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FUZZY CONGRUENCE ON THE SEMIRINGS WITH REVERSIBLE ADDITION

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Abstract: The purpose of this paper is to introduce the concept of fuzzy congruence on the semirings with reversible addition. We obtain that if S is a semirings with reversible addition and μ is the fuzzy congruence of S, then the fuzzy congruence classes of S about μ is a semirings with reversible addition. And some homomorphic properties of semirings with fuzzy congruence relation are given.

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1. Introduction

For general algebraic structures, for example, semigroup, group, ring, lattice, congruence relations and ideas play an important role in the formation of these algebraic structures. N. Kuroki [3], [4], [5] first introduced the concepts of fuzzy congruence and fuzzy idea on a semi-group. F.A. Al-Ghukair [6] also makes beneficial achievements in his research on the fuzzy congruence pairs on inverse semi-groups. Obviously, the research for the properties on fuzzy congruence relation on semiring is significant.

This paper defines the fuzzy congruence on semiring with reversible addition and discusses the homomorphic properties of semiring with reversible addition

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with fuzzy congruence relation. We obtain that if S is a semiring with reversible addition with fuzzy congruence relation, then fuzzy congruence class of S about μ forms a semiring with reversible addition. Some homomorphric properties of semiring with reversible addition with fuzzy congruence relation are also given.

2. Preliminaries

A semi-ring is a triple $(S,+,)$, among which $(S,)$ and $(S,)$ are semi-groups, and product to addition satisfies left and right distributive law. That is that, $a(b+c)=ab+ac$, $(b+c)a=ba+ca$ for each $a,b,c\in S$. A semi-ring may exist a zero element θ, which satisfies, $\theta+a=a+\theta=\alpha$, and $a\theta=\theta a=\theta$ for all $a\in S$. If a semi-group $(S,+)$ with addition of a semi-ring S is a group, and an identical element of $(S,+)$ is a zero element of S, then S is called a semiring with reversible addition.

A non-empty subset A of semi-ring S is called an idea of S, if $\forall a,b\in A,\ \forall s\in S,\ a+b\in A,\ sa\in A,\ as\in A$.

Definition 2.1. (see [1]) Let S be a semi-ring and I an idea of S. If I satisfies the following conditions:

(1) $x+I=I+x,\ \forall x\in S$,

(2) $i+I=I,\ \forall i\in I$.

Then we call I a strong idea of S, where, $x+I=\{x+i\mid i\in I\},\ I+x=\{i+x\mid i\in I\}$.

Proposition 2.1. (see [2]) *Let S be a semiring with reversible addition. Then I is a strong idea of S if and only if I is a idea of S, and $(I,+)$ is a invariant subgroup of $(S,+)$.*

Proposition 2.2. (see [2]) *Let S be a semiring with reversible addition.*

(1) If R is a congruence relation on S, then $K=\{x\in S\mid (x,\theta\in R\}$ is a strong idea of S.

(2) If I is a strong idea of S, we define a binary relation on S

$$T:\ (x,y)\in T\Leftrightarrow x+I=y+I\quad(\forall x,y\in S).$$

Then T is a congruence relation on S and the strong idea of S determined by T is K according to (1).

Proposition 2.3. (see [2]) *Let S be a semiring with reversible addition, and $A=\{I\mid I$ is a strong idea of $S\}$, $B=\{T\mid T$ is a congruence on $S\}$. Then $\varphi:\ A\to B,\ \varphi(I)=T$ is a biection, where T is defined in Proposition 2.2.*

Proposition 2.4. (see [2]) *Let S and $S^{'}$ be semirings with reversible addition, and f is a surjective homomorphish from S to $S^{'}$. If $M=\{I\mid$*

I is a strong idea of S, $I \supseteq \ker\ f\}$, $N = \{I' \mid I'$ is a strong idea of $S'\}$, then $\varphi:\ M \to N$, $\varphi(I) = f(I)$ is a bijection, and $I_1 \subseteq I_2$ if and only if $f(I_1) \subseteq f(I_2)$, $\forall I_i \in S\ (i = 1, 2)$.

3. Fuzzy Congruence Relations on a Semiring S with Reversible Addition

In this paper, we assume S to be a semiring with reversible addition.

Let S be a semiring with reversible addition. A mapping α from $S \times S$ to the unit interval [0,1] is called a fuzzy relation on S. Let α, β be two fuzzy relations on S, respectively. Then product $\alpha \circ \beta$ of α and β is defined by

$$(\alpha \circ \beta)(a, b) = \sup_{x \in S}[\min\{\alpha(a, x), \beta(x, b)\}],$$

$\alpha \leq \beta$ if and only if $\alpha(x, y) \in \beta(x, y),\ \forall (x, y) \in S \times S$.

A fuzzy set μ on S is a mapping from S to the unit interval [0,1], and all fuzzy sets of S denote as $F(S)$, $\mu_\lambda = \{x \in S \mid \mu(x) \geq \lambda\}$ is called a λ-cut set of μ.

A fuzzy relation μ on S is called fuzzy equivalence relation on S, if:

(E.1) $\mu(a, a) = 1$ (Fuzzy reflexive),

(E.2) $\mu(a, b) = \mu(b, a)$ (Fuzzy symmetric),

(E.3) $\mu \circ \mu \subseteq \mu$ (Fuzzy transitive).

Definition 3.1. Let μ be a fuzzy equivalence relation on a semiring S with reversible addition. Then μ is called a fuzzy congruence relation on S, if μ satisfies:

(c.1) $\mu(a + x, b + x) \geq \mu(a, b),\ \mu(x + a, x + b) \geq \mu(a, b)$,

(c.2) $\mu(ax, bx) \geq \mu(a, b)$ and $\mu(xa, xb) \geq \mu(a, b)$, for all $x \in S,\ a, b \in S$.

We denote χ_f as the characteristic function of a binary relation relation f on S. Then we have the following conclusions.

Proposition 3.1. *Let f be a binary relation on a semiring S with reversible addition. Then f is an equivalence (a congruence) relation on S if and only if χ_f is a fuzzy equivalence (a fuzzy congruence) relation on S.*

Proof. It is clear by the Theorem 2.4 in [5]. □

For unification, we agree on a empty set $\emptyset$ also is an equivalence (a congruence) relation on S.

Proposition 3.2. *μ is a fuzzy equivalence (congruence) relation on a semiring S with reversible addition if and only if $\forall \lambda \in [0, 1]$, μ_λ is an equivalence (a congruence) relation on S.*

Proof. (Necessity) Let μ be a fuzzy congruence relation on a semiring S with reversible addition, $\forall\lambda \in [0,1]$, $\mu_\lambda = \{(a,b) \mid (a,b) \in S \times S,\ \mu(a,b) \geq \lambda\}$.

Obviously, μ_λ is reflexive and symmetric. If $(a,b),(b,c) \in \mu_\lambda$, then

$$\mu(a,b) \geq \lambda,\ \mu(b,c) \geq \lambda,\ \mu(a,c) =\geq \mu \circ \mu(a,c)$$
$$= \sup_{x\in S}[\min\{\mu(a,x),\mu(x,c)\}] \geq \min\{\mu(a,b),\mu(b,c)\} \geq \lambda,$$

thus, $(a,c) \in \mu_\lambda$. μ is transitive.

$\forall x \in S$, $(a,b) \in \mu_\lambda$, $\mu(a+x,b+x) \geq \mu(a,b) \geq \lambda$, $\mu(x+a,x+b) \geq \mu(a,b) \geq \lambda$, $\mu(ax,bx) \geq \mu(a,b) \geq \lambda$, $\mu(xa,xb) \geq \mu(a,b) \geq \lambda$, so, $(a+x,b+x) \in \mu_\lambda$, $(x+a,x+b) \in \mu_\lambda$, $(ax,bx) \in \mu_\lambda$, $(xa,xb) \in \mu_\lambda$. Thus, μ_λ is a congruence relation on S.

(Sufficiency) If $\forall\lambda \in [0,1]$, μ_λ is congruence relation on S, then $\forall\lambda \in [0,1]$, $\forall a \in S$, $\mu(a,a) \geq \lambda$, we obtain $\mu(a,a) = 1$. From $\mu(a,b) = \lambda$, we get $(a,b) \in \mu_\lambda$, and so $(b,a) \in \mu_\lambda$, $\mu(b,a) \geq \lambda = \mu(a,b)$, according to symmetry, we have $\mu(b,a) \leq \mu(a,b)$. Thus $\mu(b,a) = \mu(a,b) = \lambda$; $\forall(a,c) \in S \times S$, $\forall x \in S$. Let $\min\{\mu(a,x),\mu(x,c)\} = \lambda$, then $(a,x) \in \mu_\lambda$, $(x,c) \in \mu_\lambda$, so we have $(a,c) \in \mu_\lambda$, *i.e.* $\mu(a,c) \geq \min\{\mu(a,x),\mu(x,c)\}$, $\forall x \in S$.

Then, $\mu(a,c) \geq \sup_{x\in S}[\min\{\mu(a,x),\mu(x,c)\}] = \mu \circ \mu(a,c)$, μ is fuzzy equivalence relation on S. For arbitrary $x \in s$, if $\mu(a,b) = \lambda$, then $(a,b) \in \mu_\lambda$, so we have $(a+x,b+x) \in \mu_\lambda$, i.e $\mu(a+x,b+x) \geq \lambda = \mu(a,b)$. Similarly, we have $\mu(x+a,x+b) \geq \lambda = \mu(a,b)$, $\mu(ax,bx) \geq \lambda = \mu(a,b)$, $\mu(xa,xb) \geq \lambda = \mu(a,b)$, μ is fuzzy congruence relation on S. □

Definition 3.2. Let υ be a fuzzy subset of the semiring S with reversible addition. If υ satisfies: $\forall x,y \in S$:

(1) $\upsilon(x-y) \geq \min\{\upsilon(x),\upsilon(y)\}$;

(2) $\upsilon(xy) \geq \max\{\upsilon(x),\upsilon(y)\}$.

Then υ is called a fuzzy idea of S.

If υ also satisfies:

(3) $\upsilon(x+y-x) \geq \upsilon(y)$.

Then υ is called a fuzzy strong idea of S.

Proposition 3.3. *Let S be a semiring with reversible addition. Then υ is a fuzzy strong idea of S if and only if υ is a fuzzy idea of S, and $(\upsilon,+)$ is a invariant fuzzy subgroup of $(S,+)$.*

Proof. It is omitted. □

Let μ be a fuzzy equivalence relation on the semiring S with reversible addition, we define $\mu_a(x) = \mu(a,x)$, $\forall x \in S$. μ_a is called a fuzzy congruence class of μ about a. Then we have the following proposition.

Proposition 3.4. *Let μ be a fuzzy congruence relation on the semiring S with reversible addition, and let θ be a zero element of S, then μ_θ is a strong idea of S.*

Proof. Let μ be a fuzzy congruence on the semiring S with reversible addition.

$$\begin{aligned}\forall x, y \in S, \quad \mu_\theta(x-y) &= \mu(\theta, x-y) = \mu(y-y, x-y) \geq \mu(y,x) = \mu(x,y) \\ &\geq (\mu \circ \mu)(x,y) = \sup_{z \in S}[\min\{\mu(x,z), \mu(z,y)\}] \\ &\geq \min\{\mu(x,\theta), \mu(\theta,y)\} = \min\{\mu_\theta(x), \mu_\theta(y)\},\end{aligned}$$

and $\forall r \in S,\ \mu_\theta(rx) = \mu(\theta, rx) = \mu(r\theta, rx) \geq \mu(\theta, x) = \mu_\theta(x)$.

Similarly, $\mu_\theta(rx) \geq \mu_\theta(r)$, and so, $\mu_\theta(rx) \geq \max\{\mu_\theta(r), \mu_\theta(x)\}$, μ_θ is a fuzzy idea of S.

$$\begin{aligned}\forall x, y \in S, \quad \mu_\theta(x+y-x) &= \mu(\theta, x+y-x) = \mu(x-x, x+y-x) \\ &\geq \mu(x, x+y) = \mu(x+\theta, x+y) \geq \mu(\theta, y) = \mu_\theta(y).\end{aligned}$$

Therefore $x+y-x \in \mu_\theta$, $(\mu_\theta, +)$ is a regular fuzzy subgroup of $(S, +,)$. Thus, μ_θ is a fuzzy strong idea of S. □

Proposition 3.5. *Let μ be a fuzzy congruence relation on the semiring S with reversible addition, for all $a \in S$, then $\forall x \in S,\ \mu_a(x) = \mu_\theta(x-a)$.*

Proof. We shall prove $\mu(a,x) = \mu(\theta, x-a)$, only.

$$\begin{aligned}\forall x \in S, \quad \mu(\theta, x-a) &= \mu(a-a, x-a) \geq \mu(a,x) \geq (\mu \circ \mu)(a,x) \\ &= \sup_{y \in S}[\min\{\mu(a,y), \mu(y,x)\}] \geq \min\{\mu(a,a), \mu(a,x)\} \\ &= \mu(a,x) = \mu(a, x-a+a) \geq \mu(\theta, x-a).\end{aligned}$$

Then $\mu_a(x) = \mu_\theta(x-a)$. □

Proposition 3.6. *Let μ be a fuzzy congruence relation on the semiring S with reversible addition ,then $\forall a, b \in S,\ \mu_a = \mu_b$ if and only if $\mu_\theta(a-b) = \mu_\theta(b-a) = 1$; Equivalently $\mu_a = \mu_b$ if and only if $\mu(a,b) = 1$.*

Proof. Let $\mu_a = \mu_b$, then $\forall x \in S$, $\mu(a,x) = \mu(b,x)$. Therefore $\mu_\theta(a-b) = \mu(a-b, \theta) = \mu(a,b) = \mu_a(b) = \mu_b(b) = \mu(b,b) = 1$. □

Conversely, if $\mu_\theta(a-b) = \mu(\theta, a-b) = 1$, then $\forall x \in S$,

$$\begin{aligned}\mu_a(x) = \mu(a,x) &\geq \mu(\theta, x-a) \geq (\mu \circ \mu)(\theta, x-a) \\ &= \sup_{y \in S}[\min\{\mu(\theta,y), \mu(y, x-a)\}] \\ &\geq \min\{\mu(\theta, b-a), \mu(b-a, x-a)\} = \mu(b-a, x-a) \geq \mu(b,x) = \mu_b(x).\end{aligned}$$

So, $\mu_b \subseteq \mu_a$. By symmetry, we have $\mu_a \subseteq \mu_b$, thus we obtain that $\mu_a = \mu_b$. $\square$

Let μ be a fuzzy congruence relation on S, we define the addition $\mu_a \oplus \mu_b$ and product $\mu_a \circ \mu_b$ of μ_a and μ_b respectively as follows:

$$\mu_a \oplus \mu_b(x) = \begin{cases} \sup_{x=y+z}[\min\{\mu_a(y), \mu_b(z)\}], & \text{if} \quad x = y+z, \\ 0 & \text{if} \quad x \neq y+z, \end{cases}$$

$$\mu_a \otimes \mu_b(x) = \begin{cases} \sup_{x=yz}[\min\{\mu_a(y), \mu_b(z)\}], & \text{if} \quad x = yz, \\ 0 & \text{if} \quad x \neq yz. \end{cases}$$

Lemma 3.7. (see Theorem 2.2 in [7]) *Let υ be a fuzzy addition subgroup of the semiring S with reversible addition, then $\forall y, z \in S$, if $\upsilon(y) \neq \upsilon(z)$, then $\upsilon(y+z) = \min\{\upsilon(y), \upsilon(z)\}$.*

Lemma 3.8. *Let μ_θ be a fuzzy strong idea of the semiring S with reversible addition, then $\forall y, z \in S,\ \mu_\theta(y+z) = \mu_\theta(z+y)$.*

Proof. $\mu_\theta(y+z) = \mu_\theta(-z+z+y+z) \geq \mu_\theta(z+y) = \mu_\theta(-y+y+z+y) \geq \mu_\theta(y+z)$. $\square$

Proposition 3.9 *Let μ be a fuzzy congruence relation on the semiring S with reversible addition, then $\mu_a \oplus \mu_b = \mu_{a+b}$.*

Proof. First, we prove that binary operations $\oplus$ are well-defined. Assume that $\mu_a = \mu_b$ and $\mu_c = \mu_d$, then by Proposition 3.6, we have $\mu(a,b) = \mu(c,d) = 1$. Thus, $\mu(a+c, b+d) \geq \mu \circ \mu(a+c, b+d) = \sup_{x\in S}[\min\{\mu(a+c, x), \mu(x, b+d)\}] \geq \min\{\mu(a+c, b+c), \mu(b+c, b+d)\} \geq \min\{\mu(a,b), \mu(c,d)\} = 1$, i.e. $\mu_{a+c} = \mu_{b+d}$. For each $y, z \in S$, if $x = y+z$, then

$$\begin{aligned} \mu_{a+b}(x) &= \mu(a+b, x) \geq \mu \circ \mu(a+b, y+z) = \sup_{t\in S}[\min\{\mu(a+b, t), \mu(t, y+z)\}] \\ &\geq \min\{\mu(a+b, y+b), \mu(y+b, y+z)\} \geq \min\{\mu(a,y), \mu(b,z)\} \\ &= \min\{\mu_a(y), \mu_b(z)\}. \end{aligned}$$

So, $\mu_{a+b}(x) \geq \sup_{x=y+z}[\min\{\mu_a(y), \mu_b(z)\}]$, i.e. $\mu_{a+b} \geq \mu_a \oplus \mu_b$. Conversely, for all $x \in S$, as $(S, +,)$ is a group and $S = S+S$, x can be expressed as $x = y+z, (y, z \in S)$. Then by Lemma 3.7,

$$\begin{aligned} \mu_a \oplus \mu_b(x) &= \sup_{x=y+z}[\min\{\mu(a,y), \mu(b,z)\}] \geq \sup_{\substack{x=y+z \\ \mu(a,y)\neq\mu(b,z)}}[\min\{\mu(a,y), \mu(b,z)\}] \\ &= \sup_{\substack{x=y+z \\ \mu(a,y)\neq\mu(b,z)}}[\min\{\mu_\theta(y-a), \mu_\theta(z-b)\}] = \sup_{\substack{x=y+z \\ \mu(a,y)\neq\mu(b,z)}}[\min\{\mu_\theta(-a+y), \mu_\theta(z-b)\}] \\ &= \sup_{\substack{x=y+z \\ \mu(a,y)\neq\mu(b,z)}}[\mu_\theta(z-b-a+y)] = \sup_{\substack{x=y+z \\ \mu(a,y)\neq\mu(b,z)}}[\mu_\theta(-y+y+z-b-a+y)] \end{aligned}$$

$$\geq \sup_{\substack{x=y+z\\ \mu(a,y)\neq\mu(b,z)}} [\mu_\theta(y+z-b-a)] \geq \sup_{\substack{x=y+z\\ \mu(a,y)\neq\mu(b,z)}} [\mu_\theta((y+z)-(a+b))]$$
$$= \mu(a+b, y+z) = \mu(a+b, x) = \mu_{a+b}(x).$$

Then, $\mu_a \oplus \mu_b = \mu_{a+b}$. □

Similarly, we have the following proposition.

Proposition 3.10. *Let μ be a fuzzy congruence relation on the semiring S with reversible addition, then $\mu_a \otimes \mu_b \subseteq \mu_{ab}$.*

Therefore, we can define the binary operation $*$ on $S/\mu = \mu_a \mid a \in S\}$ as follows:

$$\mu_a * \mu_b = \mu_{ab}.$$

By Proposition 3.9 and Proposition 3.10 we get the following theorem.

Theorem 3.11. *Let μ be a fuzzy congruence relation on the semiring S with reversible addition, then $(S/\mu, \oplus, *)$ is a semiring with reversible addition, where μ_θ is zero element of $(S/\mu, \oplus, *)$, μ_{-a} is a negative element of μ_a.*

Proposition 3.12. *Let μ be a fuzzy congruence relation on the semiring S with reversible addition, then $\mu^{-1}(1) = \{(a,b) \mid \mu(a,b) = 1, a, b \in S\}$ is a congruence relation on the semiring S with reversible addition.*

Proof. By Lemma 3.4 in [3], μ is a fuzzy equivalence relation on a semiring with reversible addition, $(ax, bx) \in \mu^{-1}(1)$, $(xa, xb) \in \mu^{-1}(1)$. Moreover, for all $x \in S$, $\mu(a+x, b+x) \geq \mu(a,b) = 1$, which implies that $\mu(a+x, b+x) = 1$, that is $(a+x, b+x) \in \mu^{-1}(1)$. By the same way, $(x+a, x+b) \in \mu^{-1}(1)$, $\mu^{-1}(1)$ is a congruence relation on S. □

4. Homomorphism Theorems

Let S and $\overline{S}$ be two semirings with reversible addition, and f is a homomorphism from S to $\overline{S}$. Then the relation $\ker(f) = \{(a,b) \mid f(a) = f(b), a, b \in S\}$ is a congruence relation on S. Then, the characteristic function $\chi_{\ker(f)}$ is a fuzzy congruence relation on S, and

$$\chi_{\ker(f)}(a,b) = \begin{cases} 1 & \text{if } \ f(a) = f(b)\,, \\ 0 & \text{if } \ f(a) \neq f(b)\,. \end{cases}$$

Theorem 4.1. *Let μ be a fuzzy congruence relation on the semiring S with reversible addition. And let $(S/\mu, \oplus, *)$ be a semiring with reversible addition. The mapping $\mu^{\#} : S \to S/\mu$ defined by $\mu^{\#}(a) = \mu_a$ for all $a \in S$, then $\mu^{\#}$ is a homomorphic mapping.*

Proof. It is clear. □

Theorem 4.2. *Let S and $\overline{S}$ be two semirings with reversible addition and $f : S \to \overline{S}$ is a homomorphism. Then the fuzzy kernel $\chi_{\ker(f)}$ is a fuzzy congruence relation on S, and there is a monomorphism $g : S/\chi_{\ker(f)} \to \overline{S}$ such that $f = g \circ (\chi_{\ker(f)})^{\#}$.*

Proof. Let $a, b \in S$, we have $\mu^{\#}(a+b) = \mu_{a+b} = \mu_a \oplus \mu_b = \mu^{\#}(a) \oplus \mu^{\#}(b)$

$$\mu^{\#}(ab) = \mu_{ab} = \mu_a * \mu_b = \mu^{\#}(a) * \mu^{\#}(b).$$

Now we define $g : S/\chi_{\ker(f)} \to \overline{S},\ g((\chi_{\ker(f)})_a) = f(a),\ \forall a \in S$. If for each $a, b \in S,\ (\chi_{\ker(f)})_a = (\chi_{\ker(f)})_b$, then $\chi_{\ker(f)}(a,b) = 1,\ (a,b) \in \ker(f)$. Thus,

$$g((\chi_{\ker(f)})_a) = f(a) = f(b) = g((\chi_{\ker(f)})_b).$$

If $f(a) = f(b)$, then $(a,b) \in \ker(f),\ \chi_{\ker(f)}(a,b) = 1,\ (\chi_{\ker(f)})_a = (\chi_{\ker(f)})_b$, g is one to one. For arbitrary $a, b \in S$,

$$\begin{aligned} g((\chi_{\ker(f)})_a \oplus (\chi_{\ker(f)})_b) &= g((\chi_{\ker(f)})_{a+b}) = f(a+b) = f(a) + f(b) \\ &= g((\chi_{\ker(f)})_a) + g((\chi_{\ker(f)})_b), \end{aligned}$$

$$\begin{aligned} g((\chi_{\ker(f)})_a * (\chi_{\ker(f)})_b) &= g((\chi_{\ker(f)})_{ab}) = f(ab) = f(a)f(b) \\ &= g((\chi_{\ker(f)})_a) * g((\chi_{\ker(f)})_b). \end{aligned}$$

Then, g is a homomorphism. Let $a \in S$,

$$g \circ (\chi_{\ker(f)})^{\#}(a) = g((\chi_{\ker(f)})^{\#}(a)) = g((\chi_{\ker(f)})_a) = f(a).$$

So we obtain that $g \circ (\chi_{\ker(f)})^{\#} = f$. □

Theorem 4.3. *Let μ and υ be two fuzzy congruence relations on the semiring S with reversible addition such that $\mu \subseteq \upsilon$, then there is a unique homomorphism $g : s/\mu \to s/\upsilon$, such that $g \circ \mu^{\#} = \upsilon^{\#}$, and $S/\mu \diagup \chi_{\ker(g)} \cong S/\upsilon$.*

Proof. Define $g : S/\mu \to S/\upsilon,\ g(\mu_a) = \upsilon_a$, for each $a, b \in S$. Assume that $\mu_a = \mu_b$, then $1 = \mu(a,b) \leq \upsilon(a,b)$. So $\upsilon(a,b) = 1$, that is $\upsilon_a = \upsilon_b$, then g is well-defined. The remainder of the proof is clear. □

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