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UNIFIED QUALITATIVE PROPERTIES OF SOLUTION OF CERTAIN THIRD ORDER NON-LINEAR DIFFERENTIAL EQUATIONS

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Abstract: In this paper, we give sufficient conditions for the existence of a unique solution which is bounded together with its derivatives on the real line, globally asymptotically stable and periodic for a certain third order non-linear differential equation.

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1. Introduction

In this paper, we study the third order differential equation

$$\dddot{x} + a\ddot{x} + b\dot{x} + h(x) = P(t), \tag{1.1}$$

where a and b are positive constants, the functions h and P are continuous in the respective argument displayed explicitly.

Interesting results have been obtained by several authors on the bounded-

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ness, stability and other qualitative properties of solutions for higher order nonlinear differential equations. Some of these results have been summarized in [12]. Andres [2-4], Chukwu [6], Ezeilo [7-10], and Tejumola [13] have studied the equation (1.1) by using Lyapunov functions to investigate the boundedness and ultimate boundedness of solution on one side and stability and asymptotic stability on the other side. In their works, they employed incomplete Lyapunov functions for their studies, except for Chukwu[6] who made an attempt and indeed used a complete Lyapunov like (Yoshizawa) function with the aid of a signum function. Each of these works discussed just one qualitative property of solution.

In [1] a complete Lyapunov function was used to establish the boundedness and stability of solution of the equation (1.1). Burton et al [5] developed a theory to discuss some qualitative properties of solution (boundedness, stability and periodicity) in a unified way. In their work [5], they established conditions in which the considered equation has a solution which is bounded on the real line, stable as well as periodic. With this method, one can discuss with the aid of a suitable Lyapunov function, more than one qualitative properties of solution.

The purpose of this paper is to adapt the general theory of Burton [5] to the third order nonlinear differential equation of the form (1.1), and give sufficient conditions on the nonlinear term $h(x)$ that will guarantee the existence of a solution which is bounded together with its derivatives on the real line, globally stable and periodic.

Definition 1.1. Let

$$\dot{x} = f(t, x) \tag{1.2}$$

be a system of n-first order differential equations, a Lyapunov function V defined as $V : I \times \Re^n \to \Re$ is said to be *complete* if for $X \in \Re^n$:

(i) $V(t, X) \geq 0$;

(ii) $V(t, X) = 0$, if and only if $X = 0$; and

(iii) $\dot{V}|_{1.2}(t, X) \leq -c\,|X|$,

where c is any positive constant and $|X|$ given by $|X| = \left(\sum_{i=1}^{n} (x_i^2)\right)^{\frac{1}{2}}$ such that $|X| \to \infty$ as $X \to \infty$.

Definition 1.2. A Lyapunov function V defined as $V : I \times \Re^n \to \Re$ is said to be *incomplete* if for $X \in \Re^n$, (i) and (ii) in the above definition are satisfied, and in addition:

(iii) $\dot{V}(t,X)|_{1.2} \leq -c\,|X|_*$, where c is any positive constant and $|X|_*$ given by

$$|X|_* = \left(\sum_{i=1}^{j} x^2\right)^{\frac{1}{2}}$$

(by j we mean that not all the variables otherwise called the trajectories are neccesarily involved) such that $|X|_* \to \infty$ as $X \to \infty$.

Remark. In particular the equation (1.1) is better handled as a system of three-coupled first order equations by letting:

$$\begin{aligned} \dot{x} &= y\,, \\ \dot{y} &= z\,, \\ \dot{z} &= -az - by - h(x) + P(t)\,. \end{aligned} \tag{1.3}$$

Definition 1.3. A Lyapunov function V defined as $V : I \times \Re^3 \to \Re$ is said to be *complete* if for $X \in \Re^3$:
(i) $V(t,X) \geq 0$,
(ii) $V(t,X) = 0$, if and only if $X = 0$, and
(iii) $\dot{V}|_{1.3}(t,X) \leq -c\,|X|$ where c is any positive constant and $|X|$ given by $|X| = \left(\sum_{i=1}^{3} (x_i^2)\right)^{\frac{1}{2}}$ such that $|X| \to \infty$ as $X \to \infty$.

Definition 1.4. A Lyapunov function V defined as $V : I \times \Re^3 \to \Re$ is said to be *incomplete* if for $X \in \Re^3$ (i) and (ii) of the above definition are satisfied and in addition:
(iii) $\dot{V}(t,X)|_{1.3} \leq -c\,|X|_*$, where c is any positive constant and $|X|_*$ given by

$$|X|_* = \left(x^2 + y^2\right)^{\frac{1}{2}}\,,$$

or $|X|_* = \left(x^2 + z^2\right)^{\frac{1}{2}}$, or $|X|_* = \left(y^2 + z^2\right)^{\frac{1}{2}}$, or $|X|_* = \left(x^2\right)^{\frac{1}{2}}$, or $|X|_* = \left(y^2\right)^{\frac{1}{2}}$, or $|X|_* = \left(z^2\right)^{\frac{1}{2}}$ such that $|X|_* \to \infty$ as $X \to \infty$.

2. Generalized Theorems (Burton et al)

In an attempt to discuss the unified theory of periodicity of dissipative ordinary differential equations, Burton et al [5] considered the general differential equation

$$\dot{X} = F(t,X). \tag{2.1}$$

When the equation (2.1) is linear, it is written as

$$\dot{X} = A(t) + P(t), \tag{2.2}$$

with the homogeneous systems

$$\dot{X} = A(t)X, \tag{2.3}$$

where $A(t)$ is an $n \times n$ matrix.

The use of Lyapunov functions which led to the formulation of the following scheme was employed:

i) If $F(t, 0) = 0$, and if there is a function $V : [0, \infty) \times \Re^3 \to \Re$ such that

$$W_1(|X|) \leq V(t, X) \leq W_2(|X|)$$

and

$$\dot{V}(t, X)|_{1.3} \leq -W_3(|X|),$$

where $W_i(i = 1, 2, 3)$ are strictly increasing continuous functions defined as $W_i : [0, \infty) \to [0, \infty)$ with $W(s) > 0$ and $W(0) = 0$ as wedges. Then the solutions of the equation (2.1) is uniformly asymptotically stable (UAS).

ii) If there is a function $V : [0, \infty) \times \Re^3 \to \Re$ such that

$$W_1(|X|) \leq V(t, X) \leq W_2(|X|)$$

and

$$\dot{V}(t, X)|_{1.3} \leq -W_3(|X|) + M(M > 0),$$

then the solutions of the equation (2.1) are ultimately bounded (UB) and uniformly ultimately bounded (UBB).

iii) If the solution of the equation (2.1) and the equation (2.2) are unique, UB and UUB, then the equation (2.1) has a periodic solution.

iv) If the zero solutions of the equation (2.3) are uniformly asymptotically stable (UAS), then the equation (2.2) has a globally stable periodic solution.

We shall now state without proof of the theorems of Burton et al [5].

Theorem A. (see [5]) *If F is Lipschitz in X and periodic in t with period T and if the solutions are uniformly bounded and uniformly ultimately bounded for any given bound (say) B, then the equation (2.1) has a $T-$ periodic solution.*

Theorem B. (see [5]) *Let the following conditions hold:*

a) $F(t + T, X) = F(t, X)$ for all t and some $T > 0$, *b) all solutions of the equation (2.1) are bounded,*

c) each solution of the equation (2.1) is equi-asymptotically stable,

d) the zero solution of the homogeneous system corresponding to the equation (2.1) is uniformly asymptotically stable (UAS).

Then the equation (2.1) has a globally stable T-periodic solution.

3. Main Result

The following is the main result in this paper.

Theorem 3.1. *Let h be continuous and periodic with period T, and the following conditions hold:*

(i) $H_0 = \frac{h(x)-h(0)}{x} \in I_0$, $x \neq 0$ with $I_0 = [\delta, ab]$, $\delta > 0$,

(ii) $ab \geq H_0$, $\forall x \in \Re$,

(iii) $h(0) = 0$,

(iv)$|P(t)| \leq M$ (constant) for all $t \geq 0$,

Then the equation (1.1) has a globally stable T-periodic solution.

Notations. Throughout this paper $K, K_0, K_1, \ldots K_{12}$ will denote finite positive constants whose magnitudes depend only on the functions h and P as well as constants a, b and δ but are independent of solutions of the equation (1.1). $K_i's$ are not neccessarily the same for each time they occur, but each $K_i, i = 1, 2...$ retains its identity throughout.

4. Preliminary Results

The main tool besides the equation (1.1) itself in the proof of Theorem 3.1 is the function $V = V(x, y, z)$ defined by

$$\begin{aligned}2V(x,y,z) = \left(\frac{\delta b\ell}{\ell-1}\right)x^2 &+ \delta\left\{\frac{b(b+1)(\ell-1)+a^2[(\ell-1)+a\ell]}{ab(\ell-1)}\right\}y^2 \\ &+ \delta\left\{\frac{(b+1)(\ell-1)+a\ell}{ab(\ell-1)}\right\}z^2 \\ &+ \frac{2a\delta\ell}{(\ell-1)}xy + \frac{2\delta\ell}{(\ell-1)}xz + 2\delta\left\{\frac{(\ell-1)+a\ell}{b(\ell-1)}\right\}yz, \quad (4.1)\end{aligned}$$

where $\delta > 0$, $\ell > 1$ and $m^2 > 1$ for all x, y, z.

Lemma 4.1. *Subject to the assumptions of Theorem 3.1 there exist positive constants* $K_i = K_i(a, b, \ell, m, \delta), i = 1, 2$ *such that*

$$K_1(x^2 + y^2 + z^2) \leq V(x, y, z) \leq K_2(x^2 + y^2 + z^2). \tag{4.2}$$

Proof. By rearranging (4.1) we have

$$\begin{aligned} 2V(x,y,z) = {} & \left(\frac{\delta b\ell}{\ell-1}\right)\left(\frac{b}{m}x + my + \frac{m}{a}z\right)^2 + \frac{\delta(m^2-1)\ell}{m^2(\ell-1)}x^2 \\ & + \frac{\delta\ell}{b(\ell-1)}\left\{z + \frac{[(\ell-1)+a\ell(1-m^2)]}{b\ell(\ell-1)}y\right\}^2 \\ & + \delta\left\{(4b^2\ell(b+1)(\ell-1)^3 + 4a^2b\ell(\ell-1)^2[(\ell-1)+a\ell] + a(\ell-1)[2a\ell(m^2-1)\right. \\ & \left. -(\ell-1)] - a^3\ell[4bm^2(\ell-1)-(1-m^2)^2])/[4ab^2\ell(\ell-1)^2]\right\}y^2 \\ & + \delta\left\{\frac{(b+1)(\ell-1)-am^2\ell}{ab(\ell-1)}\right\}z^2. \end{aligned} \tag{4.3}$$

From which we obtain,

$$\begin{aligned} 2V(x,y,z) \geq {} & \frac{\delta(m^2-1)\ell}{m^2(\ell-1)}x^2 \\ & + \delta\left\{(4b^2\ell(b+1)(\ell-1)^3 + 4a^2b\ell(\ell-1)^2[(\ell-1)+a\ell] + a(\ell-1)[2a\ell(m^2-1))\right. \\ & \left. -(\ell-1)] - a^3\ell[4bm^2(\ell-1)-(1-m^2)^2]/[4ab^2\ell(\ell-1)^2]\right\}y^2 \\ & + \delta\left\{\frac{(b+1)(\ell-1)-am^2\ell}{ab(\ell-1)}\right\}z^2 \geq K_1(x^2+y^2+z^2), \end{aligned} \tag{4.4}$$

where

$$K_1 = \delta \min\left\{\left(\frac{(m^2-1)\ell}{m^2(\ell-1)}\right), \left(\frac{(b+1)(\ell-1)-am^2\ell}{ab(\ell-1)}\right), \Delta\right\},$$

and

$$\begin{aligned} \Delta = {} & \left\{(4b^2\ell(b+1)(\ell-1)^3 + 4a^2b\ell(\ell-1)^2[(\ell-1)+a\ell])\right. \\ & +a(\ell-1)[2a\ell(m^2-1)-(\ell-1)]) \\ & \left. -a^3\ell[4bm^2(\ell-1)-(1-m^2)^2]/[4ab^2\ell(\ell-1)^2])\right\}. \end{aligned}$$

Therefore,

$$2V(x,y,z) \geq K_1(x^2+y^2+z^2).$$

We have from the equation (4.1) and on using the Schwartz inequality $|xy| \leq \frac{1}{2}\left|x^2+y^2\right|$,

$$
\begin{aligned}
2V \leq (\frac{b\delta\ell}{\ell-1})x^2 + \delta\left\{\frac{b(b+1)(\ell-1)+a^2[(\ell-1)+a\ell}{ab(\ell-1)}\right\}y^2 \\
+\left\{\frac{(b+1)(\ell-1)+a\ell}{ab(\ell-1)}\right\}z^2 + (\frac{a\delta\ell}{\ell-1})(x^2+y^2) + (\frac{\delta\ell}{\ell-1})(x^2+z^2) \\
+(\frac{\delta\ell}{b(\ell-1)})\{(\ell-1)+a\ell\}\left(y^2+z^2\right) \\
\leq (\frac{\delta}{\ell-1})(b\ell+a\ell+\ell)x^2 \\
+(\frac{\delta}{\ell-1})\left\{\frac{b(b+1)(\ell-1)+a^2[(\ell-1)+a\ell}{ab}+\frac{(\ell-1)+a\ell}{b}+a\ell\right\}y^2 \\
+(\frac{\delta}{\ell-1})\left\{\frac{(b+1)(\ell-1)+a\ell}{ab}+\ell+\frac{(\ell-1)+a\ell}{b}\right\}z^2. \quad (4.6)
\end{aligned}
$$

Hence,

$$2V \leq K_2(x^2+y^2+z^2), \tag{4.7}$$

where

$$
\begin{aligned}
K_2 = (\frac{\delta}{\ell-1})\max\left\{\ell(a+b+1), \frac{b(b+1)(\ell-1)+a^2[(\ell-1)+a\ell]}{ab}\right. \\
\left.+\frac{(\ell-1)+a\ell}{b}+a\ell, \frac{(b+1)(\ell-1)+a\ell}{ab}+\ell+\frac{(\ell-1)+a\ell}{b}\right\} > 0.
\end{aligned}
$$

From inequalities (4.5) and (4.7), we have

$$K_1(x^2+y^2+z^2) \leq V(x,y,z) \leq K_2(x^2+y^2+z^2). \tag{4.8}$$

This proves Lemma 4.1. □

Lemma 4.2. *Suppose that the conditions of Theorem 3.1 hold and in addition let $\epsilon_i > 0 (i=1,2)$ and σ be constant such that*

$$H_0 = \frac{h(x)-h(0)}{x} \leq H_1 = \min\left\{\frac{4(\sigma-1)^2}{\sigma^2\epsilon_2^2}, \frac{4(\sigma-1)^2}{\sigma^2\epsilon_1^2}\right\},$$

then there are positive constants $K_j = K_j(a,b,\ell,m,\delta) (j=3,4)$ such that for any solution (x,y,z) of system (1.3),

$$\dot{V}|_{(1.3)} \equiv \frac{d}{dt}V|_{(1.3)}(x,y,z)$$
$$\leq -K_3(x^2+y^2+z^2)+K_4(|x|+|y|+|z|)\,|P(t)|\,. \quad (4.9)$$

Proof. From equations (1.1) and (1.3) we have,

$$\dot{V}|_{(1.3)} = \frac{\partial V}{\partial x}\dot{x}+\frac{\partial V}{\partial y}\dot{y}+\frac{\partial V}{\partial z}\dot{z} = \frac{\partial V}{\partial x}y+\frac{\partial V}{\partial y}z+\frac{\partial V}{\partial z}(-az-by-h(x)+P(t))$$
$$= -h(x)x-y^2-z^2-\left\{\frac{(b+1)(\ell+1)+a\ell}{ab(\ell-1)}z+\frac{(\ell-1)+a\ell}{b(\ell-1)}y+\frac{\ell}{\ell-1}x\right\}h(x)$$
$$+\left\{\frac{(b+1)(\ell+1)+a\ell}{ab(\ell-1)}z+\frac{(\ell-1)+a\ell}{b(\ell-1)}y+\frac{\ell}{\ell-1}x\right\}P(t). \quad (4.10)$$

Set $\left(\frac{(b+1)(\ell+1)+a\ell}{ab(\ell-1)}\right)=\epsilon_1$, $\left(\frac{(\ell-1)+a\ell}{b(\ell-1)}\right)\epsilon_2, \left(\frac{\ell}{\ell-1}\right)=\epsilon_3$, then by the condition on $h(x)$, we have that

$$\dot{V}|_{(1.3)} = -\left\{H_0x^2+y^2+z^2+H_0\epsilon_1xz+H_0\epsilon_2xy+H_0\epsilon_3x^2\right\}$$
$$-\,h(0)\left(\epsilon_1z+\epsilon_2y+(1+\epsilon_3)x\right)+\left(\epsilon_1z+\epsilon_2y+\epsilon_3x\right)P(t), \quad (4.11)$$

which can be written as

$$\dot{V}|_{(1.3)} = -\left\{\left(H_0x^2+H_0\epsilon_2xy+y^2\right)+\left(H_0\epsilon_3x^2+H_0\epsilon_1xz+z^2\right)\right\}$$
$$-\,h(0)\left(\epsilon_1z+\epsilon_2y+(1+\epsilon_3)x\right)+\left(\epsilon_1z+\epsilon_2y+\epsilon_3x\right)P(t). \quad (4.12)$$

By re-writing the equation (4.12), we have,

$$\dot{V}|_{(1.3)} = -\Bigg\{\left(\frac{H_0}{\tau}x^2+\frac{1}{\tau}y^2+\frac{1}{\tau}z^2\right)$$
$$+(H_0\epsilon_3x^2+H_0\epsilon_1xz+\frac{(\tau-1)}{\tau}z^2)+(\frac{\tau-1}{\tau}H_0x^2+H_0\epsilon_2xy+\frac{\tau-1}{\tau}y^2)\Bigg\}$$
$$-\,h(0)(\epsilon_1z+\epsilon_2y+(1+\epsilon_3)x)+(\epsilon_1z+\epsilon_2y+\epsilon_3x)P(t). \quad (4.13)$$

Let

$$U = U_1+U_2+U_3+U_4-U_5, \quad (4.14)$$

where

$$U_1 = \frac{1}{\tau}\left(H_0x^2+y^2+z^2\right), \quad (4.15)$$

$$U_2 = \left(H_0\epsilon_3x^2+H_0\epsilon_1xz+\frac{1}{\tau}z^2\right), \quad (4.16)$$

$$U_3 = \left(\frac{\tau - 1}{\tau} H_0 x^2 + H_0 \epsilon_2 xy + \frac{\tau - 1}{\tau} y^2 \right), \tag{4.17}$$

$$U_4 = h(0) \left(\epsilon_1 z + \epsilon_2 y + \epsilon_3 x \right), \tag{4.18}$$

$$U_5 = \left(\epsilon_1 z + \epsilon_2 y + \epsilon_3 x \right) P(t). \tag{4.19}$$

From the equation (4.15), we have

$$U_1 \leq K_3(x^2 + y^2 + z^2),$$

where $K_3 = \frac{1}{\tau} \cdot \max\left(H_0, 1\right).$

We also have from (4.19) that,

$$U_5 \leq K_4(|x| + |y| + |z|)P(t),$$

where $K_4 = \max\left(\epsilon_1, \epsilon_2, \epsilon_3 \ell\right).$

But $U \geq U_1 + U_4 - |U_5|$ and by the hypothesis $h(0) = 0, U_4$ vanishes, hence

$$U \geq U_1 - |U_5|.$$

Remark. Let us remark that U is positive definite, since U_2 and U_3 are quadratic forms in x and z, and x and y respectively. Since it is known that any quadratic form $AX^2 + BX + C$ is positive definite if $4AC - B^2 \geq 0$. Therefore,

$$\frac{dV}{dt} = \dot{V} = -U \leq -K_3(x^2 + y^2 + z^2) + K_4(|x| + |y| + |z|) \left| P(t) \right|. \tag{4.20}$$

Since

$$(|x| + |y| + |z|) \leq \sqrt{3}(x^2 + y^2 + z^2)^{\frac{1}{2}},$$

inequality (4.20) becomes

$$\frac{dV}{dt} \leq -K_3(x^2 + y^2 + z^2) + K_5(x^2 + y^2 + z^2)^{\frac{1}{2}} \left| P(t) \right|, \tag{4.21}$$

where $K_5 = \sqrt{3} K_4$.

This completes the proof of Lemma 4.2. □

5. Proof of Main Result

We shall now give the proof of the main result.

Proof of Theorem 3.1. From Lemma 4.1 and Lemma 4.2 we establish condition (d) of the Theorem B. By the hypothesis of the Theorem 3.1, condition (a) of the Theorem B is also satisfied.

We need now to show that under the same conditions of the Theorem 3.1, condition (b) of the Theorem B is also satisfied.

Indeed from the inequality (4.5), we have

$$(x^2+y^2+z^2)^{\frac{1}{2}} \le (\frac{2V}{K_1})^{\frac{1}{2}}.$$

Thus the inequality (4.21) becomes

$$\frac{dV}{dt} \le -K_6 V + K_7 V^{\frac{1}{2}} |P(t)|. \tag{5.1}$$

We note that $K_3(x^2+y^2+z^2) = K_3 \cdot \frac{V}{K_1}$, and

$$\frac{dV}{dt} \le -K_6 V + K_7 V^{\frac{1}{2}} |P(t)|, \tag{5.2}$$

where $K_6 = \frac{K_3}{K_2}$ and $K_7 = \frac{K_5}{K_2^{\frac{1}{2}}}$.

This imply that

$$\dot{V} \le -K_6 V + K_7 V^{\frac{1}{2}} |P(t)|,$$

and this can be written as

$$\dot{V} \le -2K_8 V + K_7 V^{\frac{1}{2}} |P(t)|, \tag{5.3}$$

where $K_8 = \frac{1}{2}K_6$.

Therefore

$$\dot{V} + K_8 V \le -K_8 V + K_7 V^{\frac{1}{2}} |P(t)| \tag{5.4}$$

$$\le K_7 V^{\frac{1}{2}} \left\{ |P(t)| - K_9 V^{\frac{1}{2}} \right\}, \tag{5.5}$$

where $K_9 = \frac{K_8}{K_7}$.

Thus the inequality (5.5) becomes

$$\dot{V} + K_8 V \le K_7 V^{\frac{1}{2}} V^* \tag{5.6}$$

where

$$V^* = |P(t)| - K_9 V^{\frac{1}{2}} \tag{5.7}$$
$$\leq V^{\frac{1}{2}} |P(t)|$$
$$\leq |P(t)|. \tag{5.8}$$

When $|P(t)| \leq K_9 V^{\frac{1}{2}}$,

$$V^* \leq 0 \tag{5.9}$$

and when $|P(t)| \geq K_9 V^{\frac{1}{2}}$,

$$V^* \leq |P(t)| \cdot \frac{1}{K_9}. \tag{5.10}$$

Substituting the inequality (5.9) into the inequality (5.5), we have

$$\dot{V} + K_8 V \leq K_{10} V^{\frac{1}{2}} |P(t)|,$$

where

$$K_{10} = \frac{K_7}{K_9}.$$

This implies that

$$V^{-\frac{1}{2}} \dot{V} + K_8 V^{\frac{1}{2}} \leq K_{10} |P(t)|. \tag{5.11}$$

Multiplying both sides of the inequality (5.11) by $e^{\frac{1}{2}K_8 t}$ we have,

$$e^{\frac{1}{2}K_8 t} \left\{ V^{-\frac{1}{2}} \dot{V} + K_8 V^{\frac{1}{2}} \right\} \leq e^{\frac{1}{2}K_8 t} K_{10} |P(t)|, \tag{5.12}$$

i.e

$$2\frac{d}{dt} \left\{ V^{\frac{1}{2}} e^{\frac{1}{2}K_8 t} \right\} \leq e^{\frac{1}{2}K_8 t} K_{10} |P(t)|. \tag{5.13}$$

Integrating both sides of the inequality (5.13) from t_0 to t, we have,

$$\left\{ V^{\frac{1}{2}} e^{\frac{1}{2}K_8 \gamma} \right\}_{t0}^{t} \leq \int_{t0}^{t} \frac{1}{2} e^{\frac{1}{2}K_8 \tau} K_{10} |P(\tau) d\tau|, \tag{5.14}$$

which implies that

$$\left\{ V^{\frac{1}{2}}(t) \right\} e^{\frac{1}{2}K_8 t} \leq V^{\frac{1}{2}}(t_0) e^{\frac{1}{2}K_8 t_0} + \frac{1}{2} K_{10} \int_{t0}^{t} |P(\tau)| e^{\frac{1}{2}K_8 \tau} d\tau,$$

or

$$V^{\frac{1}{2}}(t) \leq e^{-\frac{1}{2}K_8 t} \left\{ V^{\frac{1}{2}}(t_0) e^{\frac{1}{2}K_8 t_0} + \frac{1}{2} K_{10} \int_{t0}^{t} |P(\tau)| e^{\frac{1}{2}K_8 \tau} d\tau \right\}.$$

Using the inequalities(4.5) and (4.7), we have

$$K_1(x^2(t)+\dot{x}^2(t)+\ddot{x}^2(t)) \leq e^{-\frac{1}{2}K_8 t}\left\{K_2(x^2(t_0)\right.$$
$$\left.+\dot{x}^2(t_0)+\ddot{x}^2(t_0))e^{\frac{1}{2}K_8 t_0}+\frac{1}{2}K_{10}\int_{t0}^{t}|P(\tau)|\,e^{\frac{1}{2}K_8\tau}d\tau\right\}^2 \quad (5.15)$$

for all $t \geq t_0$.

Thus,

$$x^2(t)+\dot{x}^2(t)+\ddot{x}^2(t) \leq \frac{1}{K_1}\left\{e^{-\frac{1}{2}K_8 t}\left\{K_2(x^2(t_0)+\dot{x}^2(t_0)+\ddot{x}^2(t_0))e^{\frac{1}{2}K_8 t_0}\right.\right.$$
$$\left.\left.+\frac{1}{2}K_{10}\int_{t0}^{t}|P(\tau)|\,e^{\frac{1}{2}K_8\tau}d\tau\right\}^2\right\}$$
$$\leq \left\{e^{-\frac{1}{2}K_8 t}\left\{A_1+A_2\int_{t0}^{t}|P(\tau)|\,e^{\frac{1}{2}K_8\tau}d\tau\right\}^2\right\}, \quad (5.16)$$

where A_1, and A_2 are constants depending on $\{K_1, K_2, (x^2(t_0)+\dot{x}^2(t_0)+\ddot{x}^2(t_0))\}$ and $\{K_1, K_{10}\}$ respectively.

Therefore

$$x^2(t)+\dot{x}^2(t)+\ddot{x}^2(t)$$
$$\leq \left\{e^{-\frac{1}{2}K_8 t}\left\{A_1+A_2\int_{t0}^{t}|P(\tau)|\,e^{\frac{1}{2}K_8\tau}d\tau\right\}^2\right\} \leq K \quad (5.17)$$

for sufficiently large t where K is a constant.

By the inequality (5.17) and Lemma 4.1 and Lemma 4.2 we have established conditions (a), (b) and (d) of the Theorem B. Condition (c) follows from (d) and hence the completion of the proof of Theorem 3.1. □

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