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IYENGAR TYPE ESTIMATE OF ERROR IN TRAPEZOIDAL RULE

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Abstract: Using Hayashi's inequality, weighted Iyengar type inequality for functions with bounded higher derivatives is derived. This result generalizes results obtained by P. Cerone in [3] and H. Gauchman in [7]. It also improves estimation obtained by Feng Qi in [10].

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1. Introduction

In 1938 K.S.K. Iyengar proved the following inequality (see [8]):

Theorem 1. *Let function f be differentiable on $[a,b]$ and $|f'(x)| \leq M$. Then*

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - \frac{f(a)+f(b)}{2}\right| \leq \frac{M(b-a)}{4} - \frac{(f(b)-f(a))^2}{4M(b-a)}. \qquad (1.1)$$

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Through the years, Iyengar's inequality (1.1) has been generalized in various ways. Generalizations that are of interest in this paper are the ones obtained using Hayashi's modification of the well-known Steffensen's inequality. In [2], for example, R.P. Agarwal and S.S. Dragomir first proved the following result.

Theorem 2. *Let function f be differentiable on $[a,b]$ and $m \le f'(x) \le M$. Then*

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - \frac{f(a)+f(b)}{2}\right| \le \frac{[f(b)-f(a)-m(b-a)][M(b-a)-f(b)+f(a)]}{2(M-m)(b-a)}. \quad (1.2)$$

(1.2) reduces to (1.1) upon taking $m=-M$.

In [3], P. Cerone proved the following result for the trapezoidal rule.

Theorem 3. *Let $f : I \subseteq \mathbf{R} \to \mathbf{R}$ be such that $f^{(n-1)}$ is absolutely continuous on I° (I° being the interior od I) and $[a,b] \subset I^\circ$. Assume $m = \inf_{x\in[a,b]} f^{(n)}(x) > -\infty$ and $M = \sup_{x\in[a,b]} f^{(n)}(x) < \infty$. Then*

$$\left|\int_a^b f(x)dx - \sum_{k=1}^{n} E_k(\Theta;a,b) + R - \frac{M-m}{2(n+1)!}(U+L)\right| \le \frac{M-m}{2(n+1)!}(U-L)\,, \quad (1.3)$$

where

$$E_k(\Theta;a,b) = \frac{1}{k!}[(\Theta-a)^k f^{(k-1)}(a) - (\Theta-b)^k f^{(k-1)}(b)]\,, \quad (1.4)$$

$$R = \frac{m}{(n+1)!}\left[(\Theta-b)^{n+1} - (\Theta-a)^{n+1}\right]\,, \quad (1.5)$$

$$L = \begin{cases} (\lambda_n^a)^{n+1} + (\lambda_n^b)^{n+1}, & n \text{ even}\,, \\ (\Theta-b+\lambda_n^0)^{n+1} - (\Theta-b)^{n+1}, & n \text{ odd}\,, \end{cases} \quad (1.6)$$

$$U = \begin{cases} (\Theta-b+\lambda_n^b)^{n+1} - (\Theta-a-\lambda_n^a)^{n+1} + (\Theta-a)^{n+1} - (\Theta-b)^{n+1}, & n \text{ even}\,, \\ (\Theta-a)^{n+1} - (\Theta-a-\lambda_n^0)^{n+1}, & n \text{ odd}\,, \end{cases} \quad (1.7)$$

$$\lambda_n^0 = \frac{1}{M-m}\left[f^{(n-1)}(b) - f^{(n-1)}(a) - m(b-a)\right]\,, \quad (1.8)$$

$$\lambda_n^a = \frac{1}{M-m}\left[f^{(n-1)}(\Theta) - f^{(n-1)}(a) - m(\Theta-a)\right]\,, \quad (1.9)$$

$$\lambda_n^b = \frac{1}{M-m}\left[f^{(n-1)}(b) - f^{(n-1)}(\Theta) - m(b-\Theta)\right]. \tag{1.10}$$

Taking $n = 1$ and $\Theta = (a+b)/2$ in (1.3), produces (1.2).

In [7], H. Gauchman proved two inequalities involving Taylor's remainder. He denotes by $R_{n,f}(c,x)$ the nth Taylor's remainder of function $f(x)$ with center c:

$$R_{n,f}(c,x) = f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k.$$

Theorem 4. *Let $f : I \to \mathbf{R}$ and $w : I \to \mathbf{R}$ be two functions, $a, b \in I^\circ$, $a < b$ and let $f \in C^{n+1}([a,b])$ and $w \in C([a,b])$. Assume that $m \le f^{(n+1)}(x) \le M$, $m \ne M$ and $w(x) \ge 0$ for each $x \in [a,b]$. Then:*

$$\begin{aligned}
(i) \qquad & \frac{1}{(n+1)!}\int_{b-\lambda_n^0}^{b} (x-b+\lambda_n^0)^{n+1} w(x)dx \\
\le\ & \frac{1}{M-m}\int_a^b \left[R_{n,f}(a,x) - m\frac{(x-a)^{n+1}}{(n+1)!}\right] w(x)dx \\
\le\ & \frac{1}{(n+1)!}\int_a^b [(x-a)^{n+1} - (x-a-\lambda_n^0)^{n+1}]w(x)dx \\
+\ & \frac{(-1)^{n+1}}{(n+1)!}\int_a^{a+\lambda_n^0} (a+\lambda_n^0-x)^{n+1} w(x)dx; \qquad (1.11)
\end{aligned}$$

$$\begin{aligned}
(ii) \qquad & \frac{1}{(n+1)!}\int_a^{a+\lambda_n^0} (a+\lambda_n^0-x)^{n+1} w(x)dx \\
\le\ & \frac{(-1)^{n+1}}{M-m}\int_a^b \left[R_{n,f}(b,x) - m\frac{(x-b)^{n+1}}{(n+1)!}\right] w(x)dx \\
\le\ & \frac{1}{(n+1)!}\int_a^b [(b-x)^{n+1} - (b-\lambda_n^0-x)^{n+1}]w(x)dx \\
+\ & \frac{(-1)^{n+1}}{(n+1)!}\int_{b-\lambda_n^0}^{b} (x-b+\lambda_n^0)^{n+1} w(x)dx; \qquad (1.12)
\end{aligned}$$

where λ_n^0 is defined by (1.8).

Addition of (1.11) and (1.12) upon taking $n = 0$ and $w(x) = 1$ followed by division by 2, produces (1.2) again. Of course, as a special case we get Iyengar's inequality once more.

In this paper we shall prove a generalization of both Theorem 3 and Theorem 4 in a sense that we shall obtain an inequality involving both the weight $w(x)$ and the parameter Θ.

Before we proceed with our main result, it should be mentioned that using same technique similar inequalities were proved in a number of papers. In [1], R.P. Agarwal, V. Čuljak and J. Pečarić derived inequality (1.3) for an odd n. For an even n, using a somewhat different technique, they obtained a result which involves only the midpoint. In [6], only the case $n = 2$ was considered.

In [5], an even more special case was considered. The results obtained there follow from (1.3) by taking $\Theta = (a+b)/2$ again and assuming function f satisfies $f^{(k)}(a) = (-1)^{k+1} f^{(k)}(b)$, for $1 < k < n$.

Results obtained in [4] by P. Cerone and S.S. Dragomir are special cases of Theorem 4 produced by taking $n = 0$.

2. Main Results

For the proof of our main result we shall also use the Hayashi modification of the well-known Steffensen's inequality, so let us state it first (cf. [9]).

Theorem 5. *Let $F : [a, b] \to \mathbf{R}$ be a nonincreasing function and $G : [a, b] \to \mathbf{R}$ an integrable function such that $0 \le G(x) \le A$ for each $x \in [a, b]$. Then*

$$A \int_{b-\lambda}^{b} F(x)dx \le \int_a^b F(x)G(x)dx \le A \int_a^{a+\lambda} F(x)dx, \tag{2.1}$$

where $\lambda = \frac{1}{A}\int_a^b G(x)dx$.

We introduce:

$$h_k(s, t) = \frac{1}{k!}\int_s^t (x - s)^k w(x)dx \tag{2.2}$$

for $s, t \in [a, b]$ and $k \in \mathbf{N}$.

Now we state our main result.

Theorem 6. *Let $f : [a, b] \to \mathbf{R}$ be such that $f^{(n-1)}$ is absolutely continuous on $[a, b]$. Assume that $m \le f^{(n)}(x) \le M$ for each $x \in [a, b]$. Let $w : I \to \mathbf{R}$ be integrable and such that $w(x) \ge 0$ for each $x \in [a, b]$. Let $\Theta \in [a, b]$. Then, when n is odd we have*

$$\begin{aligned} &(M - m)h_n(b - \lambda_n^0, \Theta) - Mh_n(b, \Theta) + mh_n(a, \Theta) \\ &\quad \le \int_a^b f(x)w(x)dx + \sum_{k=0}^{n-1} \Big[f^{(k)}(b)h_k(b, \Theta) - f^{(k)}(a)h_k(a, \Theta) \Big] \\ &\quad \le Mh_n(a, \Theta) - mh_n(b, \Theta) - (M - m)h_n(a + \lambda_n^0, \Theta) \end{aligned} \tag{2.3}$$

and when n is even we have

$$
\begin{aligned}
&(M-m)[h_n(\Theta-\lambda_n^a,\Theta)-h_n(\Theta+\lambda_n^b,\Theta)]+m[h_n(a,\Theta)-h_n(b,\Theta)]\\
&\quad\leq\int_a^b f(x)w(x)dx+\sum_{k=0}^{n-1}\left[f^{(k)}(b)h_k(b,\Theta)-f^{(k)}(a)h_k(a,\Theta)\right] \qquad (2.4)\\
&\quad\leq M[h_n(a,\Theta)-h_n(b,\Theta)]+(M-m)[h_n(b-\lambda_n^b,\Theta)-h_n(a+\lambda_n^a,\Theta)],
\end{aligned}
$$

where λ_n^0, λ_n^a and λ_n^b are defined by (1.8), (1.9) and (1.10), respectively.

Proof. For $\Theta\in[a,b]$, set

$$
\begin{aligned}
G_k(x)&=f^{(k)}(x)-m\,,\quad k=0,1,\dots n\,,\\
F_k(x)&=\frac{1}{k!}\int_x^\Theta(t-x)^k w(t)dt\,,\quad k=0,1,\dots n-1\,,
\end{aligned}
$$

for each $x\in[a,b]$. Now we have: $0\leq G_n(x)\leq M-m$, for each $x\in[a,b]$, so $G_n(x)$ satisfies the conditions of Theorem 5. It is easy to prove that

$$F_k'(x)=-F_{k-1}(x)$$

and from there we conclude that for $x\leq\Theta$, function $F_{n-1}(x)$ is nonincreasing. For $x\geq\Theta$ and odd n, $F_{n-1}(x)$ is again nonicreasing. However, for $x\geq\Theta$ and even n, $F_{n-1}(x)$ is nondecreasing. Therefore, inequality (2.1) is in that case reversed.

Let us assume first that n is odd. From (2.1) we get

$$
\begin{aligned}
(M-m)\int_{b-\lambda_n^0}^b F_{n-1}(x)dx\leq\int_a^b F_{n-1}(x)G_n(x)dx&\\
\leq(M-m)\int_a^{a+\lambda_n^0}F_{n-1}(x)dx.&
\end{aligned}
$$

where

$$\lambda_n^0=\frac{1}{M-m}\int_a^b(f^{(n)}(x)-m)dx$$

as defined in (1.8). Using integration by parts and the fact that $F_{n-1}'(x)=-F_{n-2}(x)$, we easily obtain

$$I_n \;=\; \int_a^b F_{n-1}(x)G_n(x)dx \qquad (2.5)$$

$$= \int_a^b f(x)w(x)dx + \sum_{k=0}^{n-1}[f^{(k)}(b)h_k(b,\Theta) - f^{(k)}(a)h_k(a,\Theta)] \\ -mh_n(a,\Theta) + mh_n(b,\Theta).$$

The upper bound is

$$U_o = \frac{M-m}{(n-1)!}\int_a^{a+\lambda_n^0}\left[\int_x^\Theta (t-x)^{n-1}w(t)dt\right]dx.$$

Assume first that $\Theta \le a + \lambda_n^0$. Changing the order of integration, we obtain

$$U_o = (M-m)[h_n(a,\Theta) - h_n(a+\lambda_n^0,\Theta)]. \tag{2.6}$$

Assuming $\Theta \ge a + \lambda_n^0$, we get the same expression for the upper bound again.

Analogously, after changing the order of integration in the case when $\Theta \ge b - \lambda_n^0$, the lower bound equals

$$\begin{aligned} L_o &= \frac{M-m}{(n-1)!}\int_{b-\lambda_n^0}^{b}\left[\int_x^\Theta (t-x)^{n-1}w(t)dt\right]dx \\ &= (M-m)[h_n(b-\lambda_n^0,\Theta) - h_n(b,\Theta)]. \end{aligned} \tag{2.7}$$

For $\Theta \le b - \lambda_n^0$, we get the same expression and thus, once again, obtain the same expression in both cases. Inequality (2.3) is produced by combining (2.5), (2.6) and (2.7), so the statement is proved for the case when n is odd.

Assume now n is even. $F_{n-1}(x)$ is nonincreasing on $[a,\Theta]$ so inequality (2.1) gives us:

$$L_e^a \le \int_a^\Theta F_{n-1}(x)G_n(x)dx \le U_e^a. \tag{2.8}$$

It is easy to check that $a + \lambda_n^a \le \Theta$. We calculate both lower and upper bound by changing the order of integration:

$$U_e^a = (M-m)\int_a^{a+\lambda_n^a} F_{n-1}(x)dx \\ = (M-m)[h_n(a,\Theta) - h_n(a+\lambda_n^a,\Theta)], \tag{2.9}$$

$$L_e^a = (M-m)\int_{\Theta-\lambda_n^a}^{\Theta} F_{n-1}(x)dx = (M-m)h_n(\Theta-\lambda_n^a,\Theta), \tag{2.10}$$

where

$$\lambda_n^a = \frac{1}{M-m}\int_a^\Theta (f^{(n)}(x)-m)dx$$

as defined in (1.9).

On $[\Theta, b]$, $F_{n-1}(x)$ is nondecreasing so inequality (2.1) is reversed. We have:

$$L_e^b \leq \int_\Theta^b F_{n-1}(x) G_n(x) dx \leq U_e^b. \tag{2.11}$$

This time $b - \lambda_n^b \geq \Theta$, so it follows

$$U_e^b = (M-m) \int_{b-\lambda_n^b}^b F_{n-1}(x) dx = (M-m)[h_n(b-\lambda_n^b, \Theta) - h_n(b, \Theta)], \tag{2.12}$$

$$L_e^b = (M-m) \int_\Theta^{\Theta+\lambda_n^b} F_{n-1}(x) dx = -(M-m) h_n(\Theta + \lambda_n^b, \Theta), \tag{2.13}$$

where

$$\lambda_n^b = \frac{1}{M-m} \int_\Theta^b (f^{(n)}(x) - m) dx$$

as defined in (1.10).

Addition of (2.8) and (2.11) gives:

$$L_e \leq I_n \leq U_e\,,$$

where

$$U_e = U_e^a + U_e^b \quad \text{and} \quad L_e = L_e^a + L_e^b,$$

and thus inequality (2.4) is produced. The proof of this theorem is now complete. □

Remark 1. If we take $w(x) = 1$ in Theorem 6, we shall obtain Theorem 3. Taking $\Theta = b$ produces inequality (1.11) and $\Theta = a$ produces inequality (1.12).

Naturally, for $w(x) = 1$, $n = 1$ and $\Theta = (a+b)/2$, we shall get inequality (1.2) again.

Next, we shall prove an alternative inequality for an even n and generalize results from [1]. Taking $\Theta = (a+b)/2$ and $w(x) = 1$ will produce results from there.

Theorem 7. *Assume assumptions of Theorem 6 are valid. Then, for $\Theta \in [a, b]$ and even n, we have*

$$m(h_n(a, \Theta) - h_n(b, \Theta)) + (M-m)|h_n(b - \lambda_n, \Theta)|$$

$$\begin{aligned} &\leq \int_a^b f(x)w(x)dx + \sum_{k=0}^{n-1}\left[f^{(k)}(b)h_k(b,\Theta) - f^{(k)}(a)h_k(a,\Theta)\right] \\ &\leq M(h_n(a,\Theta) - h_n(b,\Theta)) - (M-m)|h_n(a+\lambda_n,\Theta)|, \end{aligned} \tag{2.14}$$

where $\lambda_n = \lambda_n^a - \lambda_n^b + b - \Theta,\ \ 0 \leq \lambda_n \leq b - a.$

Proof. We shall use Hayashi's modification of Steffensen's inequality again. Set

$$F_{n-1}(x) = \begin{cases} \frac{1}{(n-1)!}\int_x^{\Theta}(t-x)^{n-1}w(t)dt, & a \leq x \leq \Theta, \\ \frac{1}{(n-1)!}\int_{\Theta}^{x}(t-x)^{n-1}w(t)dt, & \Theta \leq x \leq b. \end{cases} \tag{2.15}$$

From the proof of Theorem 6 it is clear that F_{n-1} is decreasing on $[a,b]$. Taking

$$G_n(x) = \begin{cases} f^{(n)}(x) - m, & a \leq x \leq \Theta, \\ M - f^{(n)}(x), & \Theta \leq x \leq b, \end{cases} \tag{2.16}$$

produces our statement. □

Remark 2. Estimations for an even n from Theorem 6 are better then the ones from Theorem 7. To prove this, we have to check that

$$|h_n(a+\lambda_n,\Theta)| \leq h_n(a+\lambda_n^a,\Theta) - h_n(b-\lambda_n^b,\Theta), \tag{2.17}$$

$$|h_n(b-\lambda_n,\Theta)| \leq h_n(\Theta-\lambda_n^a,\Theta) - h_n(\Theta+\lambda_n^b,\Theta). \tag{2.18}$$

After introducing notation

$$c_1 = a + \lambda_n^a, \qquad c_2 = b - \lambda_n^b, \qquad d_1 = \Theta - \lambda_n^a, \qquad d_2 = \Theta + \lambda_n^b,$$

(2.17) and (2.18) become

$$|h_n(c_1+c_2-\Theta,\Theta)| \leq h_n(c_1,\Theta) - h_n(c_2,\Theta), \tag{2.19}$$

$$|h_n(d_1+d_2-\Theta,\Theta)| \leq h_n(d_1,\Theta) - h_n(d_2,\Theta). \tag{2.20}$$

We already know that $c_1 \leq \Theta$ and $c_2 \geq \Theta$ and it is clear that $d_1 \leq \Theta$ and $d_2 \geq \Theta$, so we have $c_1 \leq c_1 + c_2 - \Theta \leq c_2$ and $d_1 \leq d_1 + d_2 - \Theta \leq d_2$. For an even n, function $h_n(x,\Theta)$ is decreasing. Also, $h_n(\Theta,\Theta) = 0$. Let us consider (2.19). First assume $c_1 + c_2 - \Theta \leq \Theta$. Then $h_n(c_1+c_2-\Theta,\Theta) \geq 0$ and

$$h_n(c_1+c_2-\Theta,\Theta) \leq h_n(c_1,\Theta) \leq h_n(c_1,\Theta) - h_n(c_2,\Theta)$$

since $h_n(c_2,\Theta) \leq 0$. Next, suppose $c_1+c_2-\Theta \geq \Theta$. Then $h_n(c_1+c_2-\Theta,\Theta) \leq 0$ and

$$h_n(c_1+c_2-\Theta,\Theta) \geq h_n(c_2,\Theta) \geq h_n(c_2,\Theta) - h_n(c_1,\Theta)$$

since $h_n(c_1,\Theta) \geq 0$. Proof of (2.20) is analogous.

3. Comparison with a Similar Result Obtained Using Taylor's Formula

In [10], F. Qi obtained the following result.

Theorem 8. *Let $f \in C^{n-1}([a,b])$ be such that $m \le f^{(n)}(x) \le M$ for each $x \in (a,b)$ and let $w(x) \ge 0$ for $x \in [a,b]$. Then, when n is odd, for any $\Theta \in (a,b)$, we have*

$$\begin{aligned} & mh_n(a,\Theta) - Mh_n(b,\Theta) \\ & \le \int_a^b w(x)f(x)dx + \sum_{k=0}^{n-1}[f^{(k)}(b)h_k(b,\Theta) - f^{(k)}(a)h_k(a,\Theta)] \\ & \le Mh_n(a,\Theta) - mh_n(b,\Theta); \end{aligned} \tag{3.1}$$

and when n is even we have

$$\begin{aligned} & m[h_n(a,\Theta) - h_n(b,\Theta)] \\ & \le \int_a^b w(x)f(x)dx + \sum_{k=0}^{n-1}[f^{(k)}(b)h_k(b,\Theta) - f^{(k)}(a)h_k(a,\Theta)] \\ & \le M[h_n(a,\Theta) - h_n(b,\Theta)]. \end{aligned} \tag{3.2}$$

Connection with Theorem 6 is obvious – under same assumptions, same expression is being estimated. The idea is now to compare those two estimations. We will prove that the estimation we obtained in Theorem 6 is better then the one in Theorem 8.

First let us consider the case when n is odd. Upper bound in (2.3) is

$$U_S = Mh_n(a,\Theta) - mh_n(b,\Theta) - (M-m)h_n(a+\lambda_n^0,\Theta)$$

and in (3.1)

$$U_T = Mh_n(a,\Theta) - mh_n(b,\Theta)\,.$$

Now, $D_U = U_S - U_T \le 0$.

Similarly, lower bound in (2.3) is

$$L_S = (M-m)h_n(b-\lambda_n^0,\Theta) - Mh_n(b,\Theta) + mh_n(a,\Theta)$$

and in (3.1)

$$L_T = mh_n(a,\Theta) - Mh_n(b,\Theta),$$

so $\quad D_L = L_S - L_T \ge 0.$

Next, consider the case when n is even. Upper bound in (2.4) is

$$U_S = M[h_n(a,\Theta) - h_n(b,\Theta)] + (M-m)[h_n(b-\lambda_n^b,\Theta) - h_n(a+\lambda_n^a,\Theta)]$$

and in (3.2)

$$U_T = M[h_n(a,\Theta) - h_n(b,\Theta)].$$

As before, $D_U = U_S - U_T \leq 0$.

Finally, lower bound in (2.4) is

$$L_S = (M-m)[h_n(\Theta-\lambda_n^a,\Theta) - h_n(\Theta+\lambda_n^b,\Theta)] + m[h_n(a,\Theta) - h_n(b,\Theta)]$$

and in (3.2)

$$L_T = m[h_n(a,\Theta) - h_n(b,\Theta)],$$

so $D_L = L_S - L_T \geq 0$.

This proves that bounds in Theorem 6 are better then bounds in Theorem 8.

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