax = \begin{cases} 2 & \text{if } x = 2, \\ 3 & \text{if } x > 2, \end{cases} \quad Sx = \begin{cases} 2 & \text{if } x = 2, \\ 6 & \text{if } x > 2, \end{cases}$$

$$Bx = \begin{cases} 2 & \text{if } x = 2 \text{ or } x > 5, \\ 6 & \text{if } 2 < x \le 5, \end{cases} \quad Tx = \begin{cases} 2 & \text{if } x = 2, \\ 12 & \text{if } 2 < x \le 5, \\ x - 3 & \text{if } x > 5. \end{cases}$$

Then we find here that $AX = \{2,3\}$, $SX = \{2,6\}$, $BX = \{2,6\}$ and $TX = \{2,12\} \cup (2,17] = [2,17] \subset X$. $AX \subseteq TX$, $BX \subseteq SX$. TX is complete as it is an infinite bounded subset of R satisfying the Bolzano-Weierstrass property. Pair (A,S) is weak-compatible as $A2 = S2$ implies $AS2 = SA2$. Pair (B,T) is also weak-compatible at $x = 2$ as $BT2 = TB2$ whenever $B2 = T2 = 2$.

If $x = 2$ and $y = 2$ then from (1.5) we have for $\phi(t) = \frac{1}{2}t$ (for $t > 0$)

$$ad(A2, B2) + (1-a)d(B2, T2) \le \phi(ad(S2, T2) + (1-a)d(A2, S2)),$$

that is, $0 \le \phi(0)$.

For $x = 2$ and $y = 5$, the condition (1.5)(c) yields

$$\epsilon \le d(B5, T5) < \delta(\epsilon) \Rightarrow \phi(d(A2, S2)) < \epsilon,$$

that is, $\epsilon \le 6 < \delta(\epsilon) \Rightarrow \phi(0) < \epsilon$, which is true.

If we put $x = 2$ and $y > 5$ then (1.5) yields

$$a0 + (1-a)|y-5| \le \phi(a|y-5| + (1-a)0) < \frac{a|y-5|}{2},$$

that is, $1 - a < \frac{1}{2}a$ or, $a > \frac{2}{3}$, which is true as $a \in (\frac{1}{2}, 1]$.

For $x = 2$ and $y > 5$, (1.5)(c) is also true. Since

$$\epsilon \leq d(By, Ty) = |y - 5| < \delta(\epsilon) \Rightarrow 0 = \phi(0) = \phi(d(Ax, Sx)) < \epsilon.$$

This inequality is true for y, where $5 + \epsilon \leq y < 5 + \delta(\epsilon)$ as $\delta(\epsilon) > \epsilon > 0$.

Other conditions for y, i.e., $5 - \delta(\epsilon) < y \leq 5 - \epsilon$ is also true for $x = 2$ and $y < 5$ as from (1.5)

$$ad(2, 6) + (1 - a)d(6, 12) \leq \phi(ad(2, 12) + (1 - a)d(2, 2))$$

and hence $4a + 6(1 - a) \leq \frac{1}{2}(10a)$, i.e., $\frac{6}{7} \leq a$, which is true as $a \in (\frac{1}{2}, 1]$ and (1.5)(c) yields

$$\epsilon \leq d(By, Ty) = 6 < \delta(\epsilon) \Rightarrow 0 = \phi(0) = \phi(d(Ax, Sx)) < \epsilon,$$

which is also true.

Similarly for other values of x and y all the inequalities may be test. Thus our example satisfy all the conditions of (1.5) with (a)-(d).

Therefore the pair (A, B) is parametrically $\phi(\epsilon, \delta; a)$-contraction with respect to (S, T). We may observe that $u = 2$, $v_1 = 5$ and $w = 2$ are such that $Au = Su = w = 2 = Bv_1 = Tv_1$.

Here (A, S) is weak compatible at the coincidence point $w = 2$, that is $ASw = SAw$ whenever $Aw = Sw$. Hence $Aw = Sw = w = 2$.

Again (B, T) is weak compatible at the coincidence point $w = 2$, that is $Bw = Tw = w = 2$. Thus we may choose $w = 2$ such that $Aw = Sw = w$ (resp. $Bw = Tw = w$).

The common coincidence point for both weak compatible pairs (A, S) and (B, T) is $w = 2$. Thus the unique common fixed point of A, B, S and T is $w = 2$. This verifies the proof of Theorem 2.1.

Example 2.2. In the previous Example 2.1 we may take the mappings A, B, S and T and $d(x, y) = \|x - y\|$, $X = C \subseteq R$. Let us take $q = 2 \in C$ as star-center and $y = 2$ such that $Ty = q$. Then all the conditions of Theorem 2.2 satisfy. We can observe that $Aw = Sw$ (for $w = 2$), $Bw = Tw$ (for $w = 2$). Therefore $w = 2$ is the unique common fixed point of mappings A, B, S and T. This verifies the proof of Theorem 2.2.

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