International Journal of Pure and Applied Mathematics

Volume 26 No. 2 2006, 211-216

EDGE SIGNED DOMINATION NUMBERS OF A GRAPH AND ITS COMPLEMENT

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Abstract: Let G be a finite connected simple graph with vertex set $V(G)$ and edge set $E(G)$. An edge signed domination function of G is a function f: $E(G) \to \{-1, 1\}$ such that $f[e] = \sum_{e' \in N[e]} f(e') \geq 1$ for all $e \in E(G)$, where $N[e]$ is the closed edge neighborhood of the edge e. The edge signed domination number $\gamma'_s(G)$ of G is $\min\{\sum_{e \in E(G)} f(e) | f$ is an edge signed domination function $\}$. An edge signed domination function of weight $\gamma'_s(G)$, we call a γ'_s-function of G. Let $\overline{G}$ be the complement of the graph G. In this paper we establish upper and lower bounds on $\gamma'_s(G) + \gamma'_s(\overline{G})$.

AMS Subject Classification: 26A33
Key Words: edge signed domination function, edge signed domination number

1. Introduction

Let G be a finite connected simple graph with vertex set $V(G)$ and edge set $E(G)$. We denote by $|X|$ The cardinality of a set X. For an edge $e = uv \in E(G)$, let $N(e) = \{e' | e' = uw \ or \ e' = vw, e' \in E(G)\}$ and $N[e] = N(e) \cup \{e\}$, and call them the edge neighborhood and closed edge neighborhood of e in G. For a set S of edges of $E(G), G[S]$ denote the subgraph of G induced by S. For each $v \in V(G), d_G(v)$ is the degree of v in G and for each $e \in E(G), d_G(e) = |N(e)|$

Received: November 20, 2005

is the edge-degree of e in G. Let $\overline{G}$ be the complement of the graph G. For an edge $e = uv \in E(\overline{G})$, let $N_{\overline{G}}(e) = \{e'|e' = uw \ or \ e' = vw, e' \in E(\overline{G})\}$ and $N_{\overline{G}}[e] = N_{\overline{G}}(e) \cup \{e\}$, and call them the edge neighborhood and closed edge neighborhood of e in $\overline{G}$. For each $v \in V(\overline{G}), d_{\overline{G}}(v)$ is the degree of v in $\overline{G}$ and for each $e \in E(\overline{G}), d_{\overline{G}}(e) = |N_{\overline{G}}(e)|$ is the edge-degree of e in $\overline{G}$.

Let $f : E(G) \to \{-1, 1\}$ be a function. The weight of f is $w(f) = \sum_{e \in E(G)} f(e)$. For an edge $e \in E(G)$, we define $f[e] = \sum_{e' \in N[e]} f(e')$. An edge signed domination function of G is a function $f : E(G) \to \{-1, 1\}$ such that $f[e] = \sum_{e' \in N[e]} f(e') \geq 1$ for all $e \in E(G)$. The edge signed domination number $\gamma'_s(G)$ of G is the minimum weight of an edge signed domination function of G. An edge signed domination function of $\gamma'_s(G)$, we call a $\gamma'_s(G)$-function of G.

Finding bounds on the sum or product of a parameter on a graph and its complement has become a useful lens for looking at the behavior of graph parameter, starting with the results of Nordhaus and Gaddum for the chromatic number [2]. For signed domination number it has been shown that for any graph G on n vertices such that $\gamma_s(G) + \gamma_s(\overline{G}) \leq 2n$ (see [4]). For any graph G on n vertices and m edges, since the edge signed domination number $\gamma'_s G)$ and $\gamma'_s(\overline{G})$ are at most m and $\binom{n}{2} - m$, respectively, we have $\gamma'_s(G) + \gamma'_s(\overline{G}) \leq \binom{n}{2}$. In this paper we show that this trivial bound is in fact achieved in exactly five graphs, and more generally, we establish upper and lower bounds on $\gamma'_s(G) + \gamma'_s(\overline{G})$.

2. Upper Bounds on $\gamma'_s(G) + \gamma'_s(\overline{G})$

We present a lemma characterizing graphs G on m edges for which $\gamma'_s(G) = m$.

Lemma 1. *A graph G on m edges has $\gamma'_s(G) = m$ if and only if every $e \in E(G)$ is either $d_G(e) = 1$ or e is adjacent to an edge e' of $d_G(e') = 1$.*

Proof. Let G be a graph contains an edge e such that $d_G(e) \geq 2$, and for each edge y adjacent to $e, d_G(y) \geq 2$. Consider the function $f : E(G) \to \{-1, 1\}$ for which $f(e) = -1$ and for any other edge $x, f(x) = 1$. Clearly this is an edge signed domination function of G. Therefore, the edge signed domination number of G is at most $m - 2$. Now suppose G be a graph with every $e \in E(G)$ is either $d_G(e) = 1$ or e adjacent to an edge e' of $d_G(e') = 1$. An edge signed domination function of G must assign 1 to every edge of edge-degree 1, and it must assign 1 to every edge that is adjacent to an edge of edge-degree 1. Hence, every edge in $E(G)$ must assign 1, and $\gamma'_s(G) = m$. □

From Lemma 1 we can seen that for a graph G on m edges if has $\gamma'_s(G) = m$, then G contains neither P_t (where P_t is a path on t vertices, $t \geq 5$) nor C_t (where

C_t is a cycle on t vertices, $t \geq 3$). Given a graph G, we denote the subdivision G^* of G is a graph obtained from G by subdividing each edge of G exactly once. Obviously, $|V(G^*)| = |V(G)| + |E(G)|$ and $|E(G^*)| = 2|E(G)|$.

Lemma 2. (see [7]) *Let G be a connected graph, then $\gamma_s'(G) = |E(G)|$ if and only if either $G \cong P_n$ for some n ($1 \leq n \leq 5$) or G is the subdivision of some star $K_{1,n}(n \geq 3)$.*

Lemma 3. *If G is a graph containing a vertex v of degree k then $\gamma_s'(G) \geq 1 + k - m$.*

Proof. Suppose $f : E(G) \to \{-1, 1\}$, v is a vertex of degree k in G and for some $u \in V(G)$ such that $vu \in E(G)$. The edge-degree of vu is $d_G(vu) = d_G(v) + d_G(u) - 2$ and $|N[vu]| = d_G(v) + d_G(u) - 1$. If f is an edge signed domination function of G, then $f[vu] \geq 1$. The least possible weight for f will now be achieved if $f(e') = -1$ for all $e' \notin N[vu]$. In this case the weight for f is $f(G) = 1 - (m - N[vu]) = 1 - [m - (d_G(v) + d_G(u) - 1)] \geq 1 + k - m$. □

In [8] X.Z. Lu investigated the edge signed domination number of complete graph K_n, an exact expression for the edge signed domination number of K_n was obtained.

Theorem 4. (see [8]) *For any complete graph K_n, we have*

$$\gamma_s'(K_n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even,} \\ \frac{n-1}{2} & \text{if } n \text{ is odd.} \end{cases}$$

Theorem 5. *If G is a connected graph on m edges such that $\gamma_s'(G) = m$ then $0 \leq \gamma_s'(\overline{G}) \leq \frac{m}{2}$.*

Proof. By Lemma 2, if $\gamma_s'(G) = |E(G)|$ if and only if either $G \cong P_n$ ($1 \leq n \leq 5$) or G is the subdivision of some star $K_{1,n}(n \geq 3)$.

(1) If $G \cong P_n$ ($1 \leq n \leq 5$), we can easily to check that the conclusions are true.

(2) If G is the subdivision of $K_{1,n}(n \geq 3)$. For convenience let $V(G) = \{v_i : i = 0, 1, \cdots, n; u_j : j = 1, \cdots, n\}$ and $E(G) = \{v_o u_j : j = 1, \cdots, n;\ u_i v_i : i = 1, \cdots, n\}$, $|V(G)| = 2n + 1$ and $|E(G)| = 2n = m$. In this case $V(G) = V(\overline{G})$ and $E(\overline{G}) = \{v_i v_j : i, j = 0, 1, \cdots, n, i \neq j\} \bigcup \{u_i u_j : i, j = 1, \cdots, n, i \neq j\} \bigcup \{u_i v_j : i, j = 1, \cdots, n, i \neq j\}$, $|V(\overline{G})| = 2n + 1$ and $|E(\overline{G})| = \binom{2n+1}{2} - 2n$. Let $E_1 = \{v_i v_j : i, j = 0, 1, \cdots, n, i \neq j\}, E_2 = \{u_i u_j : i, j = 1, \cdots, n, i \neq j\}$ and $E_3 = \{u_i v_j : i, j = 1, \cdots, n, i \neq j\}$. $\overline{G}[E_1], \overline{G}[E_2]$ and $\overline{G}[E_3]$ denote the subgraphs of $\overline{G}$ induced by E_1, E_2 and E_3, respectively.

For the lower bound, let f be an edge signed domination function of G, let $V_1 = \{u, v | e = uv \in E(\overline{G}), f(e) = 1\}, E_1 = \{e | e \in E(\overline{G}), f(e) = 1\}, V_2 = \{u, v | e = uv \in E(\overline{G}), f(e) = -1\}, E_2 = \{e | e \in E(\overline{G}), f(e) = -1\}, G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$. Let $|E(\overline{G})| = m'$ and $|E(G_1)| = |E_1| = t$. Then $|E(G_2)| = |E_2| = m' - t$ and $\gamma_s'(\overline{G}) = |E(G_1)| - |E(G_2)| = 2t - m'$. We define that $d^*(u) = d_{G_1}(u) - d_{G_2}(u)$ for each vertex $u \in V(\overline{G})$. By using this notation, we can see that

$$\gamma_s'(\overline{G}) = |E_1| - |E_2| = \frac{1}{2} \sum_{u \in V(\overline{G})} d_{G_1}(u) - \frac{1}{2} \sum_{u \in V(\overline{G})} d_{G_2}(u) = \frac{1}{2} \sum_{u \in V(\overline{G})} d^*(u).$$

If for each $u \in V(\overline{G})$ such that $d^*(u) \geq 0$ then $\gamma_s'(\overline{G}) = \frac{1}{2} \sum_{u \in V(\overline{G})} d^*(u) \geq 0$.

If there exist some $u \in V(\overline{G})$ such that $d^*(u) \leq -1$ then any two these vertices u and $v, uv \notin E(\overline{G})$. Otherwise, $f[uv] = \sum_{e \in N[uv]} f(e) = d^*(u) + d^*(v) - f(uv) \leq -1$, it is contradiction on f being an edge signed domination function. From the construction of $\overline{G}$, there exist at most two vertices u and v in $\overline{G}$ such that $d^*(u) \leq -1$ and $d^*(v) \leq -1$. If exists only one vertex $u \in V(\overline{G})$ such that $d^*(u) \leq -1$, then there exists a vertex $w \in V(\overline{G})$ such that $uw \in E(\overline{G})$. Since $\gamma_s'(\overline{G}) = \sum_{e \in N[uw]} f(e) = d^*(u) + d^*(w) - f(uw) \geq 1$, then $d^*(u) + d^*(w) \geq 0$, hence

$$\gamma_s'(\overline{G}) = \frac{1}{2} \sum_{v \in V(\overline{G})} d^*(v) = \frac{1}{2} \left(\sum_{v \in V(\overline{G}) - \{u, w\}} d^*(v) + d^*(u) + d^*(w) \right) \geq 0.$$

If exist two vertices $u, v \in V(\overline{G})$ such that $d^*(u) \leq -1, d^*(v) \leq -1$, then there exist two vertices $w_1 \in V(\overline{G}), w_2 \in V(\overline{G})$ such that $uw_1 \in E(\overline{G}), vw_2 \in E(\overline{G})$ and $d^*(u) + d^*(w_1) \geq 0, d^*(v) + d^*(w_2) \geq 0$. Since for each $v \in V(\overline{G})$ has $d_{\overline{G}}(v) \geq n$, so we can choose that $w_1 \neq w_2$, hence

$$\gamma_s'(\overline{G}) = \frac{1}{2} \left(\sum_{x \in V(\overline{G}) - \{u, v, w_1, w_2\}} d^*(x) + d^*(u) + d^*(w_1) + d^*(v) + d^*(w_2) \right) \geq 0.$$

For the upper bound, to complete the proof, it suffices to construct an edge signed domination function f of $\overline{G}$.

If n is odd, at this time $\overline{G}[E_3]$ is a $(n-1)$-regular graph and $|\overline{G}[E_3]| = n(n-1)$. Since $(n-1)$ is even, $\overline{G}[E_3]$ has an Euler circuit $C_{n(n-1)} = \{e_1, e_2, \cdots, e_{n(n-1)}\}$. Let $V(\overline{G}[E_3]) = \{u_i | 1 \leq i \leq \frac{n-1}{2}\} \cup \{u_j | \frac{n-1}{2} < j \leq n-1\} \cup \{u_n\}$. Define $f : E(\overline{G}) \to \{-1, 1\}$ by:

$$f(v_i v_j) = \begin{cases} 1 & \text{if } i + j \text{ is odd,} \\ -1 & \text{if } i + j \text{ is even,} \end{cases} \quad f(e) = \begin{cases} 1 & \text{if } e = u_i u_n \text{ or } e = u_i u_j, \\ -1 & \text{otherwise,} \end{cases}$$

$$f(e_t) = \begin{cases} 1 & \text{if } e_t \in \overline{G}[E_3] = \{e_1, e_2, \cdots, e_{n(n-1)}\}, t \text{ is odd,} \\ -1 & \text{if } e_t \in \overline{G}[E_3] = \{e_1, e_2, \cdots, e_{n(n-1)}\}, t \text{ is even.} \end{cases}$$

Then f is an edge signed domination function of $\overline{G}$, by Theorem 4 we have $\gamma'_s(\overline{G}) \le w(f) = \frac{n-1}{2} + \frac{n+1}{2} = n = \frac{m}{2}$.

If n is even, at this time the graph $G^* = \overline{G}[E_3]\backslash\{u_n, v_n\}$ is a $(n-2)$-regular graph and $|G^*| = (n-1)(n-2)$. Since $(n-2)$ is even, G^* has an Euler circuit $C_{(n-1)(n-2)} = \{e_1, e_2, \cdots, e_{(n-1)(n-2)}\}$. Let $V(\overline{G}[E_1]) = \{v_i | 1 \le i \le \frac{n}{2}\} \cup \{v_j | \le \frac{n}{2} < j \le n\} \cup \{v_0\}$. Define $f : E(\overline{G}) \to \{-1, 1\}$ by:

$$f(u_iu_j) = \begin{cases} 1 & \text{if } i+j \text{ is odd,} \\ -1 & \text{if } i+j \text{ is even,} \end{cases} \quad f(e) = \begin{cases} 1 & \text{if } e = v_iv_0 \text{ or } e = v_iv_j, \\ -1 & \text{otherwise,} \end{cases}$$

$$f(e_t) = \begin{cases} 1 & \text{if } e_t \in G^* = \{e_1, e_2, \cdots, e_{(n-1)(n-2)}\}, t \text{ is odd,} \\ -1 & \text{if } e_t \in G^* = \{e_1, e_2, \cdots, e_{(n-1)(n-2)}\}, t \text{ is even,} \end{cases}$$

$$f(v_nu_j) = \begin{cases} 1 & \text{if } n+j \text{ is odd, } 1 \le j \le n-1, \\ -1 & \text{if } n+j \text{ is even, } 1 \le j \le n-1, \end{cases}$$

$$f(u_nv_i) = \begin{cases} 1 & \text{if } 1 \le i \le \frac{n}{2}, \\ -1 & \text{if } \frac{n}{2} < i \le n-1. \end{cases}$$

Then f is an edge signed domination function of $\overline{G}$, by Theorem 4 we have $\gamma'_s(\overline{G}) \le w(f) = \frac{n}{2} + \frac{n+1-1}{2} = n = \frac{m}{2}$.

This completes the proof. □

For any graph G on n vertices and m edges, since the edge signed domination number $\gamma'_sG)$ and $\gamma'_s(\overline{G})$ are at most m and $\binom{n}{2} - m$, respectively, we have $\gamma'_s(G) + \gamma'_s(\overline{G}) \le \binom{n}{2}$. From Lemma 1 and Theorem 5 we notices that $\gamma'_s(G) + \gamma'_s(\overline{G}) = \binom{n}{2}$ if and only if $\gamma'_s(G) = m$ and $\gamma'_s(\overline{G}) = \binom{n}{2} - m$, that is

Corollary 6. *Let G be a graph with n vertices and m edges, then $\gamma'_s(G) + \gamma'_s(\overline{G}) = \binom{n}{2}$ if and only if $G \in \{P_1, P_2, \overline{P_2}, P_3, \overline{P_3}, P_4\}$, where P_i is a path on i vertices.*

Acknowledgements

This project is supported by the NNSF (19871036).

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