International Journal of Pure and Applied Mathematics
Volume 26 No. 2 2006, 265-273

AN ALGORITHM FOR GENERATING MULTI-PARAMETER WEIGHT FUNCTIONS FOR THE q^{-1}-HERMITE POLYNOMIALS

N.M. Atakishiyev

Instituto de Matemáticas
UNAM – Universidad Nacional Autonoma de Mexico
Apartado Postal 273–3, C.P. 62210
Cuernavaca, Morelos, MEXICO
e-mail: natig@matcuer.unam.mx

Abstract: We discuss an algorithm for generating multi-parameter orthogonality weight functions, both continuous and discrete, for the q^{-1}-Hermite polynomials. The key idea is to exploit the general property of these weight functions whereby they are in fact defined only up to multiplication by an arbitrary positive periodic function.

AMS Subject Classification: 33D45, 39A13, 42C30
Key Words: orthogonality, q^{-1}-Hermite polynomials, weight function, theta functions, indeterminate moment problem

*

The q^{-1}-Hermite polynomials $h_n(x|q)$, $0 < q < 1$, were first studied by Askey [1] and shown to be orthogonal with respect to the measure $w_A(x)dx$, $x \in \mathbb{R}$, where

$$\begin{aligned} w_A^{-1}(x;q) &:= \sqrt{1+x^2}\,\prod_{k=0}^{\infty}\Big[1+2(1+2x^2)\,q^{k+1}+q^{2k+2}\Big] \\ &= \cosh\xi\,(-qe^{2\xi},\,-qe^{-2\xi};q)_\infty \simeq \theta_2(\mathrm{i}\,\xi,q^{1/2})\,, \qquad x=\sinh\xi\,. \end{aligned} \tag{1}$$

The symbol $\simeq$ in (1) indicates that the left side is a multiple of the right side

Received: December 12, 2005

by the ξ-independent factor. It is well known that the moment problem associated with the q^{-1}-Hermite polynomials $h_n(x|q)$ is indeterminate and therefore one can construct in fact infinitely many orthogonality measures for them. In Atakishiyev et al [2] another example of the weight function for $h_n(x|q)$ was derived, which has the form

$$w_{AFW}(x;q) := \exp\left(\frac{2}{\ln q}\left[\ln\left(x+\sqrt{x^2+1}\right)\right]^2\right) \equiv \exp\left(\frac{2\xi^2}{\ln q}\right). \qquad (2)$$

The next important step was made by Ismal and Masson [5], who found a family of weight functions

$$w_{IM}^{-1}(x,a;q) := \left|\left(ae^{\xi}, -qe^{\xi}/a, -ae^{-\xi}, qe^{-\xi}/a; q\right)_\infty\right|^2$$

$$\simeq \theta_3\left(\frac{\mathrm{i}}{2}\left[\xi - \ln a\, q^{-1/2}\right], q^{1/2}\right)\theta_3\left(\frac{\mathrm{i}}{2}\left[\xi - \ln \bar{a}\, q^{-1/2}\right], q^{1/2}\right)$$

$$\times\,\theta_4\left(\frac{\mathrm{i}}{2}\left[\xi + \ln a\, q^{-1/2}\right], q^{1/2}\right)\theta_4\left(\frac{\mathrm{i}}{2}\left[\xi + \ln \bar{a}\, q^{-1/2}\right], q^{1/2}\right), \qquad (3)$$

which depend on a complex parameter a, such that $\Im a \neq 0$. Moreover, Ismail and Masson [5] explicitly defined all the N-extremal solutions to the q^{-1}-Hermite moment problem.

This circumstance that all three weight functions (1)-(3) may serve as orthogonality measures for the q^{-1}-Hermite polynomials $h_n(x|q)$ stems from the mutual property of these weight functions: they satisfy the same Pearson-type difference equation on the q-quadratic lattice (see, for example, Atakishiyev et al [3]):

$$e^{\partial_s}\,\tilde{w}(x;q) = q^{2(2s+1)}\,\tilde{w}(x;q), \quad x = \frac{1}{2}\left(q^s + q^{-s}\right), \quad \partial_s \equiv \frac{d}{ds}. \qquad (4)$$

The linear change of the variable $s = \xi/\ln q$ converts (4) into a difference equation

$$\exp\left(\ln q\,\partial_\xi\right)\tilde{w}(x;q) = q^2\,e^{4\xi}\,\tilde{w}(x;q), \quad x = \sinh\xi, \qquad (5)$$

which we shall frequently use. As it was already recalled above, there exist infinitely many solutions of the Pearson-type difference equation (4) (or its equivalent (5)), which may serve as orthogonality weight functions for the $h_n(x|\,q)$. Nevertheless, after 1994, when both $w_{AFW}(x;\,q)$ and $w_{IM}(x,a;q)$ became known, no other explicit examples of such solutions have appeared in the literature. Therefore our goal here is to provide a plain illustration of how to construct solutions, which depend on arbitrary number of parameters. It turns out that by using the well-known characteristic and transformation properties of the theta-functions one may formulate an algorithm for generating other explicit forms of weight functions from $w_{AFW}(x;q)$.

It is important to observe that the Pearson-type difference equations (4) and (5) interrelate values of the weight functions (1)-(3) at two points: s and $s+1$ in the case of (4) and ξ and $\xi+\ln q$ in the case of (5). This means that any solution of the equation (4) or (5) admits multiplication by an arbitrary unit-periodic function $c(s)$, $c(s\pm 1) = c(s)$ in the case of (4) and by an arbitrary $(\ln q)$-periodic function $c(\xi)$, $c(\xi\pm\ln q) = c(\xi)$ in the case of (5). This important characteristic property of solutions of the Pearson-type difference equations (4) and (5) will be essentially used subsequently.

We need first to recall Jacobi's imaginary transformations

$$\begin{aligned}
\theta_1(z\,|\tau) &= \frac{\mathrm{i}}{\sqrt{-\mathrm{i}\tau}}\, e^{z^2/\pi\mathrm{i}\tau}\,\theta_1\left(\frac{z}{\tau}\,|-\frac{1}{\tau}\right)\,, \\
\theta_{2(4)}(z|\tau) &= \frac{1}{\sqrt{-\mathrm{i}\tau}}\, e^{z^2/\pi\mathrm{i}\tau}\,\theta_{4(2)}\left(\frac{z}{\tau}\,|-\frac{1}{\tau}\right)\,, \\
\theta_3(z|\tau) &= \frac{1}{\sqrt{-\mathrm{i}\tau}}\, e^{z^2/\pi\mathrm{i}\tau}\,\theta_3\left(\frac{z}{\tau}\,|-\frac{1}{\tau}\right)\,,
\end{aligned} \tag{6}$$

for the four theta functions $\theta_j(z,q) \equiv \theta_j(z|\tau)$, where $q = e^{\pi\mathrm{i}\tau}$, $\Im\tau > 0$, and $j = 1, 2, 3, 4$ (see Whittaker et al [7], p. 475). Let us consider, as an example, the case of

$$\theta_2(\mathrm{i}\,\xi, q) = \sqrt{\frac{\pi}{\ln q^{-1}}}\; e^{-\xi^2/\ln q}\,\theta_4\left(\frac{\pi\xi}{\ln q}\,,\, e^{\pi^2/\ln q}\right); \tag{7}$$

all other theta functions of imaginary argument will transform similarly. It is plain that the theta function $\theta_4\left(\frac{\pi\xi}{\ln q}\,,\, e^{\pi^2/\ln q}\right)$ on the right side of (7) is a $(\ln q)$-periodic function in the variable ξ on account of the well-known periodicity property $\theta_4(z+\pi, q) = \theta_4(z, q)$. So if one rescales $q \to q^{1/2}$ in (7), then this formula exhibits the fact that both $w_A(x;q)$ and $w_{AFW}(x;q)$ satisfy the same Pearson-type difference equation.

Similar to (7) formulas are valid also for other theta functions, θ_1, θ_3 and θ_4. But in the cases of θ_1 and θ_4 these formulas lead to the θ_1 and θ_2 functions of the argument $\pi\xi/\ln q$, which change their signs under the shift $\xi \to \xi + \ln q$ (we recall that $\theta_{1(2)}(z+\pi, q) = -\,\theta_{1(2)}(z, q)$); so these two cases should be discarded. To the contrary, the remaining case with θ_3 function fits well and yields a weight function $\theta_3^{-1}(\mathrm{i}\,\xi, q^{1/2})$. But we are not getting a novel solution to the moment problem at this stage: it is not hard to verify that this weight function represents a particular member of the family $w_{IM}(x,a;q)$ with $a = \mathrm{i}\, q^{1/4}$.

Nevertheless, it is now clear how to find solutions, which are distinct from (1)-(3). Indeed, by combining (2) and (7) one obtains that

$$
\begin{aligned}
w_{AFW}(x;q) &= \sqrt{\frac{\pi}{\ln q^{-1}}}\; \frac{\theta_{4(3)}\left(\frac{\pi\xi}{\ln q}, e^{\pi^2/\ln q}\right)}{\theta_{2(3)}(\mathrm{i}\xi, q)}\; e^{\xi^2/\ln q} \\
&\sim \theta_{2(3)}^{-1}(\mathrm{i}\xi, q)\, e^{\xi^2/\ln q}\,, \qquad (8)
\end{aligned}
$$

where the symbol $\sim$ between two expressions denotes that one of them is a $(\ln q)$-periodic function multiple of another. So we arrived at two new weight functions in (8) of the form

$$
w_{1,2}(x;q) := \theta_{2,3}^{-1}(\mathrm{i}\,\xi, q)\; e^{\xi^2/\ln q}\,. \qquad (9)
$$

Observe that the first weight function in (9), $w_1(x;q)$, is a product of two already known weight functions (1) and (2), in which the base q is changed to q^2, namely,

$$
w_1(x;q) \simeq w_A(x;q^2)\, w_{AFW}(x;q^2)\,. \qquad (10)
$$

It should be also noted that when one finds two weight functions, $\theta_2^{-1}(\mathrm{i}\,\xi, q^{1/2})$ and $\theta_3^{-1}(\mathrm{i}\,\xi, q^{1/2})$, say, then it is plain that their quotient, $\theta_2\,(\mathrm{i}\,\xi, q^{1/2})/\theta_3\,(\mathrm{i}\,\xi, q^{1/2})$ is a $(\ln q)$-periodic function in the variable ξ. So, these two weight functions give rise to two infinite sequences of weight functions of the form $\theta_2^n\,(\mathrm{i}\,\xi, q^{1/2})/\theta_3^{n+1}\,(\mathrm{i}\,\xi, q^{1/2})$ and $\theta_3^n\,(\mathrm{i}\,\xi, q^{1/2})/\theta_2^{n+1}\,(\mathrm{i}\,\xi, q^{1/2})$ with integer values of $n \in \mathbb{N}$. This general observation remains valid for all subsequent examples of weight functions in this exposition.

The question now presents itself: Can we diversify the foregoing procedure of getting weight functions in order to bring in them some extra parameters (similar to the parameter a in $w_{IM}(x,a;q)$)? The answer is affirmative and it seems that theoretically one may introduce an arbitrary number of parameters in the following way.

Jacobi's imaginary transformation (7) signifies that

$$
\theta_2(\mathrm{i}\,\xi, q) \sim e^{-\,\xi^2/\ln q}\,. \qquad (11)
$$

Let us now make the change of variable $\xi \to \frac{2}{N}\,\xi$, N is a positive integer, $N \geq 2$, and rescale the base $q \to q^{\,2/N}$ in (7). This yields

$$\theta_2\left(\frac{2\,\mathrm{i}}{N}\,\xi\,,\,q^{2/N}\right)=\sqrt{\frac{\pi N}{2\ln q^{-1}}}e^{2\xi^2/N\ln q}\theta_4\left(\frac{\pi\xi}{\ln q},e^{N\pi^2/w\ln q}\right)$$
$$\sim\exp(-2\xi^2/N\ln q)\,, \quad (12)$$

on account of the $(\ln q)$-periodicy of the θ_4 function from (12) in the variable ξ. Consequently,

$$w_{AFW}^{-1}(x;q)=\exp\left(-N\frac{2\xi^2}{N\ln q}\right)\sim\theta_2^n\left(\frac{2i}{N}\xi,q^{2/N}\right)\,. \quad (13)$$

Along the same lines one can show that similar to (13) relation holds also for the $\theta_3(\,2\,\mathrm{i}\,\xi/N,q^{2/N})$ function. Moreover, "similarity" relation (13) remains valid for any product of the type $\theta_2^k(\,2\,\mathrm{i}\,\xi/N,q^{2/N})\,\theta_3^{N-k}(\,2\,\mathrm{i}\,\xi/N,q^{2/N})$, $0\le k\le N$. Note that not all of the weight functions of this type are novel. For instance, when $N=2$ and $k=1$ one obtains that

$$w(x;q):=\theta_2^{-1}(\mathrm{i}\,\xi,q)\,\theta_3^{-1}(\mathrm{i}\,\xi,q)\simeq\theta_2^{-1}(\mathrm{i}\,\xi,q^{1/2})\,, \quad (14)$$

where in the last step we employed Landen's type of transformation connecting theta functions with parameter τ and 2τ, namely, $\theta_2(z,q)\theta_3(z,q)\simeq\theta_2(z,q^{1/2})$ (see Whittaker et al [7], p. 476). So in this case we reproduce the Askey weight function (1). But relation (13), or any of its variety with some number of θ_3 factors in it, is very important because it admits pairwise shifts in the arguments of theta functions involved. For instance, let us consider the above-mentioned simple case with $N=2$, $k=1$, and take

$$\theta_2(\mathrm{i}\,(\xi+a),q)\;\theta_3(\mathrm{i}\,(\xi-a),q)\sim\;e^{-(\xi+a)^2/\ln q}\,e^{-(\xi-a)^2/\ln q}$$
$$\simeq e^{-2\xi^2/\ln q}=w_{AFW}^{-1}(x;q)\,, \quad (15)$$

where a is a real parameter. In this way one gets a one-parameter family of weight functions

$$w(x,a;q):=\theta_2^{-1}\left(\mathrm{i}\,(\xi+a),q\right)\;\theta_3^{-1}\left(\mathrm{i}\,(\xi-a),q\right)\,. \quad (16)$$

It is evident from (14) that this family contains the Askey weight function (1) as its particular case when $a=0$.

By a derivation similar to that of (16), one arrives at two more one-parameter weight functions of the form

$$w_{3,4}(x,a;q):=\theta_{2,3}^{-1}\left(\mathrm{i}\,(\xi+a),q\right)\;\theta_{2,3}^{-1}\left(\mathrm{i}\,(\xi-a),q\right)\,. \quad (17)$$

The next step is to take $N = 3$. This leads to five more weight functions with Gaussian exponential factors,

$$w_{5,6}(x;q) := \theta_{2,3}^{-1}\left(\frac{2\mathrm{i}}{3}\,\xi\,,\,q^{\,2/3}\right)\,e^{\,4\,\xi^2/3\ln q}\,, \tag{18}$$

$$w(x;q) := \theta_2^{-1}\left(\frac{2\mathrm{i}}{3}\,\xi\,,\,q^{\,1/3}\right)\,e^{\,2\,\xi^2/3\ln q}\,, \tag{19}$$

$$w_{7,8}(x;q) := \theta_{2,3}^{-2}\,\left(\frac{2\mathrm{i}}{3}\,\xi\,,\,q^{\,2/3}\right)\,e^{\,2\,\xi^2/3\ln q}\,. \tag{20}$$

For more weight functions in terms of the theta functions θ_2 and θ_3 in this case are

$$w_{9,10}(x;q) := \theta_{2,3}^{-2}\,\left(\frac{2\mathrm{i}}{3}\,\xi\,,\,q^{\,2/3}\right)\,\theta_{3,2}^{-1}\,\left(\frac{2\mathrm{i}}{3}\,\xi\,,\,q^{\,2/3}\right)\,, \tag{21}$$

$$w_{11,12}(x;q) := \theta_{2,3}^{-3}\,\left(\frac{2\mathrm{i}}{3}\,\xi\,,\,q^{\,2/3}\right)\,. \tag{22}$$

Evidently, for each of the cases (19)–(22) one can also construct one-parameter extensions of the form

$$w(x,a;q) := \theta_2^{-1}\,\left(\frac{2\mathrm{i}}{3}\,(\xi + a)\,,q^{\,2/3}\right)\,\theta_3^{-1}\,\left(\frac{2\mathrm{i}}{3}\,(\xi - a)\,,\,q^{\,2/3}\right)e^{\,2\,\xi^2/3\ln q}\,, \tag{23}$$

with a real parameter a.

Finally, at the next step with $N = 4$ there are again some weight functions with Gaussian exponential factors,

$$w_{13,14}^{(k)}(x;q) := \theta_{2,3}^{-k}\left(\frac{\mathrm{i}}{2}\,\xi\,,\,q^{\,1/2}\right)\,e^{\,(4-k)\,\xi^2/2\ln q}\,,\quad k = 1,2,3, \tag{24}$$

$$w^{(k,l)}(x;q) := \theta_2^{-k}\left(\frac{\mathrm{i}\,\xi}{2}\,,q^{1/2}\right)\theta_3^{-l}\left(\frac{\mathrm{i}\,\xi}{2}\,,q^{1/2}\right)\,e^{\,(4-k-l)\,\xi^2/2\ln q}\,,$$
$$1 \le k + l \le 3, \tag{25}$$

and more weight functions in terms of θ_2 and θ_3 only:

$$w^{(k,l)}(x;q) := \theta_2^{-k}\left(\frac{\mathrm{i}\,\xi}{2}\,,\,q^{1/2}\right)\theta_3^{-l}\left(\frac{\mathrm{i}\,\xi}{2}\,,\,q^{1/2}\right),\quad 0 \le k + l \le 4. \tag{26}$$

One-parameter extensions for $N = 4$ can be constructed as in (23); for instance, from (24) with $k = 2$ one derives that

$$w(x,a;q) := \theta_2^{-1}\left(\frac{\mathrm{i}}{2}\,(\xi + a)\,,\,q^{1/2}\right)\theta_2^{-1}\left(\frac{\mathrm{i}}{2}\,(\xi - a))\,,\,q^{1/2}\right)e^{\,\xi^2/\ln q}\,, \tag{27}$$

where a is a real number. Similarly, from (26) with $k = l = 2$ one obtains a two-parameter family (both of a and b are real)

$$w(x,a,b;q) := \theta_2^{-1}\left(\frac{\mathrm{i}}{2}\,(\xi+a)\,,\,q^{1/2}\right)\theta_2^{-1}\left(\frac{\mathrm{i}}{2}\,(\xi-a))\,,\,q^{1/2}\right)$$
$$\times\,\theta_3^{-1}\left(\frac{\mathrm{i}}{2}\,(\xi+b)\,,\,q^{1/2}\right)\theta_3^{-1}\left(\frac{\mathrm{i}}{2}\,(\xi-b)\,,\,q^{1/2}\right)\,. \qquad (28)$$

The weight functions, constructed up to this point, are expressed in terms of the θ_2 and θ_3 functions of imaginary argument. The reason for discarding two other theta functions, $\theta_1(\mathrm{i}\,\xi,q)$ and $\theta_4(\mathrm{i}\,\xi,q)$, is that they have zeroes for real values of ξ at the points $\xi = 0, \pm\,(k+1)\,\ln q$ and $\xi = \pm\,(k+1/2)\,\ln q$, $\quad k = 0,1,2,...,$ respectively. But in the case when $N = 4$ (and, consequently, in all other cases when $N > 4$) it is possible to avoid these singularities of the functions $\theta_1^{-1}(\mathrm{i}\,\xi,q)$ and $\theta_4^{-1}(\mathrm{i}\,\xi,q)$ by moving them out of the real axis $\xi \in \mathbb{R}$. This can be achieved by pairwise shifts of the variable ξ to $\xi \pm\, c$ (c is a complex constant, such that $\Im\, c \neq 0$) in a product of two θ_1 or θ_4 functions.

This point is well illustrated by the weight function

$$w\,(x,a;q) := \theta_1^{-1}\left(\frac{\mathrm{i}}{2}\,[\,\xi+\ln a\,]\,,\,q^{1/2}\right)\theta_1^{-1}\left(\frac{\mathrm{i}}{2}\,[\,\xi+\ln\bar a\,]\,,\,q^{1/2}\right)$$
$$\times\,\theta_2^{-1}\left(\frac{\mathrm{i}}{2}\,[\,\xi-\ln a\,]\,,\,q^{1/2}\right)\theta_2^{-1}\left(\frac{\mathrm{i}}{2}\,[\,\xi-\ln\bar a\,]\,,\,q^{1/2}\right)\,, \qquad (29)$$

where $\Im a \neq 0$. Other possibilities are those which may be obtained from (29) by replacing in it: a) two θ_1 functions by θ_4 functions with the same arguments as in (29); b) two θ_2 functions by θ_3 (or θ_4) functions; c) two θ_1 and two θ_2 functions by θ_4 and θ_3 functions, respectively. The last case reproduces the Ismail–Masson weight function (3).

So, to incorporate a real parameter into an orthogonality weight function for the q^{-1}-Hermite polynomials, one needs a weight function, which contains a product of two θ_2 or θ_3 functions (*cf.* the passage from (24) to (27)); for introducing a complex parameter (with a nonzero imaginary part) it suffices to have a product of two pairs of θ_2 and θ_3 functions, and so on.

In our exposition up to the present only continuous weight functions have been discussed. We now turn to the question of deriving discrete weight functions and show that a similar treatment is applicable in this case as well. Indeed, let us consider a weight function

$$w_1(x,b;q) := \sum_{k=-\infty}^{\infty} \delta\left(\xi-\xi_k^{(b)}\right)\,w_{AFW}(x;q)\,,$$

$$\xi := \ln\left(x + \sqrt{x^2+1}\right) , \tag{30}$$

where $\delta\left(\xi - \xi_k^{(b)}\right)$ is the Dirac delta function. In conformity with the foregoing strategy, the sum

$$\sum_{k=-\infty}^{\infty} \delta\left(\xi - \xi_k^{(b)}\right)$$

must be invariant with respect to the shifts $\xi \to \xi \pm \ln q$ in order that the $w_1(x, b; q)$ may serve as another orthogonality weight function for the q^{-1}-Hermite polynomials, generated by the $w_{AFW}(x; q)$. This requirement is met if one chooses in (30) a sequence of points $\xi_k^{(b)} := \ln(b\, q^k)$, $k \in \mathbb{Z}$, with arbitrary parameter $b > 0$. Since

$$\int_{-\infty}^{\infty} f(x)\, w_1(x, b; q)\, dx = \sum_{k=-\infty}^{\infty} f(x_k^{(b)})\, w_{AFW}(x_k^{(b)}; q)\, \cosh \xi_k^{(b)}$$

$$= \sum_{k=-\infty}^{\infty} f(x_k^{(b)})\, w_1^{(d)}(k, b; q) , \qquad x_k^{(b)} := \sinh \xi_k^{(b)} , \tag{31}$$

this means that the $w_1^{(d)}(k, b; q) := w_{AFW}\,(x_k^{(b)}; q)\, \cosh \xi_k^{(b)}$ represents a one-parameter family of discrete orthogonality weight functions, supported on the points $\xi_k^{(b)}$. By using the explicit form (2) of the $w_{AFW}(x; q)$, it is not hard to evaluate that the values of $w_1^{(d)}(k, b; q)$ at those points are equal to

$$w_1^{(d)}(k, b; q) = \frac{1}{2}\, e^{\,2(\ln b)^2/\ln q}\, (1 + b^2\, q^{2k})\, b^{4k-1}\, q^{k(2k-1)} . \tag{32}$$

One could use in (30) the Askey weight function (1) instead of the $w_{AFW}(x; q)$ and to define

$$w_2^{(d)}(k, b; q) := w_A\,(x_k^{(b)}; q)\, \cosh \xi_k^{(b)}$$

$$= \frac{1 + b^2\, q^{2k}}{(-b^2, -q/b^2; q)_\infty}\, b^{4k}\, q^{k(2k-1)} . \tag{33}$$

This is a one-parameter family of discrete N-extremal measures for the q^{-1}-Hermite polynomials, derived in Ismail et al [5]. Apart from a b-dependent normalization factor, the $w_1^{(d)}(k, b; q)$ is the same. It is important to observe that one could use *any* other continuous weight function in the defining relation (30) - the result would be the same as in (32) or (33) up to a multiplicative normalization factor $w_2\left(x_0^{(b)}; q\right) / w_1\left(x_0^{(b)}; q\right)$, which depends on the parameter b.

We shall conclude with the following remark. The main motivation for this study comes from the simple observation that instances of weight functions of type (1)–(3) sometimes appear in solving various problems of mathematical and theoretical physics, not directly related to the problem of moments in the theory of special functions. For example, the weight function (2) emerged in Atakishiyev et al [2] from an attempt to construct an explicit realization of the Heisenberg q-algebra

$$b\, b^{\dagger} - q\, b^{\dagger}\, b = I$$

in terms of finite difference operators, acting on square-integrable functions on the full real line $x \in \mathbb{R}$. Therefore we hope that it will be of use to exhibit some explicit forms of multi-parameter orthogonality weight functions, associated with the q^{-1}-Hermite polynomials $h_n(x|q)$. Of course, we do not claim that all possible orthogonality weight functions are exhausted by the foregoing examples; but the archetypical pattern of a weight function with an arbitrary number of parameters seems to be quite clear from them.

Discussions with J. Christiansen, M. Ismail, and A. Klimyk are gratefully acknowledged.

References

[1] R. Askey, In: *q-Series and Partitions*, (Ed. D. Stanton), The IMA Volumes in Mathematics and its Applications, **18**, 151-158, Springer-Verlag, New York (1989).

[2] N.M. Atakishiyev, A. Frank, K.B. Wolf, *J. Math. Phys.*, **35** (1994), 3253-3260.

[3] N.M. Atakishiyev, M. Rahman, S.K. Suslov, *Constructive Approximation*, **11** (1995), 181-226.

[4] J.S. Christiansen, M.E.H. Ismail, *Trans. Amer. Math. Soc.*, To Appear.

[5] M.E.H. Ismail, D.R. Masson, *Trans. Amer. Math. Soc.*, **346** (1994), 63-116.

[6] M.E.H. Ismail, M. Rahman, *Pacific J. Math.*, **170** (1995), 497-515.

[7] E.T. Whittaker, G.N. Watson, *A Course of Modern Analysis*, Cambridge University Press, Cambridge (1944).