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MULTIPLE COVERINGS OF CURVES AND COHOMOLOGICAL INVARIANTS

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Abstract: Let $f : X \to Y$ a degree k covering between smooth and projective curves. Here (following [2]) we introduce some cohomological invariants of the rank $k-1$ vector bundle $f_*(\mathcal{O}_X)/\mathcal{O}_Y$ and give a condition which implies the existence of a smooth curve Z and morphisms $u : X \to Z$, $v : Z \to Y$ such that $f = v \circ u$ and $1 < \deg(u) < k$.

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1. Multiple Coverings of Curves and Cohomological Invariants

Let X, Y be smooth and connected projective curves and $f : X \to Y$ a degree $k \geq 2$ covering. We work over an algebraically closed field $\mathbb{K}$ such that either $\mathrm{char}(\mathbb{K}) = 0$ or $\mathrm{char}(\mathbb{K}) > k$. Under one of these assumptions on $\mathrm{char}(\mathbb{K})$ the trace map gives a splitting of the natural map $\mathcal{O}_Y \to f_*(\mathcal{O}_X)$. Hence $f_*(\mathcal{O}_X) \cong \mathcal{O}_Y \oplus E_f$ with E_f a rank $k-1$ vector bundle on Y. As in [2] we will say that E_f is the vector bundle associated to f. Set $q := p_a(Y)$ and $g := p_a(X)$. We have $\deg(E_f) = \chi(E_f) + (k-1)(q-1) = q - g + (k-1)(q-1) = kq - k - g + 1$. Let $\tau(f)$ denote the minimal integer t such that there is an effective degree t divisor on Y such that $h^0(Y, E(D)) > 0$. Let $\sigma(f)$ denote the minimal integer t such that there exist $L \in \mathrm{Pic}^t(Y)$ with $h^0(Y, E \otimes L) > 0$. Hence $\sigma(f) \leq \tau(f)$.

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Let $\rho(f)$ be the minimal integer t such that there is an inclusion $E \subseteq F$ with F rank $k-1$ vector bundle on Y, $h^0(Y,F) > 0$ and $\deg(F) = t + \deg(E)$. Notice that a rank $k-1$ vector bundle on Y with $\deg(F) = t + \deg(E)$ contains E if and only if it is obtained from E making t positive elementary transformations (or, equivalently, if and only if E is obtained from F making t negative elementary transformations).

Remark 1. Since X is connected, we have $h^0(X,\mathcal{O}_X) = 1$ and hence $h^0(Y,E_f) = 0$. Hence $\tau(f) > 0$. There are coverings f such that $\sigma(f) = 0$. Here we give an example. Take $k = 2$, any Y of genus $q > 0$ and any $M \in \mathrm{Pic}^0(Y)$ such that $M^{\otimes 2} = \mathcal{O}_Y$, but $M \neq \mathcal{O}_Y$. There are $2^q - 1$ such line bundles M. To the pair (Y,M) there is associated a unique degree 2 connected étale covering $f : X \to Y$ such that $E_f \cong M$. Since $M^{\otimes 2} = \mathcal{O}_Y$, we have $h^0(Y,E_f \otimes M) = 1$, while $h^0(Y,E_f \otimes L) = 0$ if $\deg(L) < 0$. Obviously, we have $\tau(f) \le \rho(f) \le (k-1)\cdot\tau(f)$.

Remark 2. It is easy to compute the integer $\tau(f)$ for the coverings which are cohomologically balanced in the sense of [2], Definition 1.2. For the coverings (with X singular) considered in [2], Theorem 1.7, we have $\tau(f) = \sigma(f) = \rho(f)$.

Remark 3. Since $\chi(E\otimes L) = \chi(E)+\deg(L)\cdot(k-1) = q-g+\deg(L)\cdot(k-1)$ for every $L \in \mathrm{Pic}(Y)$, we have $h^0(Y,E\otimes L) > 0$ for every $L \in \mathrm{Pic}^t(Y)$ such that $t(k-1) \ge q-g+1$, i.e. for all $t \ge \lceil (q-g+1)/(k-1)\rceil$. Hence $\sigma(f) \le \lceil (q-g+1)/(k-1)\rceil$. This is very rough bound, although it is sharp in some cases (e.g. the ones considered in [2], Theorem 1.2). In the general case the knowledge of the integer $\sigma(f)$ is equivalent to the knowledge of the instability degree $s_1(E_f)$ with respect to line bundles of E_f.

Here we prove the following result (the case $q = 0$ being proved in [4] a different way when $\mathrm{char}(\mathbb{K}) = 0$

Theorem 1. *Fix integers $k \ge 2$, $g \le q > 0$ and a degree k covering $f : X \to Y$ of smooth and connected projective curves such that $p_a(X) = g$ and $p_a(Y) = q$. Assume $g > 1 + kq + qk\cdot\tau(f) - k\cdot\tau(f) - k$. Then there is a smooth and connected projective curve Z and morphisms $u : X \to Z$, $v : Z \to Y$ such that $f = v \circ u$ and $1 < \deg(u) < k$. Furthermore, we may find Z,u,v such that $p_a(Z) \le 1 + \deg(v)\cdot q + q\cdot\deg(v)\tau(v) - \deg(v)\cdot\tau(v) - k$.*

Proof. By assumption there is an effective degree $k\cdot\tau(f)$ effective divisor D on Y such that $h^0(X,\mathcal{O}_X(f^*(D))) = h^0(Y,\mathcal{O}_Y(D)) + h^0(Y,E(D)) \ge 1 + h^0(Y,\mathcal{O}_Y(D)) \ge 2$. Hence a general 2-dimensional linear subspace of $H^0(X,\mathcal{O}_X(f^*(D)))$ is not the pull-back of a pencil of 2-dimensional linear subspace of ${}^0(Y,\mathcal{O}_Y(D))$. Deleting the base points (if any) of this linear system we get a positive integer $d \leq k \cdot \tau(f)$ and a degree d morphism $h : X \to \mathbf{P}^1$ which does not factor through f. By Castelnuovo-Severi inequality ([3] or [1], p. 21) the induced map $(f, h) : X \to C \times \mathbf{P}^1$ is not birational onto its image. Taking the normalization Z of its image we get the triple (Z, u, v). If this triple does not satisfy the last assertion, then we may continue and get in finitely many steps a triple (Z, u, v) satisfying the "furthermore" part. $\square$

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