International Journal of Pure and Applied Mathematics
Volume 26 No. 3 2006, 397-406

ON CIRIC TYPE MAPPINGS WITH NONUNIQUE FIXED AND PERIODIC POINTS

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Abstract: In this paper, we introduce several classes of Ciric type mappings in metric spaces, prove the existence of fixed and periodic points for these mappings and give some nonuique fixed and periodic point theorems, two of which extend properly the result in [7].

AMS Subject Classification: 54H25
Key Words: fixed point, periodic point, Ciric type mapping, metric space

1. Introduction and Preliminaries

In 1974, Ciric [1] proved the existence of fixed points for the following mappings:

$$\min\{d(Tx,Ty),d(x,Tx),d(y,Ty)\} - \min\{d(x,Ty),d(y,Tx)\} \le rd(x,y), \quad \forall x,y \in X, \quad (1.1)$$

Received: January 13, 2006

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$$\min\{d(Tx,Ty),d(x,Tx),d(y,Ty)\}\le rd(x,y),\qquad \forall x,y\in X \text{ with } 0<d(x,y)<\varepsilon, \tag{1.2}$$

where r and ε are positive constants with $r < 1$. Later on, many researchers introduced a few kinds of Ciric type mappings with single-valued and multi-valued and obtained the existence of fixed points, periodic points and common fixed points for these mappings. In 1979, Pachpatte [7] established some sufficient conditions, which guarantee the existence of nonunique fixed points for the following mappings:

$$\begin{aligned}&\min\{d^2(Tx,Ty),d(x,y)d(Tx,Ty),d^2(y,Ty)\}\\&\quad-\min\{d(x,Tx)d(y,Ty),d(x,Ty)d(y,Tx)\}\\&\le rd(x,Tx)d(y,Ty),\quad \forall x,y\in X,\end{aligned}\tag{1.3}$$

where r is a constant in $(0,1)$. In 1986, Liu [4] studied the following mappings:

$$\begin{aligned}&\min\{d^2(Tx,Ty),d(x,y)d(Tx,Ty),d(x,y)d(y,Ty),d(x,Tx)d(Tx,Ty),\\&\quad d^2(y,Ty)\}-\min\{d(x,Tx)d(y,Ty),d(x,Ty)d(y,Tx)\}\\&\le r\max\{d(x,Tx)d(y,Ty),d(x,Ty)d(y,Tx)\},\quad \forall x,y\in X,\end{aligned}\tag{1.4}$$

where r is a constant in $(0,1)$, and gave some results about the existence of nonunique fixed points for the mapping (1.4).

In this paper, we introduce some new classes of Ciric type mappings and establish some nonunique fixed and periodic point theorems for these mappings, two of which generalize properly Theorem 1 in [7].

Throughout this paper, we assume that (X,d) is a metric space, T is a self mapping of X, $\mathbf{R}^+$ denotes the set of nonnegative real numbers, and N denotes the set of positive integers. Recall that T is said to be orbitally continuous if $\lim_{i\to\infty}T^{n_i}x=u$ implies that $\lim_{i\to\infty}TT^{n_i}x=Tu$ for x in X, where $\{n_i\}_{i\ge1}\subset N$, and X is said to be orbitally complete if every Cauchy sequence of the form $\{T_i^n x\}_{i\ge1}$ converges in X for x in X. For any sequence $\{x_n\}_{n\ge0}\subset X$, put $d_n=d(x_n,x_{n+1}),\forall n\ge0$.

2. Fixed and Periodic Point Theorems

Our main results are as follows.

Theorem 2.1. *Let T be an orbitally continuous self mapping of an orbitally complete metric space (X,d) such that*

$$\begin{aligned}&\min\{d^2(Tx,Ty),d^2(x,Tx),d^2(y,Ty),d(x,y)d(Tx,Ty),\\&\quad d(x,y)d(x,Tx),d(x,y)d(y,Ty)\}\\&\quad-\min\{d(x,Tx)d(y,Ty),d(x,Ty)d(y,Tx)\}\\&\le r\max\{d^2(x,y),d^2(x,Tx),d^2(y,Tx),d(x,Ty)d(y,Tx),\\&\quad d(y,Tx)d(y,Ty),d(y,Tx)d(Tx,Ty)\},\quad \forall x,y\in X,\end{aligned}\tag{2.1}$$

where r is a constant in $(0,1)$. Then T has a fixed point in X and the sequence $\{T^n x_0\}_{n\ge 0}$ converges to a fixed point of T for each x_0 in X.

Proof. Let x_0 be in X. Define a sequence $\{x_n\}_{n\ge0}$ by $x_{n+1}=Tx_n$ for $n\ge 0$. If $x_n=x_{n+1}$ for some $n\ge 0$, the assertion follows immediately. Now suppose that $x_n\ne x_{n+1}$ for each $n\ge 0$. By (2.1) we get that

$$\begin{aligned}&\min\{d^2(Tx_n,Tx_{n+1}),d^2(x_n,Tx_n),d^2(x_{n+1},Tx_{n+1}),\\&\quad d(x_n,x_{n+1})d(Tx_n,Tx_{n+1}),d(x_n,x_{n+1})d(x_n,Tx_n),\\&\quad d(x_n,x_{n+1})d(x_{n+1},Tx_{n+1})\}\\&\quad-\min\{d(x_n,Tx_n)d(x_{n+1},Tx_{n+1}),d(x_n,Tx_{n+1})d(x_{n+1},Tx_n)\}\\&\le r\max\{d^2(x_n,x_{n+1}),d^2(x_n,Tx_n),d^2(x_{n+1},Tx_n),\\&\quad d(x_n,Tx_{n+1})d(x_{n+1},Tx_n),d(x_{n+1},Tx_n)d(x_{n+1},Tx_{n+1}),\\&\quad d(x_{n+1},Tx_n)d(Tx_n,Tx_{n+1})\},\quad \forall\ n\ge 0.\end{aligned}$$

That is,

$$\begin{aligned}&\min\{d_{n+1}^2,d_n^2,d_{n+1}^2,d_nd_{n+1},d_n^2,d_nd_{n+1}\}-\min\{d_nd_{n+1},0\}\\&\le r\max\{d_n^2,d_n^2,0,0,0,0\},\quad \forall\ n\ge 0,\end{aligned}$$

which implies that

$$\min\{d_n^2,d_nd_{n+1},d_{n+1}^2\}\le rd_n^2,\quad \forall\ n\ge 0.$$

It is not difficult to infer that $d_{n+1}\le\sqrt{r}d_n$ for $n\ge 0$, from which we obtain that

$$d_n\le\sqrt{r}d_{n-1}\le(\sqrt{r})^2d_{n-2}\le\cdots\le(\sqrt{r})^nd_0,\quad \forall\ n\ge 0.$$

Hence for any $n\ge 0,p\in N$, we get that

$$d(x_n,x_{n+p})\le\sum_{k=n}^{n+p-1}d(x_k,x_{k+1})\le\sum_{k=n}^{n+p-1}(\sqrt{r})^kd_0\le\frac{(\sqrt{r})^n}{1-\sqrt{r}}d_0,$$

which shows that $\{x_n\}_{n\geq 0}$ is a Cauchy sequence. Since X is orbitally complete, there exists an u in X such that $u = \lim_{n\to\infty} T^n x$. It follows from the orbital continuity of T that $Tu = \lim_{n\to\infty} TT^n x = u$. This completes the proof. □

Theorem 2.2. *Let T be an orbitally continuous self mapping of an orbitally complete metric space (X,d) such that*

$$\begin{aligned} &\min\{d^2(Tx,Ty), d^2(x,Tx), d^2(y,Ty), d^2(x,y)\} \\ &\quad - \min\{d(x,Tx)d(y,Ty), d(x,Ty)d(y,Tx)\} \\ &\leq r\max\{d^2(x,y), d^2(x,Tx), d^2(y,Tx), d(x,Ty)d(y,Tx), \\ &\qquad d(y,Tx)d(y,Ty), d(y,Tx)d(Tx,Ty)\}, \quad \forall x,y\in X, \end{aligned} \tag{2.2}$$

where r is a constant in $(0,1)$. Then T has a fixed point in X and the sequence $\{T^n x_0\}_{n\geq 0}$ converges to a fixed point of T for each x_0 in X.

The proof is similar to the proof of Theorem 2.1, so we omit it.

Remark 2.1. Theorem 1 in [7] is a special case of Theorem 2.1 and Theorem 2.2, respectively. The following example reveals that Theorem 2.1 and Theorem 2.2 extend properly Theorem 1 in [7].

Example 2.1. Let $X = \{0,1,2\}$ with usual metric d. Obviously, (X,d) is a complete metric space. Now let $T : X \to X$ by $T0 = 0$, $T1 = 0$, $T2 = 2$. It is easy to verify that the conditions of Theorem 2.1 and Theorem 2.2 are satisfied for $r = 0.5$. But Theorem 1 in [7] is not applicable, because T does not satisfy (1.3) for $x = 2$, $y = 1$.

As consequences of Theorem 2.1 and Theorem 2.2, we have the following results.

Corollary 2.1. *Let T be an orbitally continuous self mapping of an orbitally complete metric space (X,d) such that*

$$\begin{aligned} &a_1 d^2(Tx,Ty) + a_2 d^2(x,Tx) + a_3 d^2(y,Ty) + a_4 d(x,y)d(Tx,Ty) \\ &\qquad + a_5 d(x,y)d(x,Tx) + a_6 d(x,y)d(y,Ty) \\ &\quad - \min\{d(x,Tx)d(y,Ty), d(x,Ty)d(y,Tx)\} \\ &\leq r\max\{d^2(x,y), d^2(x,Tx), d^2(y,Tx), d(x,Ty)d(y,Tx), \\ &\qquad d(y,Tx)d(y,Ty), d(y,Tx)d(Tx,Ty)\}, \quad \forall x,y\in X, \end{aligned} \tag{2.3}$$

where a_i is in $\mathbf{R}^+$ for $i \in \{1,2,\cdots,6\}$, r is a constant with $0 < r < \sum_{i=1}^6 a_i$. Then T has a fixed point in X and the sequence $\{T^n x_0\}_{n\geq 0}$ converges to a fixed point of T for each x_0 in X.

Corollary 2.2. *Let T be an orbitally continuous self mapping of an orbitally complete metric space (X,d) such that*

$$\begin{aligned}&a_1d^2(Tx,Ty)+a_2d^2(x,Tx)+a_3d^2(y,Ty)+a_4d^2(x,y)\\&\quad-\min\{d(x,Tx)d(y,Ty),d(x,Ty)d(y,Tx)\}\\&\le r\max\{d^2(x,y),d^2(x,Tx),d^2(y,Tx),d(x,Ty)d(y,Tx),\\&\qquad d(y,Tx)d(y,Ty),d(y,Tx)d(Tx,Ty)\},\quad\forall x,y\in X,\end{aligned}\tag{2.4}$$

where a_i is in $\mathbf{R}^+$ for $i\in\{1,2,3,4\}$, r is a constant with $0<r<\sum_{i=1}^4 a_i$. Then T has a fixed point in X and the sequence $\{T^nx_0\}_{n\ge0}$ converges to a fixed point of T for each x_0 in X.

Theorem 2.3. *Let T be an orbitally continuous self mapping of a metric space (X,d) such that*

$$\begin{aligned}&\min\{d^2(Tx,Ty),d^2(x,Tx),d^2(y,Ty),d(x,y)d(Tx,Ty),\\&\qquad d(x,y)d(x,Tx),d(x,y)d(y,Ty)\}\\&\quad-\min\{d(x,Tx)d(y,Ty),d(x,Ty)d(y,Tx)\}\\&<\max\{d^2(x,y),d^2(x,Tx),d^2(y,Tx),\\&\qquad d(x,Ty)d(y,Tx),d(y,Tx)d(y,Ty),\\&\qquad d(y,Tx)d(Tx,Ty)\},\quad\forall x,y\in X\text{ with }x\ne y.\end{aligned}\tag{2.5}$$

If there exists some x_0 in X satisfying that the sequence $\{T^nx_0\}_{n\ge0}$ has a cluster point u in X, then u is a fixed point of T.

Proof. Set $x_n=T^nx_0$ for $n\ge0$. If $T^{p-1}x_0=T^px_0$ for some $p\ge1$, then $T^nx_0=T^px_0$ for all $n\ge p$, which is impossible. Therefore $T^{p-1}x_0\ne T^px_0$ for all $p\ge1$. Since u is a cluster point of the sequence $\{T^nx_0\}_{n\ge0}$, there exists a subsequence $\{T^{n_i}x_0\}_{i\ge1}$ of $\{T^nx_0\}_{n\ge0}$ satisfying that $\lim_{i\to\infty}T^{n_i}x_0=u$. By (2.5) we deduce that

$$\begin{aligned}&\min\{d_n^2,d_{n-1}^2,d_n^2,d_{n-1}d_n,d_{n-1}d_{n-1},d_{n-1}d_n\}-\min\{d_{n-1}d_n,0\}\\&<\max\{d_{n-1}^2,d_{n-1}^2,0,0,0,0\},\quad\forall\ n\ge1,\end{aligned}$$

which implies that $d_n<d_{n-1}$ for $n\ge1$. Consequently, $d_n\to l\ge0$ as $n\to\infty$. Note that $\lim_{i\to\infty}d_{n_i}=d(u,Tu)$ and $\lim_{i\to\infty}d_{n_i+1}=d(Tu,T^2u)$. It is easy to verify that

$$d(u,Tu)=\lim_{n\to\infty}d_n=\lim_{n\to\infty}d_{n+1}=d(Tu,T^2u).$$

Assume that $d(u,Tu)>0$. (2.5) ensures that

$$\min\{d^2(Tu,T^2u),d^2(u,Tu),d^2(Tu,T^2u),d(u,Tu)d(Tu,T^2u),$$

$$
\begin{aligned}
&d(u,Tu)d(u,Tu), d(u,Tu)d(Tu,T^2u)\}\\
&-\min\{d(u,Tu)d(Tu,T^2u), d(u,T^2u)d(Tu,Tu)\}\\
&< \max\{d^2(u,Tu), d^2(u,Tu), d^2(Tu,Tu), d(u,T^2u)d(Tu,Tu),\\
&\quad d(Tu,Tu)d(Tu,T^2u), d(Tu,Tu)d(Tu,T^2u)\},
\end{aligned}
$$

which implies that $d(Tu,T^2u) < d(u,Tu)$. This is a contradiction. Hence $Tu = u$. The proof is completed. □

Theorem 2.4. *Let T be an orbitally continuous self mapping of a metric space (X,d) such that*

$$
\begin{aligned}
&\min\{d^2(Tx,Ty), d^2(x,Tx), d^2(y,Ty), d^2(x,y)\}\\
&\quad -\min\{d(x,Tx)d(y,Ty), d(x,Ty)d(y,Tx)\}\\
&< \max\{d^2(x,y), d^2(x,Tx), d^2(y,Tx),\\
&\qquad d(x,Ty)d(y,Tx), d(y,Tx)d(y,Ty),\\
&\qquad d(y,Tx)d(Tx,Ty)\}, \quad \forall x,y \in X \text{ with } x \neq y. \qquad (2.6)
\end{aligned}
$$

If there exists some x_0 in X satisfying that the sequence $\{T^n x_0\}_{n\geq 0}$ has a cluster point u in X, then u is a fixed point of T.

Proof. Being similar to the proof of Theorem 2.3, the proof is omitted. □

Theorem 2.5. *Let T be an orbitally continuous self mapping of an orbitally complete metric space (X,d). If there exist x_0 in X, $k \in N$, $r \in (0,1)$ and $\varepsilon > 0$ such that $d(x_0, T^k x_0) < \varepsilon$ and*

$$
\begin{aligned}
&\min\{d^2(Tx,Ty), d^2(x,Tx), d^2(y,Ty), d(x,y)d(Tx,Ty),\\
&\qquad d(x,y)d(x,Tx), d(x,y)d(y,Ty)\}\\
&\leq rd^2(x,y), \quad \forall x,y \in X \text{ with } 0 < d(x,y) < \varepsilon. \qquad (2.7)
\end{aligned}
$$

Then T has a periodic point in X.

Proof. Obviously the subset $K = \{k : d(x,T^k x) < \varepsilon \text{ for some } x \text{ in } X\} \subset N$ is non-void. Put $m = \min K$ and let $x \in X$ be such that $d(x,T^m x) < \varepsilon$. Suppose that $m = 1$. If $x \neq Tx$, by (2.7) we infer that

$$
\begin{aligned}
&\min\{d^2(Tx,T^2x), d^2(x,Tx), d^2(Tx,T^2x), d(x,Tx)d(Tx,T^2x),\\
&\quad d(x,Tx)d(x,Tx), d(x,Tx)d(Tx,T^2x)\}\\
&\leq rd^2(x,Tx),
\end{aligned}
$$

that is ,

$$\min\{d^2(Tx,T^2x),d(x,Tx)d(Tx,T^2x),d^2(x,Tx)\}\le rd^2(x,Tx),$$

which implies that $d(Tx,T^2x)\le\sqrt{r}d(x,Tx)$. As in the proof of Theorem 2.1, we conclude that $Tu=u$ for some $u\in X$.

Suppose that $m\ge 2$, that is, $d(y,Ty)\ge\varepsilon$ for each y in X. Without loss of generality we assume that $T^py\ne T^{m+p}y$ for each $p\ge 0$ and $y\in X$. Now we claim that $d(T^nx,T^{n+m}x)\to 0$ as $n\to\infty$. Using $0<d(x,T^mx)<\varepsilon$ and (2.7), we see that

$$\begin{aligned}&\min\{d^2(Tx,T^{m+1}x),d^2(x,Tx),d^2(T^mx,T^{m+1}x),\\&\qquad d(x,T^mx)d(Tx,T^{m+1}x),d(x,T^mx)d(x,Tx),\\&\qquad d(x,T^mx)d(T^mx,T^{m+1}x)\}\\&\le rd^2(x,T^mx).\end{aligned}$$

From $d(x,Tx)\ge\varepsilon$ and $d(T^mx,T^{m+1}x)\ge\varepsilon$, it follows that

$$\min\{d^2(Tx,T^{m+1}x),d(x,T^mx)d(Tx,T^{m+1}x)\}\le rd^2(x,T^mx),$$

which reveals that $d(Tx,T^{m+1}x)\le\sqrt{r}d(x,T^mx)<\sqrt{r}\varepsilon$. Similarly,

$$d(T^nx,T^{m+n}x)\le\sqrt{r}d(T^{n-1}x,T^{m+n-1}x)\le\cdots<(\sqrt{r})^n\varepsilon\to 0$$

as $n\to\infty$. Put $x_{n+1}=T^mx_n$ for $n\ge 0$, where $x_0=x$. Obviously

$$d(x_n,x_{n+1})=d(T^{nm}x,T^{m+nm}x)\le(\sqrt{r})^{nm}d(x,T^mx)<(\sqrt{r})^{nm}\varepsilon$$

for each $n\ge 0$. It is easy to show that $\{x_n\}_{n\ge 0}$ is a Cauchy sequence. Since $\{x_n\}_{n\ge 0}\subseteq\{T^nx\}_{n\ge 0}$ and X is orbitally complete, there exists some $u\in X$ such that $u=\lim_{n\to\infty}x_n=\lim_{n\to\infty}T^{nm}x$. By the orbital continuity of T, we obtain that $T^mu=\lim_{n\to\infty}T^{(n+1)m}x=u$. This completes the proof. □

Theorem 2.6. *Let T be an orbitally continuous self mapping of an orbitally complete metric space (X,d). Assume that there exist x_0 in X, $k\in N$ and $\varepsilon>0$ such that $d(x_0,T^kx_0)<\varepsilon$ and*

$$\begin{aligned}&\min\{d^2(Tx,Ty),d^2(x,Tx),d^2(y,Ty),d(x,y)d(Tx,Ty),\\&\qquad d(x,y)d(x,Tx),d(x,y)d(y,Ty)\}\\&<d^2(x,y),\quad\forall x,y\in X\text{ with }0<d(x,y)<\varepsilon.\end{aligned}\tag{2.8}$$

If there exists some x_0 in X such that the sequence $\{T^nx_0\}_{n\ge 0}$ has a cluster point u in X, then u is a periodic point of T.

Proof. Since the sequence $\{T^n x_0\}_{n\geq 0}$ has a cluster point u in X, there exists a sequence $\{n_i\}_{i\geq 1}\subset N$ satisfying that $\lim_{i\to\infty} T^{n_i}x_0 = u$. Thus we can find $N_1 \in N$ such that $d(T^{n_i}x_0, u) < \frac{\varepsilon}{2}$ for $i > N_1$. Consequently, $d(T^{n_i}x_0, T^{n_{i+1}}x_0) < \varepsilon$ for $i > N_1$ and the set $K = \{k \in N : d(T^p x_0, T^{p+k}x_0) < \varepsilon$ for some $p \geq 0\}$ is non-void. Put $m = \min K$. Note that $\{T^n x_0\}_{n\geq 1}$ has a cluster point. Therefore $d(T^s x_0, T^{s+m}x_0) > 0$ for each $s \geq 0$. Let $p \geq 0$ be with $d(T^p x_0, T^{p+m}x_0) < \varepsilon$.

Suppose that $m = 1$. As in the proof of Theorem 2.5, by (2.8) we infer that $\{d(T^n x_0, T^{n+1}x_0)\}_{n\geq 0}$ is decreasing for $n \geq p$ and it follows that $Tu = u$.

Suppose that $m \geq 2$. Obviously,

$$d(T^n x_0, T^{n+1}x_0) \geq \varepsilon, \quad \forall\, n \geq 0. \tag{2.9}$$

From the orbital continuity of T and (2.9), we get that

$$d(T^s u, T^{s+1}u) = \lim_{i\to\infty} d(T^{n_i+s}x_0, T^{n_i+s+1}x_0) \geq \varepsilon, \quad \forall\, s \geq 0. \tag{2.10}$$

In view of $0 < d(T^p x_0, T^{p+m}x_0) < \varepsilon$ and (2.8), it follows that

$$\begin{aligned}
&\min\{d^2(T^{p+1}x_0, T^{p+m+1}x_0), d^2(T^p x_0, T^{p+1}x_0),\\
&\quad d^2(T^{p+m}x_0, T^{p+m+1}x_0),\\
&\quad d(T^p x_0, T^{p+m}x_0)d(T^{p+1}x_0, T^{p+m+1}x_0),\\
&\quad d(T^p x_0, T^{p+m}x_0)d(T^p x_0, T^{p+1}x_0),\\
&\quad d(T^p x_0, T^{p+m}x_0)d(T^{p+m}x_0, T^{p+m+1}x_0)\}\\
&< d^2(T^p x_0, T^{p+m}x_0).
\end{aligned}$$

By virtue of (2.9), we gain that

$$\begin{aligned}
&\min\{d^2(T^{p+1}x_0, T^{p+m+1}x_0), d(T^p x_0, T^{p+m}x_0)d(T^{p+1}x_0, T^{p+m+1}x_0)\}\\
&< d^2(T^p x_0, T^{p+m}x_0),
\end{aligned}$$

which implies that

$$d(T^{p+1}x_0, T^{p+m+1}x_0) < d(T^p x_0, T^{p+m}x_0) < \varepsilon.$$

Similarly we obtain the following inequalities

$$\begin{aligned}
d(T^{p+n+1}x_0, T^{p+n+1+m}x_0) &< d(T^{p+n}x_0, T^{p+n+m}x_0)\\
&< \varepsilon, \quad \forall\, n \geq 0.
\end{aligned} \tag{2.11}$$

Therefore the sequence $\{d(T^n x_0, T^{n+m} x_0)\}_{n \geq p}$ is decreasing and converges to some $l \geq 0$. It is easy to verify that

$$\{d(T^{n_i} x_0, T^{n_i+m} x_0)\}_{i \geq 1}$$

and $\{d(T^{n_i+1} x_0, T^{n_i+1+m} x_0)\}_{i \geq 1}$ converge to $d(u, T^m u)$ and $d(Tu, T^{m+1} u)$, respectively, from which we infer that

$$l = d(Tu, T^{m+1} u) = d(u, T^m u) = \lim_{n \to \infty} d(T^n x_0, T^{n+m} x_0). \tag{2.12}$$

Now we show that $T^m u = u$. Suppose that $d(u, T^m u) > 0$. (2.11) and (2.12) imply that $d(u, T^m u) < \varepsilon$. By (2.8), we derive that

$$\begin{aligned}
&\min\{d^2(Tu, T^{m+1} u), d^2(u, Tu), d^2(T^m u, T^{m+1} u), \\
&\quad d(u, T^m u) d(Tu, T^{m+1} u), d(u, T^m u) d(u, Tu), \\
&\quad d(u, T^m u) d(T^m u, T^{m+1} u)\} \\
&< d^2(u, T^m u).
\end{aligned}$$

Hence $d(Tu, T^{m+1} u) < d(u, T^m u)$, which is a contradiction. This completes the proof. □

Acknowledgments

This work was supported by the Science Research Foundation of Educational Department of Liaoning Province (2004C063).

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