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THE INFINITE VECTORS AND INFINITE MATRICES OVER A FIELD

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Abstract: The linear relations of infinite vectors over a field are discussed, some new necessary and sufficient conditions for the existence of the inverse of a row (or column) – finite infinite matrix A over a field are given by the left (or right) cancellation law for multiplication of infinite matrices.

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1. Introduction

Let F be an arbitrary field. Then all the matrices we consider below are over F.

Definition 1. (see [2]) Let A be an infinite matrix over F. A is called row-finite (or column-finite) if every row (or column) of A contains only finitely many non-zero elements. If A is not only row-finite but also column-finite, then A is called row-column-finite (briefly, rcf).

Definition 2. (see [2]) Let A be an infinite matrix over F. The rows (or columns) of A are linearly dependent if there are finite many row-vectors (or column-vectors) which are linearly dependent over F. Otherwise, the rows (or columns) of A are called linearly independent over F.

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Definition 3. (see [2]) Let A be a column-finite matrix over F. For the row-vectors of A, $r_1, r_2, r_3, \cdots$, and arbitrary group of elements $a_1, a_2, a_3, \cdots$ of F, $a_1r_1+a_2r_2+a_3r_3+\cdots$ is called the infinite linear combinations of the columns of A in a natural way. The row-vectors of A are infinite linearly dependent if $a_1r_1+a_2r_2+a_3r_3+\cdots=0$ when $a_1, a_2, a_3, \cdots$ are not all zero. Otherwise, the row-vectors of A are called infinite linearly independent over F.

Dually, we can define the concepts of infinite linearly dependent and infinite linearly independent of the column-vectors of A.

Lemma A. (see [1]) *Let S be the set of infinite linear equations over F (the number of variables can be infinite). S is solvable in F if any finite subset of S is solvable in F.*

2. The Infinite Vectors over F

Definition 4. Let A be an infinite matrix over F, $r_1, r_2, r_3, \cdots$ and $c_1, c_2, c_3, \cdots$ the row-vectors and column-vectors of A, respectively. If $r_i = (a_{i1}, a_{i2}, a_{i3}, \cdots), i = 1, 2, 3, \cdots$, then

$$r_1(n) = (a_{11}, a_{12}, \cdots, a_{1n}), r_2(n) = (a_{21}, a_{22}, \cdots, a_{2n}),$$

$$r_3(n) = (a_{31}, a_{32}, \cdots, a_{3n}), \cdots$$

is called a n-cut of $r_1, r_2, r_3, \cdots$.

Similarly, the n-cut of $c_1, c_2, c_3, \cdots$ can be defined.

Lemma 1. *(1) Let A be a row-finite matrix over F and $c_1, c_2, c_3, \cdots$ the column vectors of A. If there exists a natural number i such that $c_i(n)$ can be linear represented by $\{c_j(n)|j \neq i\}$, for all $n \in N$, (N is the set of natural numbers), the c_i can be infinite linear represented by $\{c_j|j \neq i\}$.*

(2) Dual conclusion.

Proof. (1) Without loss generality, we only prove that c_1 can be infinite linear represented by $\{c_j|j \neq 1\}$ if $c_1(n)$ can be linear represented by $\{c_j(n)|j \neq 1\}$.

Assume that

$$c_1 = x_2c_2 + x_3c_3 + x_4c_4 + \cdots \tag{1}$$

Since A is row-finite, then (1) can be written as

$$S = \begin{cases} a_{11} = a_{12}x_2 + a_{13}x_3 + \cdots, \\ a_{21} = a_{22}x_2 + a_{23}x_3 + \cdots, \\ a_{31} = a_{32}x_2 + a_{33}x_3 + \cdots, \vdots \end{cases}$$

From the conditions of Lemma 1, any finite subset S_n of (S) is solvable in F. Hence (S) is solvable in F by Lemma A.

(2) Similar to the proof of (1). □

Lemma 2. *(1) Let A be a row-finite matrix over F , $r_1, r_2, r_3 \cdots$ and $c_1, c_2, c_3, \cdots$ the row vectors and the column vectors of A respectively. If $r_1, r_2, r_3, \cdots$ are linear independent over F, then S_c is infinite set. Here S_c is the maximal linear independent set of $c_1, c_2, c_3, \cdots$.*

(2) Dual conclusion.

Proof. (1) Assume that there are only finite many vectors $c_1, c_2, \cdots, c_n$ in S_c. Let $k > n$, since $r_1, r_2, \cdots, r_k$ are linear independent, then we have a matrix $A(r_1, r_2, \cdots, r_k)$ which is composed by $(r_1, r_2, \cdots, r_k)$ as the row vectors.

Since A is row-finite, then there is a function $r(i)$ (i$\in N$) satisfies that

$$a_{i,r(i)+1} = a_{i,r(i)+2} = \cdots = \cdots \qquad (i = 1, 2, 3, \cdots).$$

Let $s = s(k) = \max\{r(i)|i = 1, 2, 3, \cdots, k\}$, and the $k \times s$ matrix $A_{k,s}$ be composed by the first s columns of $r_1, r_2, \cdots, r_k$. Then $\mathrm{rank}(A_{k,s}) = k$, there exists at least a nonzero k-order subdeterminant in $A_{k,s}$. Suppose that this nonzero subdeterminant is located the $i_1, i_2, \cdots, i_k$ columns in $A_{k,s}$. Then $c_{i_1}, c_{i_2}, \cdots, c_{i_k}$ are linear independent over F. This is a contradiction. Thus S_c is a infinite set.

(2) Similar to the proof of (1). □

Notes. The maximal linear independent set S_c not have to be infinite linear independent even though it is infinite. Please see the following example.

Example 1. Let A be an infinite matrix over F, and

$$a_{ii} = a_{i,i+1} = 1 \;\; (i = 1, 2, 3, \cdots), \quad a_{i,j} = 0 \;\; (j \neq i, j \neq i + 1).$$

Then A is row-finite, the row vectors of A,$r_1, r_2, r_3, \cdots$ are linear independent over F and the column vectors of A, $c_1, c_2, c_3, \cdots$, so are. Thus $S_c = (c_1, c_2, c_3, \cdots)$, but $c_1, c_2, c_3, \cdots$ are infinite linear dependent.

Lemma 3. *(1) Let A be a row-finite matrix over F. $r_1, r_2, r_3, \cdots, c_1, c_2, c_3, \cdots$ and S_c are above. If S_c is infinite linear independent over F, then $S_R$is infinite set. Where S_R is the maximal linear independent set of $r_1, r_2, r_3, \cdots$.*

(2) Dual conclusion.

Proof. Similar to Lemma 3. □

Theorem 1. *(1) Let A be a row-finite matrix over F. $r_1, r_2, r_3, \cdots$ and $c_1, c_2, c_3, \cdots$ are above.If $c_1, c_2, c_3, \cdots$ are infinite linear independent over F, then for any row-finite infinite vector α, α can be represented by finite many row vectors of A.*

(2) Dual conclusion.

Proof. (1) Let $\alpha = (u_1, u_2, \cdots, u_n, 0, 0, \cdots)$. We only need to prove that the infinite vectors

$$e_i = (0, \cdots, 0, \overset{ith}{1}, 0, \cdots) \quad (i = 1, 2, 3, \cdots) \tag{2}$$

can be linear represented by finite many row vectors of A.

Assume that e_1 can not be linear represented by $r_1, r_2, \cdots, r_n$ for all natural number n. From Lemma 3, then S_R, the maximal linear independent set of $r_1, r_2, r_3, \cdots$ infinite. Without loss generality, we can assume that $r_1, r_2, \cdots, r_n$ be linear independent, then e_1, $r_1, r_2, \cdots, r_n$ are also linear independent.

Let $s = s(n) = \max\{r(i)|i = 1, 2, 3, \cdots, n\}$ ($r(i)$ is above), and

$$\begin{bmatrix} 1 & \cdots & 0 & 0 & \cdots & 0 \\ a_{11} & \cdots & a_{1n} & a_{1,n+1} & \cdots & a_{1,s} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \cdots & a_{nn} & a_{n,n+1} & \cdots & a_{n,s} \end{bmatrix}.$$

Then $\text{rank}(B) = n + 1$. Thus the $(n + 1)$-order nonzero subdeterminant in B must contain the first column of B. It implies that there is a n-order nonzero subdeterminaet in the $2, 3, \cdots, s$ columns of B. Hence $c_1(n)$ can be linear represent by $\{c_j(n)|j \neq 1\}$, thus c_1 can be infinite linear represented by $\{c_j(n)|j \neq 1\}$. It is a contradiction. Then e_1 can be linear represented by finite many row vectors of A.

Similarly, we can prove that e_i $(i = 2, 3, \cdots)$ can be linear represented by finite many row vectors of A.

(2) Similar to (1). □

Corolllary 1. *(1) Let A be a row-finite matrix over F. $r_1, r_2, r_3, \cdots$ and $c_1, c_2, c_3, \cdots$ are above.If $c_1, c_2, c_3, \cdots$ are infinite linear independent over F, then there must exists infinite row vectors whose are infinite linear independent over F.*

(2) Dual conclusion.

3. Some Conditions for a Inversible Infinite Matrix

Lemma 4. (see [2]) *(1) Let A and B be two row-finite matrices over F. Then for any infinite matrix C,*

$$(AB)C = A(BC)\,. \tag{3}$$

(2) Let A and B be two column-finite matrices over F. Then for any infinite matrix C,

$$(CB)A = C(BA)\,. \tag{4}$$

Theorem 2. *Let A be a row-finite matrix over F and $c_1, c_2, c_3, \cdots$ be above. Then the following conclusions are equivalent:*

(1) A is left row-finite inversible, i.e. there is a row-finite matrix B such that $BA = I$;

(2) If $AD = O$,then $D = O$ for any infinite matrix D;

(3) $c_1, c_2, c_3, \cdots$ are infinite linear independent over F.

Proof. Schema of the proof: (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1).

(1) $\Rightarrow$ (2) Assume that there is a row-finite matrix B such that $BA = I$. If $AD = O$, by Lemma 4, we have $D = ID = BAD = B(AD) = O$ for any infinite matrix D.

(2) $\Rightarrow$ (3) Assume that for any infinite matrix D, If $AD = O$, then $D = O$. If $c_1, c_2, c_3, \cdots$ are infinite linear dependent over F, then there is $\alpha = (d_1, d_2, d_3, \cdots) \neq 0$ such that

$$d_1c_1 + d_2c_2 + d_3c_3 + \cdots = 0\,.$$

Let $D = (\alpha^T, \alpha^T, \alpha^T, \cdots)$, then $D \neq O$. But $AD = O$, it implies that $D = O$. This is a contradiction.

(3) $\Rightarrow$ (1) Let $c_1, c_2, c_3, \cdots$ be infinite linear independent over F. By Theorem1, e_i can be linear represented by finite many row vectors of A. Then there are finite many numbers $b_{i1}, b_{i2}, \cdots, b_{it(i)} \in F$ $(i = 1, 2, 3, \cdots)$ such that $e_i = b_{i1}r_1 + b_{i2}r_2 + \cdots + b_{it(i)}r_{ti}$ for all $i \in N$.

Let $B = (b_{ij})$ satisfy that $r_i^B = (b_{i1}, b_{i2}, \cdots, b_{it(i)}, 0, \cdots,$ where r_i^B is the i-th row vector of B. Then B is row-finite and $BA = I$, A is left row-finite inversible. □

Theorem 3. *Let A be a column-finite matrix over F and $r_1, r_2, r_3, \cdots$ be above. Then the following conclusions are equivalent:*

(1) A is right column-finite inversible, i.e. there is a column -finite matrix B such that $AB = I$;

(2) If $DA = O$ then $D = O$ for any infinite matrix D;

(3) $r_1, r_2, r_3, \cdots$ are infinite linear independent over F.

Theorem 4. *Let A be a row-finite matrix over F. Then A is left row-finite inversible if and only if the multiplicative left-cancellation law holds, i.e. for arbitrary infinite matrices C and D, if $AC = AD$ then $C = D$.*

Theorem 5. *Let A be a column-finite matrix over F. Then A is right column-finite inversible if and only if the multiplicative right-cancellation law holds, i.e. for arbitrary infinite matrices C and D, if $CA = DA$ then $C = D$.*

The proofs of Theorems 3, 4, 5 are omitted.

Theorem 6. *Let Abe a rcf-matrix over F. Then A is rcf-inversible if and only if the multiplicative left and right cancellation law holds, i.e. for arbitrary infinite matrices C and D, if $CA = DA$ or $AC = AD$, then $C = D$.*

Proof. We only need to prove the sufficiency.

By Theorem 4, there is a row-finite matrix B such that $BA = I$,then $A = A(BA) = (AB)A$, $O = (AB)A - A = (AB - I)A$ it implies that $AB - I = O$, i.e. $AB = I$.

By Theorem 5, there is a column-finite matrix C such that $AC = I$, then $B = B(AC) = (BA)C = IC = C$. The proof is completed. □

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References

[1] A. Abian, Solvability of infinite systems of linear equations, *Archive of Mathematics*, **12** (1976), 43-46.

[2] S.Q. Wang, On the inverses of infinite n-matrices over a field, *J. Beijing Normal University*, **29** (1993), 327-330, In Chinese.

[3] S.Q. Wang, Diagonalization of row-column-finite infinite matrices, *Science in China, Series A*, **40**, No. 12 (1997), 1279-1286.