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ON THE SPECTRUM OF THE BUNDLE OF SECOND ORDER DIFFERENTIAL OPERATORS WITH ALMOST PERIODIC COEFFICIENTS

A.D. Orujov
Department of Mathematics
Faculty of Science and Art
Cumhuriyet University
Sivas, 58140, TURKEY
e-mail: eorucov@cumhuriyet.edu.tr

Abstract: In this paper, spectrum and resolvent of operator L_λ which is generated by differential expression $\ell_\lambda(y) = -y'' + (p_1\lambda + q_1(x))y' + (p_2\lambda^2 + \lambda q_2(x) + q_3(x))y$ have been investigated in the space $L_2(R)$. Here the coefficients $q_j(x), q_1'(x)$ are Bohr almost periodic functions whose fourier series are absolutely convergent. Fourier exponents of coefficients are positive and only have limit point at $+\infty$. It has been shown that spectrum of operator is pure continous and it consists of two lines (they can also coincide) which pass from the origin. Moreover simple spectral singularities can exist over the continous spectrum.

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1. Introduction

In this study, we investigate the spectrum and resolvent of maximal differential operator L_λ which is generated by the linear differential expression

$$\ell_\lambda(y) = -y'' + (p_1\lambda + q_1(x))y' + (p_2\lambda^2 + \lambda q_2(x) + q_3(x))y$$

in the space $L_2(R)$, where p_1 and $p_2 \neq 0$ are complex numbers, λ complex parameter,

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$$q_j(x) = \sum_{n=1}^{\infty} q_{jn} e^{i\alpha_n x}, \quad j = 1, 2, 3 \tag{1}$$

and the series

$$\sum_{n=1}^{\infty} \alpha_n \mid q_{1n} \mid < +\infty, \quad \sum_{n=1}^{\infty} \mid q_{jn} \mid < +\infty, \quad j = 2, 3 \tag{2}$$

converge. Besides a set of exponents $G = \{\alpha_n\}$ satisfies the following conditions:

1) $0 < \alpha_1 < \alpha_2 < \ldots < \alpha_n < \ldots, \quad \alpha_n \to +\infty.$

2) if $\alpha_i, \alpha_j \in G$ then $\alpha_i + \alpha_j \in G$.

Let Q be the class of Bohr almost periodic functions $q(x) = \sum_{n=1}^{\infty} q_n e^{i\alpha_n x}$ where $\sum_{n=1}^{\infty} |q_n| < +\infty$. It is clear that $q_j(x)$, $q_1'(x) \in Q$, $j = 1, 2, 3$. If at least one of $q_j(x)$, $j = 1, 2, 3$ is not zero then operator L_λ is nonselfadjoint.

Let θ_1 and θ_2 be roots of characteristic equation $-\theta^2 + p_1\theta + p_2 = 0$. It has been proved that operator L_λ has pure continuous spectrum which consists of the lines $\mathrm{Re}(\lambda\theta_1) = 0$ and $\mathrm{Re}(\lambda\theta_2) = 0$. If $\theta_1/\theta_2 \in R$ then the lines coincide and in particular case $\theta_1 \neq \theta_2$ spectrum contains the numbers $\lambda = 0$, $\lambda = \mp i\alpha_n/(\theta_2 - \theta_1)$, $n \in N$ which can be spectral singularities (in the sense of [2]) of operator L_λ but when $\theta_1 = \theta_2$, spectral singularities of operator L_λ do not exist.

Ordinary differential operators with coefficients from Q have been investigated in [1], [3]. Spectrum and resolvent of the bundle of $2n$ order differential operators with coefficients from Q have been examined in [4] by putting very restriction conditions over coefficients and roots of characteristic equation. However structure of spectrum has not determined exactly. In this paper spectrum and resolvent of the bundle of second order differential operators have been investigated under the more general conditions. Obtained results in a manner generalize the results of study [4] for case of $n = 1$.

2. Floquet Solutions of the Equation $\ell_\lambda(y) = 0$

Theorem 1. *Let $\theta_1 \neq \theta_2$. If $q_j(x)$, $j = 1, 2, 3$ are in the form of (1) and (2) holds, then for any root θ of characterisric equation and for $\lambda \neq i\alpha_n/(p_1 - 2\theta)$, $n \in N$ the differential equation*

$$-y'' + (p_1\lambda + q_1(x))y' + (p_2\lambda^2 + \lambda q_2(x) + q_3(x))y = 0 \tag{3}$$

has solution as

$$f(x, \lambda) = e^{\lambda\theta x}\left(1 + \sum_{n=1}^{\infty} U_n(\lambda) e^{i\alpha_n x}\right), \tag{4}$$

where series

$$\sum_{n=1}^{\infty} \alpha_n^2 \mid U_n(\lambda) \mid < +\infty, \tag{5}$$

is uniform convergent in each compact set $S \subset C$ which does not contain numbers $\lambda = i\alpha_n/(p_1 - 2\theta)$, $n \in N$.

Proof. Let θ be a root of the equation $-\theta^2 + p_1\theta + p_2 = 0$. Then $p_1 - 2\theta \neq 0$ since $\theta_1 \neq \theta_2$. If we assume the existence of solution of equation (3) as (4) then according to (5) we can find the derivatives of $f(x,\lambda)$ with respect to x as follows

$$f'(x,\lambda) = e^{\lambda\theta x}\left(\lambda\theta + \sum_{n=1}^{\infty}(i\alpha_n + \lambda\theta)U_n(\lambda)e^{i\alpha_n x}\right), \tag{6}$$

$$f''(x,\lambda) = e^{\lambda\theta x}\left(\lambda^2\theta^2 + \sum_{n=1}^{\infty}(i\alpha_n + \lambda\theta)^2 U_n(\lambda)e^{i\alpha_n x}\right). \tag{7}$$

If we substitute these in (3) and divide both side by $e^{\lambda\theta x}$ and consider the uniqueness theorem for almost periodic functions then we get

$$\begin{aligned}&\left[p_2\lambda^2 + p_1\lambda(i\alpha_n + \lambda\theta) - (i\alpha_n + \lambda\theta)^2\right] U_n(\lambda) + \lambda\theta q_{1n} + \lambda q_{2n} + q_{3n} \\ &\quad + \sum_{\alpha_r + \alpha_s = \alpha_n} U_r(\lambda)\left[(i\alpha_r + \lambda\theta)q_{1s} + \lambda q_{2s} + q_{3s}\right] = 0, \quad n = 1,2,3,\dots,\end{aligned}$$

or

$$\begin{aligned}U_n(\lambda) = &-\frac{1}{\alpha_n[\alpha_n + i(p_1 - 2\theta)\lambda]}\{\lambda\theta q_{1n} + \lambda q_{2n} + q_{3n} \\ &+ \sum_{\alpha_r + \alpha_s = \alpha_n} U_r(\lambda)[(i\alpha_r + \lambda\theta)q_{1s} + \lambda q_{2s} + q_{3s}]\}, \quad n \in N.\end{aligned} \tag{8}$$

On the contrary if $\{U_n(\lambda)\}$ satisfies system of equations (8) and (5) converges then it can be shown that $f(x,\lambda)$ determined by (4) is a solution of (3). Therefore solvability of (8) and convergent of series (5) is sufficient to proof the theorem.

From (8) $\{U_n(\lambda)\}$ is found uniquely as reccurence. It is seen that $U_n(\lambda)$ is rational function which can have simple poles $\lambda = i\alpha_k/(p_1 - 2\theta)$, $k = 1,2,\dots,n$, so can be uniquely written as

$$U_n(\lambda) = U_{0n} + \sum_{k=1}^{n}\frac{U_{kn}}{\alpha_k + i(p_1 - 2\theta)\lambda}, \quad \forall n \in N.$$

Let the set $S \subset C$ be a compact set which does not contain the points $\lambda = i\alpha_n/(p_1 - 2\theta)$, $n \in N$. Let us show that series (5) is uniformly convergent with respect to λ in S for $\{U_n(\lambda)\}$ which is found from (8).

It is obvious that $c_0 > 0, m > 0$ exist such that $\alpha_n \mid \alpha_n + i(p_1 - 2\theta)\lambda \mid \geq c_0\alpha_n^2$ and $\mid \lambda \mid \leq m$, for $\forall n \in N, \forall \lambda \in S$. If we care this in (8), we get

$$c_0\alpha_n^2 \mid U_n(\lambda) \mid \leq (\mid \theta \mid\mid q_{1n} \mid + \mid q_{2n} \mid)\, m + \mid q_{3n} \mid \\ + \sum_{\alpha_r+\alpha_s=\alpha_n} \mid U_r(\lambda) \mid \{(\mid \theta \mid m + \alpha_r) \mid q_{1s} \mid + m \mid q_{2s} \mid + \mid q_{3s} \mid\},$$

for $\forall \lambda \in S$.

Denoting

$$A_n = \frac{1}{c_0}\{(\mid \theta \mid\mid q_{1n} \mid + \mid q_{2n} \mid)\, m + \mid q_{3n} \mid\},$$

$$B_n = \frac{1}{c_0}\left\{\left(\frac{\mid \theta \mid m}{\alpha_1} + 1\right) \mid q_{1n} \mid + \frac{m}{\alpha_1} \mid q_{2n} \mid + \frac{1}{\alpha_1} \mid q_{3n} \mid\right\},$$

we obtain

$$\alpha_n^2 \mid U_n(\lambda) \mid \leq A_n + \sum_{\alpha_r+\alpha_s=\alpha_n} \alpha_r \mid U_r(\lambda) \mid B_s, \quad n \in N$$

from the above inequality. Let $u_n = \sup\limits_{\lambda \in S} \mid U_n(\lambda) \mid$, we get

$$\alpha_n^2 u_n \leq A_n + \sum_{\alpha_r+\alpha_s=\alpha_n} \alpha_r u_r B_s, \quad n \in N.$$

or

$$\sum_{n=1}^{k} \alpha_n^2 u_n \leq \sum_{n=1}^{k} A_n + \sum_{n=1}^{k} \sum_{\alpha_r+\alpha_s=\alpha_n} \alpha_r u_r B_s \\ \leq A + \sum_{r=1}^{k-1} \alpha_r u_r \sum_{s=1}^{k-1} B_s \leq A + B \sum_{n=1}^{k-1} \alpha_n u_n, \quad n \in N,$$

where it is clear that $A = \sum\limits_{n=1}^{\infty} A_n < +\infty$ and $B = \sum\limits_{n=1}^{\infty} B_n < +\infty$. Therefore

$$\sum_{n=1}^{k} \alpha_n^2 u_n \leq A + B \sum_{n=1}^{k-1} \alpha_n u_n, \quad \forall k \in N$$

is satisfied. By using this inequality and $\alpha_n \to +\infty$, we can show convergent of series $\sum_{n=1}^{+\infty} \alpha_n^2 u_n$ easily. In this case, $\sum_{n=1}^{+\infty} \alpha_n^2 \mid u_n(\lambda) \mid$ is majorizing series in S. According to Weierstrass Theorem series (5) is uniform convergent in S. In this case, since $S \subset C$ is any compact set, series (5) is convergent for all $\lambda \neq i\alpha_n/(p_1 - 2\theta), n \in N$. So, the theorem was proved. □

Corollary 1. *For $\forall x \in R$ the function $f(x,\lambda)$, $f'(x,\lambda)$ and $f''(x,\lambda)$ are meromorphic functions with respect to λ and they can have only simple poles $\lambda = i\alpha_n/(p_1 - 2\theta)$, $n \in N$ and they are also continous functions of x and λ for all $x \in R$, $\lambda \neq i\alpha_n/(p_1 - 2\theta)$, $n \in N$.*

Corollary 2. *If $\theta_1 \neq \theta_2$ then for $\forall \lambda \neq \lambda_n = i\alpha_n/(\theta_2 - \theta_1)$, $n \in N$ equation (3) have Floquet solutions as*

$$f_1(x,\lambda) = e^{\lambda\theta_1 x}\left(1 + \sum_{n=1}^{\infty} U_n^{(1)}(\lambda) e^{i\alpha_n x}\right)$$

and for $\forall \lambda \neq -\lambda_n$ as

$$f_2(x,\lambda) = e^{\lambda\theta_2 x}\left(1 + \sum_{n=1}^{\infty} U_n^{(2)}(\lambda) e^{i\alpha_n x}\right),$$

where

$$U_n^{(1)}(\lambda) = U_{0n}^{(1)} + \sum_{k=1}^{n} \frac{U_{kn}^{(1)}}{\alpha_k + i(\theta_2 - \theta_1)\lambda},$$

$$U_n^{(2)}(\lambda) = U_{0n}^{(2)} + \sum_{k=1}^{n} \frac{U_{kn}^{(2)}}{\alpha_k + i(\theta_1 - \theta_2)\lambda},$$

$n \in N$.

We can see easily for the function $f_1(x,\lambda)$ points of $\lambda = \lambda_n$, $n \in N$ and for the function $f_2(x,\lambda)$ points of $\lambda = -\lambda_n$, $n \in N$ can be simple pole.

Every $q(x) \in Q$ can be extended to upper semi-plane as analytical function. By using this property, wronskian of functions $f_1(x,\lambda)$ and $f_2(x,\lambda)$ is found as $W[f_1, f_2] = W(x,\lambda) = \lambda(\theta_2 - \theta_1) w_0 e^{p_1\lambda x + \int_0^x q_1(t)dt}$, where $w_0 = e^{\sum_{n=1}^{\infty} \frac{q_{1n}}{i\alpha_n}}$, $\lambda \neq \mp\lambda_n$, $n \in N$. It is seen that $f_1(x,\lambda)$ and $f_2(x,\lambda)$ are linearly independent for $\lambda \neq 0$ and $\lambda \neq \pm\lambda_n$, $n \in N$.

Obvious that function

$$f_{1n}(x) = \lim_{\lambda \to \lambda_n} f_1(x,\lambda)(\alpha_n + i(\theta_2 - \theta_1)\lambda) = e^{\theta_1 \lambda_n x} \sum_{k=n}^{\infty} U_{nk}^{(1)} e^{i\alpha_k x} \tag{9}$$

is a solution of equation (3) for $\lambda = \lambda_n$. If $U_{nn}^{(1)} = 0$ then $f_{1n}(x) \equiv 0$ and $f_1(x,\lambda_n)$ is a solution of (3) besides functions $f_1(x,\lambda_n)$, $f_2(x,\lambda_n)$ are linearly independent. If $U_{nn}^{(1)} \neq 0$ then $f_{1n}(x)$ is not zero and solutions $f_{1n}(x)$, $f_2(x,\lambda_n)$ are linearly dependent, because $W[f_{1n}(x), f_2(x,\lambda_n)] = 0$. There exists s_n such that $f_{1n}(x) = s_n f_2(x,\lambda_n)$. If we compare analytical expressions of both side, we get $s_n = U_{nn}^{(1)}$.

Now let us show that how to obtain the linearly independent solution with $f_2(x,\lambda_n)$ of equation (3) for $\lambda = \lambda_n$. If we consider (9) then function $\widetilde{f}_{1n}(x) = \lim_{\lambda \to \lambda_n} \left(f_1(x,\lambda) - \frac{U_{nn}^{(1)} f_2(x,\lambda)}{\alpha_n + i(\theta_2 - \theta_1)\lambda} \right)$ is also solution of equation (3) corresponding to $\lambda = \lambda_n$. According to expressions of $f_1(x,\lambda)$ and $f_2(x,\lambda)$, we can say that $\widetilde{f}_{1n}(x) = e^{\theta_1 \lambda_n x}(\psi_{1n}(x) + x\phi_{1n}(x))$, where $\psi_{1n}(x)$ and $\phi_{1n}(x)$ are Bohr almost functions. Obviously $\widetilde{f}_{1n}(x)$ and $f_2(x,\lambda_n)$ are linearly independent solutions of (3) for $\lambda = \lambda_n$. In the same manner, existence of linearly independent solutions $f_1(x,-\lambda_n)$ and $\widetilde{f}_{2n}(x) = e^{-\theta_2 \lambda_n x}(\psi_{2n}(x) + x\phi_{2n}(x))$ of (3) for $\lambda = -\lambda_n$ can be shown.

Note that since wronskian of solutions $f_1(x,\lambda)$ and $f_2(x,\lambda)$ are equal to zero for $\lambda = 0$ they are linearly dependent. Linearly independent solutions of equation (3) corresponding to $\lambda = 0$ are established according to Theorem 1. It is clear that solutions of the equation

$$-y'' + q_1(x)y' + (\lambda^2 + q_3(x))y = 0 \tag{10}$$

corresponding to $\lambda = 0$ are also solutions of (3). By Theorem 1, equation (10) has solution $\widetilde{f}(x,\lambda)$ which is analytic in near neighbourhood of $\lambda = 0$. By differentiating the equation (10) with respect to λ, it is sure that function $\widetilde{f}(x) = \frac{\partial \widetilde{f}(x,\lambda)}{\partial \lambda}|_{\lambda=0}$ is also a solution of (10) and (3) corresponding to $\lambda = 0$. We see easily that $\widetilde{f}(x,0) = p(x)$, $\tilde{f}(x) = xp(x) + q(x)$ where $p(x)$ and $q(x)$ are Bohr almost periodic functions. Linearly independence of $\widetilde{f}(x,0)$ and $\widetilde{f}(x)$ is seen from their open type.

Theorem 2. *When $\theta = \theta_1 = \theta_2$, equation (3) has Floquet solutions as*

$$f(x,\lambda) = e^{\theta\lambda x} p(x,\lambda), \quad \widetilde{f}(x,\lambda) = e^{\theta\lambda x}[p_1(x,\lambda) + xp(x,\lambda)],$$

where functions $p(x,\lambda)$, $p_1(x,\lambda)$ are almost periodic functions as $p(x,\lambda) = 1 + \sum_{n=1}^{\infty} u_n(\lambda) e^{i\alpha_n x}$, $p_1(x,\lambda) = 1 + \sum_{n=1}^{\infty} \widetilde{u}_n(\lambda) e^{i\alpha_n x}$ for $\forall \lambda \in C$ and $u_n(\lambda)$, $\widetilde{u}_n(\lambda)$

are polynomials whose degrees are not exceed of n. Series of $\sum_{n=1}^{\infty} \alpha_n^2 \mid u_n(\lambda) \mid$ *and* $\sum_{n=1}^{\infty} \alpha_n^2 \mid \widetilde{u}_n(\lambda) \mid$ *are uniformly convergent in each compact set* $S \subset C$.

Theorem 2 is proved by the method in the proof of Theorem 1. In this case $f(x,\lambda)$, $\widetilde{f}(x,\lambda)$ are entire functions of λ for each fixed x and Wronskian of their is found $\widetilde{W}(x,\lambda) = w_0 e^{\lambda p_1 x + \int_0^x q_1(t)dt}$.

Since $\widetilde{W}(x,\lambda) \neq 0$ for $\forall\lambda \in C$, $x \in R$ then $f(x,\lambda)$ and $\widetilde{f}(x,\lambda)$ form the fundamental system of solutions of equation (3).

3. Spectrum and Resolvent of Operator L_λ

Theorem 3. *Operator L_λ does not have eigenvalues.*

Proof. Let us show that equation $L_\lambda y = 0$ have only trivial solution which belongs to $L_2(R)$ for $\forall\lambda \in C$. In case $\theta_1 \neq \theta_2, \lambda \neq 0, \lambda \neq \pm\lambda_n$, $n \in N$ this follows from the properties of solutions $f_1(x,\lambda)$ and $f_2(x,\lambda)$. Really, solution $y(x) = c_1 f_1(x,\lambda) + c_2 f_2(x,\lambda)$ is in $L_2(R)$ when only $c_1 = c_2 = 0$. If linearly independent solutions of (3) according to $\lambda = 0$ or $\lambda = \pm\lambda_n$, $n \in N$ and to case $\theta_1 = \theta_2$ are taken instead of $f_1(x,\lambda)$ and $f_2(x,\lambda)$ then similar result is also valid. Hence $\sigma_p(L_\lambda) = \emptyset$. Theorem is proved. □

Let function $z(x) \in L_2(R)$ is a solution of $L_\lambda^*(z) = 0$ for $\lambda \in C$ then $\overline{z(x)}$ satisfies

$$-z''(x) - (p_1\lambda + q_1(x))z'(x) + [p_2\lambda^2 + \lambda q_2(x) + q_3(x) - q_1'(x)]z(x) = 0. \quad (11)$$

Since (11) is in the type of (3), (11) cannot have a solution which belongs to $L_2(R)$. That means $\sigma_p(L_\lambda^*) = \emptyset$ or $\sigma_r(L_\lambda) = \emptyset$ so $\sigma(L_\lambda) = \sigma_c(L_\lambda)$ and L_λ^{-1} is defined in dense set in $L_2(R)$ for $\forall\lambda \in C$.

In order to find $L_\lambda^{-1}$and resolvent set $\rho(L_\lambda)$ let us investigate solution $y(x) \in L_2(R)$ of

$$-y'' + (p_1\lambda + q_1(x))y' + (p_2\lambda^2 + \lambda q_2(x) + q_3(x))y = f(x)\,, \quad (12)$$

when $f(x) \in L_2(R)$. If we apply Lagrange method by using Floquet solutions of equation (3) then we find the solution of (12) as $y(x) = \int_{-\infty}^{+\infty} G(x,t,\lambda)f(t)dt$ where expression of $G(x,t,\lambda)$ can be written obviously.

If $\theta_1/\theta_2 \notin R$ then the lines $\mathrm{Re}(\lambda\theta_1) = 0$, $\mathrm{Re}(\lambda\theta_2) = 0$ divide the complex plane λ in to the four parts and in each part, $\mathrm{Re}(\lambda\theta_1)$ and $\mathrm{Re}(\lambda\theta_2)$ have constant sign. Let $S_0 = \{\lambda \in C : \mathrm{Re}(\lambda\theta_1) > 0, \mathrm{Re}(\lambda\theta_2) > 0\}$, $S_1 = \{\lambda \in C : \mathrm{Re}(\lambda\theta_1) > 0, \mathrm{Re}(\lambda\theta_2) < 0\}$ and S_2, S_3 are symmetric of S_0 and S_1 with respect to origin respectively.

If $\lambda \neq 0$, $\lambda \neq \pm\lambda_n$, $n \in N$ then $G(x,t,\lambda)$ can be written as

$$G(x,t,\lambda) = \frac{1}{\lambda(\theta_1-\theta_2)} \begin{cases} f_1(x,\lambda)\varphi_1(t,\lambda) - f_2(x,\lambda)\varphi_2(t,\lambda), & t > x\,, \\ 0, & t \leq x\,, \end{cases}$$

for $\lambda \in S_0$, as

$$G(x,t,\lambda) = \frac{1}{\lambda(\theta_1-\theta_2)} \begin{cases} f_1(x,\lambda)\varphi_1(t,\lambda), & t > x\,, \\ f_2(x,\lambda)\varphi_2(t,\lambda), & t \leq x\,, \end{cases}$$

for $\lambda \in S_1$, as

$$G(x,t,\lambda) = \frac{1}{\lambda(\theta_2-\theta_1)} \begin{cases} f_1(x,\lambda)\varphi_1(t,\lambda) - f_2(x,\lambda)\varphi_2(t,\lambda), & t < x\,, \\ 0, & t \geq x\,, \end{cases}$$

for $\lambda \in S_2$ and as

$$G(x,t,\lambda) = \frac{1}{\lambda(\theta_2-\theta_1)} \begin{cases} f_1(x,\lambda)\varphi_1(t,\lambda), & t < x\,, \\ f_2(x,\lambda)\varphi_2(t,\lambda), & t \geq x\,, \end{cases}$$

for $\lambda \in S_3$, where

$$\varphi_1(x,\lambda) = e^{-\lambda\theta_1 x}(1 + \sum_{n=1}^{\infty} V_n^{(1)}(\lambda)e^{i\alpha_n x})\,,$$

$$\varphi_2(x,\lambda) = e^{-\lambda\theta_2 x}(1 + \sum_{n=1}^{\infty} V_n^{(2)}(\lambda)e^{i\alpha_n x})$$

are solutions of equation (11) (their existence follows from Theorem 1) and

$$V_n^{(1)}(\lambda) = V_{0n}^{(1)} + \sum_{k=1}^{n} \frac{V_{kn}^{(1)}}{\alpha_k - i(\theta_2-\theta_1)\lambda}, \quad V_n^{(2)}(\lambda) = V_{0n}^{(2)} + \sum_{k=1}^{n} \frac{V_{kn}^{(2)}}{\alpha_k - i(\theta_1-\theta_2)\lambda},$$

$n \in N$.

Note that if $\theta_1/\theta_2 \in R$ then in case $\theta_1/\theta_2 > 0$ sectors S_1, S_3 and in case $\theta_1/\theta_2 < 0$ sectors S_0, S_2 disappear.

Similarly, in the case $\theta_1 = \theta_2 = \theta$, the kernel $G(x,t,\lambda)$ can be written as

$$G(x,t,\lambda) = \begin{cases} -f(x,\lambda)\widetilde{\varphi}(t,\lambda) + \widetilde{f}(x,\lambda)\varphi(t,\lambda), & t > x\,, \\ 0, & t \leq x\,, \end{cases}$$

for $\lambda \in S_0$, as

$$G(x,t,\lambda) = \begin{cases} f(x,\lambda)\widetilde{\varphi}(t,\lambda) - \widetilde{f}(x,\lambda)\varphi(t,\lambda), & t < x\,, \\ 0, & t \geq x\,, \end{cases}$$

for $\lambda \in S_2$, where $f(x,\lambda)$ and $\widetilde{f}(x,\lambda)$ are solutions of (3) and $\varphi(x,\lambda), \widetilde{\varphi}(x,\lambda)$ are solutions of (11) according to Theorem 2.

From the expanding type of functions $f_i(x,\lambda)$ and $\varphi_i(x,\lambda)$, ($f(x,\lambda)$, $\widetilde{f}(x,\lambda)$ and $\varphi(t,\lambda)$, $\widetilde{\varphi}(t,\lambda)$) it follows that

$$|G(x,t,\lambda)| \leq Ce^{-\tau|x-t|}(1+|x-t|) \tag{13}$$

where $C = C(\lambda) > 0$, $\tau = \min\left\{|\mathrm{Re}(\lambda\theta_1)|, |\mathrm{Re}(\lambda\theta_2)|\right\}, \forall x, t \in R$. By considering of open type of function $G(x,t,\lambda)$ and (13) can be proved by the standard method (see [2], p. 302-304) that, operator L_λ^{-1} whose kernel $G(x,t,\lambda)$ is bounded for $\mathrm{Re}(\lambda\theta_1) \neq 0$, $\mathrm{Re}(\lambda\theta_2) \neq 0$ (it is $\lambda \in \rho(L_\lambda)$) and unbounded for $\mathrm{Re}(\lambda\theta_1) = 0$ or $\mathrm{Re}(\lambda\theta_2) = 0$ (it is $\lambda \in \sigma_c(L_\lambda)$). On the other hand, points $\lambda = \pm\lambda_n$, $n \in N$ can be simple pole points of L_λ^{-1}. If $\theta_1/\theta_2 \notin R$ then these points belong to $S_0 \cup S_2$ and like points must be eigenvalues of operator L_λ. Since L_λ has not got eigenvalue, there is no singularity at these points. Therefore $\pm\lambda_n \in \rho(L_\lambda), n \in N$ and $G(x,t,\lambda)$ is regular in these points too. So $\mathrm{Re}(\lambda\theta_1) = 0$ and $\mathrm{Re}(\lambda\theta_2) = 0$ consist of continous spectrum of L_λ. Note that when $\theta_1/\theta_2 \in R$ and $\theta_1 \neq \theta_2$, the analytical continuation of kernel $G(x,t,\lambda)$ has simple poles at points $\lambda = 0$, $\lambda = \pm i\alpha_n/(\theta_2 - \theta_1)$, $n \in N$ on the spectrum which are called spectral singularities (in the sense of [2], p. 306) of operator L_λ.

Thus following theorem is true.

Theorem 4. *L_λ has pure continous spectrum and it is made up of lines $\mathrm{Re}(\lambda\theta_1) = 0$ and $\mathrm{Re}(\lambda\theta_2) = 0$. If $\theta_1/\theta_2 \in R$ then these lines coincide and in particular case when $\theta_1 \neq \theta_2$, simple spectral singularities can be at points $\lambda = 0$, $\lambda = \pm i\alpha_n/(\theta_2 - \theta_1)$, $n \in N$ on spectrum. When $\theta_1 = \theta_2$, spectral singularities of L_λ do not exist. Resolvent L_λ^{-1} is an integral operator with the kernel Karleman type for $\lambda \in \rho(L_\lambda)$.*

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