International Journal of Pure and Applied Mathematics
Volume 26 No. 3 2006, 421-430

ON CONJUGACY EQUATIONS FOR ITERATIVE FUNCTIONAL EQUATIONS

Jitka Laitochová
Department of Mathematics, PdF
Palacký University
Žižkovo nám. 5, Olomouc, 771 40, CZECH REPUBLIC
e-mail: jitka.laitochova@upol.cz

Abstract: The paper studies conjugacy equations $\alpha(f(x)) = g(\alpha(x))$, which are used to convert iterative functional equations to simpler forms. The functions f and g are given, with g especially simple, e.g., $g(x) = x + 1$ in the case of Abel's functional equation. An iterative construction is given for infinitely many monotonic solutions α. The recursive algorithm can be implemented in computer algebra systems.

AMS Subject Classification: 39B05, 39B12
Key Words: conjugacy equations, iterative functional equations, phase function, iteration, fundamental central dispersion, Abel functional equation

1. Introduction

A *conjugacy equation*

$$\alpha(f(x)) = g(\alpha(x)), \quad x \in X, \tag{1.1}$$

is used to simplify iterative functional equations (see Kuczma [2], [3])

$$F(x, \psi(x), \psi(f(x))) = 0, \quad x \in X, \tag{1.2}$$

in which f and F are assumed given and the unknown function $\psi(x)$ is to be

Received: February 6, 2006

found. The bijective function α in (1.1) is used to change variables $y = \alpha(x)$, producing a hopefully solvable equation

$$F(\alpha^{-1}(y), \Psi(y), \Psi(g(y))) = 0, \quad y \in \alpha(X). \tag{1.3}$$

Then Ψ in (1.3) gives $\psi = \Psi\alpha$ in (1.2).

The *conjugacy problem* seeks to identify *conjugate pairs* f, g which satisfy (1.1) for some bijective change of variables α. The object is to find α, given f and g. Generally, g is simpler than f, for instance, $g(x) = x + a$ results in the *Abel functional equation*, which causes (1.3) to be simpler.

This article identifies conjugate pairs f, g in (1.1) and α is explicitly constructed by iterative methods. The algorithm can be implemented in a computer algebra system like *MAPLE*.

2. Construction of $\alpha(t)$

Theorem 1. *Let f and g be continuous functions from $(-\infty, \infty)$ onto $(-\infty, \infty)$, strictly increasing and fixed-point free. Assume $g(0) > 0$ and choose an arbitrary point t_0. Define $\alpha_0(t)$ to be any strictly increasing continuous function on $t_0 \le t \le f(t_0)$ such that $\alpha_0(f(t_0)) = g(\alpha_0(t_0))$. Define*

$$\alpha(t) = \begin{cases} \alpha_0(t) & t_0 \le t < f(t_0), \\ g^n\alpha_0(f^{-1})^n(t) & f^n(t_0) \le t < f^{n+1}(t_0), \quad n \ge 1, \\ (g^{-1})^n\alpha_0 f^n(t) & (f^{-1})^{n+1}(t_0) \le t < (f^{-1})^n(t_0), \quad n \ge 0. \end{cases}$$

Then:

(1) $\alpha(t)$ is defined on $(-\infty, \infty)$ and $\alpha(f(t)) = g(\alpha(t))$ for all t.

(2) $\alpha(t)$ is continuous.

(3) $\alpha(t)$ is strictly increasing.

(4) $\alpha(t)$ has image $(-\infty, \infty)$.

Lemma 1. *Let f_1, f_2 be functions defined on $(-\infty, \infty)$ and strictly monotonic. Then $f_1 f_2$ is strictly increasing if f_1 and f_2 are both increasing or both decreasing, and otherwise $f_1 f_2$ is strictly decreasing.*

Lemma 2. *If f_1 is strictly increasing on $(-\infty, \infty)$ with image $(-\infty, \infty)$, then f_1^{-1} is strictly increasing on $(-\infty, \infty)$ with image $(-\infty, \infty)$. A similar statement applies if f_1 is strictly decreasing.*

Lemma 3. *If f_1 is continuous on $(-\infty, \infty)$ and fixed-point free, then* $f_1(0)(f_1(t) - t) > 0$.

Lemma 4. *If f_1 is continuous and strictly increasing on $(-\infty, \infty)$, fixed-point free, and has image $(-\infty, \infty)$, then for any t_0,*

$$\lim_{n\to\infty} f_1^n(t_0) = \infty, \quad \lim_{n\to\infty} (f_1^{-1})^n(t_0) = -\infty.$$

Proofs of the lemmas. Lemma 1 and Lemma 2 are well-known. A function f_1 is *fixed-point free* provided $f_1(t) \neq t$ for all t. In Lemma 3, $f_1(t) - t$ is one-signed, hence $(f_1(s) - s)(f_1(t) - t) > 0$. Choose $s = 0$. In Lemma 4, the failure of a monotone sequence of iterates to converge to $\pm\infty$ implies it has a finite limit x, which in turn must satisfy $f_1(x) = x$. Because f_1 is fixed-point free, the two limits must be infinite. □

Proof of Theorem 1. Let us assume that f, g are given continuous, strictly increasing functions on $(-\infty, \infty)$ which are fixed-point free and have image $(-\infty, \infty)$. Assume further that $g(0) > 0$.

Definition of the sequence $\{t_n\}_{n=-\infty}^{\infty}$. Let t_0 be any given point and define iteratively

$$t_n = \begin{cases} f^n(t_0) & n \geq 0, \\ (f^{-1})^n(t_0) & n \geq 1. \end{cases}$$

The points are increasing in the index n, because of Lemma 2, and by Lemma 4 they limit to $-\infty$ at $n = -\infty$ and ∞ at $n = \infty$. Consequently, the intervals $[t_k, t_{k+1})$ for $k = 0, \pm 1, \pm 2, \ldots$ are disjoint and have union $(-\infty, \infty)$.

Definition of $\alpha_0(t)$. Define $\alpha_0(t)$ on $t_0 \leq t \leq t_1$ to be continuous and strictly increasing with $\alpha_0(t_1) = g(\alpha_0(t_0))$. Since $g(0) > 0$, then also $g(t) > t$ by Lemma 3, therefore there are infinitely many possible choices of $\alpha_0(t)$. In particular, $\alpha_0(t)$ can be a linear function $c_1(t - t_0) + c_2$ with $c_2 = \alpha_0(t_0)$ and $c_1 = (g(\alpha_0(t_0)) - \alpha_0(t_0))/(t_1 - t_0)$.

Definition of $\alpha(t)$. Define

$$\alpha(t) = \begin{cases} \alpha_0(t) & t_0 \leq t < t_1, \\ g^n \alpha_0 (f^{-1})^n(t) & t_n \leq t < t_{n+1}, \quad n \geq 1, \\ (g^{-1})^n \alpha_0 f^n(t) & t_{-n} \leq t < t_{-n+1}, \quad n \geq 1. \end{cases}$$

Justify $\alpha f = g\alpha$. The set formula $f([t_k, t_{k+1})) = [t_{k+1}, t_{k+2})$ justifies that

α is well-defined. Given $t_k \le t < t_{k+1}$ with $k \ge 1$, then

$$\begin{aligned} \alpha f(t) &= g^{k+1}\alpha_0(f^{-1})^{k+1}(f(t)) \\ &= gg^k\alpha_0(f^{-1})^k(t) \\ &= g\alpha(t). \end{aligned}$$

Similarly, $\alpha f(t) = g\alpha(t)$ for $t_{-k} \le t < t_{-k+1}$, $k \ge 1$.

Continuity of $\alpha(t)$. Because α_0 is continuous on $t_0 \le t \le t_1$, the compositions that define α show that α is continuous on each open interval (t_n, t_{n+1}). It suffices, therefore, to show that α is continuous at each point t_n. The details will be illustrated for $n = 1$. Then

$$\begin{aligned} \lim_{h\to 0+} \alpha(t_1 + h) &= \lim_{h\to 0+} g\alpha_0 f^{-1}(t_1 + h) \\ &= g\alpha_0 f^{-1}(t_1) \\ &= \alpha(t_1), \end{aligned}$$

$$\begin{aligned} \lim_{h\to 0+} \alpha(t_1 - h) &= \lim_{h\to 0+} \alpha_0(t_1 - h) \\ &= \alpha_0(t_1) \\ &= \alpha(t_1). \end{aligned}$$

Monotonicity of $\alpha(t)$. On each interval $[t_n, t_{n+1})$, α is a composition of increasing functions, therefore by Lemma 1 it is increasing. By continuity of α, it is increasing on $(-\infty, \infty)$.

Image of α *is* $(-\infty, \infty)$. Evaluate

$$\alpha(t_n) = g^n(\alpha_0(t_0)), \quad \alpha(t_{-n}) = g((g^{-1})^{n+1}(\alpha_0(t_0))).$$

By Lemma 4, these sequences of points belong to the image of g and limit to ∞ and $-\infty$, respectively, as $n \to \infty$. Therefore, g has image $(-\infty, \infty)$. $\square$

3. Remarks on the Construction

The requirement $g(0) > 0$ can be replaced by $g(0) < 0$, the only other possibility for a fixed-point free function g. In this case, α will be strictly decreasing, because α_0 is then chosen to be strictly decreasing.

It is not possible to choose f and g to be decreasing, because then they cannot be fixed-point free.

Additional smoothness of α is possible, by demanding more smoothness of the functions α_0, f, g. See *extensions* below.

The geometry of the construction can be imagined by constructing the curves $g^{-2}\alpha_0$, $g^{-1}\alpha_0$, α_0, $g\alpha_0$, $g^2\alpha_0$ on the interval $[t_0, t_1]$. These curves in the case $g(x) = x + 1$, $g^{-1}(x) = x - 1$ are translates of the increasing function α_0. The appearance for other choices of g is essentially the same: the curves do not cross: see Figure 1. The curve α is made by moving one of these curves on $[t_0, t_1]$ to the corresponding interval $[t_n, t_{n+1}]$, followed by a change of scale on the t-axis, which deforms the moved curve to fit. The newly placed curves join continuously at each transition point t_n.

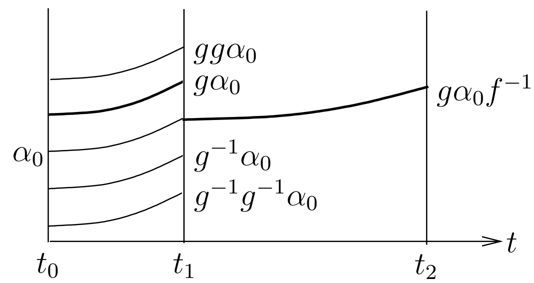

Figure 1: The curve $g\alpha_0$ is moved rigidly from $[t_0, t_1]$ to $[t_1, t_2]$, then scaled by f^{-1} to fit the domain (heights unchanged). The moved curve matches α_0 at $t = t_1$

4. Extensions

The base function α_0 on $[t_0, t_1]$, where $t_1 = f(t_0)$, can be chosen to be continuously differentiable, in hopes to find a continuously differentiable change of variables α for the iterative functional equation. The conditions required are as follows.

Theorem 2. *The result of Theorem 1 holds with the added conclusion that α is continuously differentiable, provided the following hypotheses are added:*

(a) f and g have continuous derivatives.

(b) α_0 satisfies the two requirements

$$\begin{aligned} \alpha_0(f(t_0)) &= g(\alpha_0(t_0)), \\ \alpha_0'(f(t_0)) &= g'(\alpha_0(t_0))\frac{\alpha_0'(t_0)}{f'(t_0)}. \end{aligned}$$

Proof of Theorem 2. Assume for the presentation of details that α_0 has been extended to the whole real line as a continuously differentiable function. In order to show that α is continuously differentiable, it suffices to establish this fact at the transition points t_n. We will illustrate how to do this for $t = t_2$, and indicate how the induction proceeds for the other points. Let us assume that it has already been established that α is continuously differentiable on $[t_0, t_2)$, which is implied by derivative matching in (b).

First, calculate the value of $\alpha'(t_2)$ using the formula $\alpha = g\alpha_0 f^{-1}$ on $[t_1, t_2)$. Then

$$\begin{aligned} \alpha'(t_2) &= g'(\alpha_0 f^{-1}(t_2))\alpha_0'(f^{-1}(t_2))(f^{-1})'(t_2) \\ &= g'(\alpha_0(t_1))\alpha_0'(t_1)\frac{1}{f'(t_1)}. \end{aligned}$$

Second, calculate the value of $\alpha'(t_2)$ using the formula $\alpha = gg\alpha_0 f^{-1}f^{-1}$ on $[t_2, t_3)$. It is important here to let $H = g\alpha_0 f^{-1}$ and write $\alpha = gHf^{-1}$, because this is the intuition for the induction step for the other points. Then $H(t) = \alpha(t)$ on $[t_1, t_2]$ plus the assumption that α is continuously differentiable on $[t_0, t_2)$ gives

$$\begin{aligned} \alpha'(t_2) &= g'(Hf^{-1}(t_2))H'(f^{-1}(t_2))(f^{-1})'(t_2) \\ &= g'(H(t_1))H'(t_1)\frac{1}{f'(t_1)} \\ &= g'(\alpha(t_1))H'(t_1)\frac{1}{f'(t_1)} \\ &= g'(\alpha_0(t_1))\alpha'(t_1)\frac{1}{f'(t_1)} \\ &= g'(\alpha_0(t_1))\alpha_0'(t_1)\frac{1}{f'(t_1)}. \end{aligned}$$

Standard quadratic α_0. The conditions in (b) hold for the quadratic function

$$\alpha_0(t) = \frac{g(0)}{(1+k)[f(0)]^2}\left(2f(0)t + (k-1)t^2\right), \quad k = \frac{g'(0)}{f'(0)},$$

provided we choose $t_0 = 0$, $\alpha_0(t_0) = 0$. In particular, this choice of α_0 has a positive derivative. □

5. Different Domains

The construction of α in Theorem 1 requires f and g to have a domain and image of $(-\infty, \infty)$. Therefore, the construction cannot handle a conjugacy equation like $\alpha(e^t) = 1 + \alpha(t)$. Shown here is a general method for applying Theorem 1 to special situations.

Theorem 3. *Let F and G be continuous functions from $[0, \infty)$ into $(-\infty, \infty)$, strictly increasing and fixed-point free, with $\lim_{t\to\infty} F(t) = \lim_{t\to\infty} G(t) = \infty$. Assume $F(0) > 0$ and $G(0) > 0$. Let $t_0 = 0$. Define $\alpha_0(t)$ to be any strictly increasing continuous function on $t_0 \le t \le F(t_0)$ such that $\alpha_0(t_0) \ge 0$ and $\alpha_0(F(t_0)) = G(\alpha_0(t_0))$. Define*

$$\alpha(t) = \begin{cases} \alpha_0(t) & t_0 \le t < F(t_0), \\ G^n \alpha_0 (F^{-1})^n(t) & F^n(t_0) \le t < F^{n+1}(t_0), \quad n \ge 1. \end{cases}$$

Then:

(1) $\alpha(t)$ is defined on $[0, \infty)$ and $\alpha(F(t)) = G(\alpha(t))$ for all $t \ge 0$.

(2) $\alpha(t)$ is continuous.

(3) $\alpha(t)$ is strictly increasing.

(4) $\lim_{t\to\infty} \alpha(t) = \infty$.

Proof of Theorem 3. The details of the proof can begin with the definition $f(t) = F(t)$ for $t \ge 0$, $f(t) = t + F(0)$ for $t \le 0$. A similar definition is made for g and G. Then f and g satisfy the hypotheses of Theorem 1 and α_0, α are then defined. The restriction of α to $[0, \infty)$ gives the formula for α stated in Theorem 3. The verification of (1)-(4) is left to the reader. □

6. Examples

Illustrated here is a computer algebra system solution for $\alpha(t)$ in the conjugacy problem

$$\alpha(e^{t/2}) = 1 + \alpha(t).$$

This conjugacy equation uses $F(x) = e^{x/2}$ and $G(x) = 1 + x$. Considered are two functions to start the iterative solution:

$$\begin{aligned} \alpha_0(t) &= \frac{G(0)t}{F(0)}, \\ \alpha_0(t) &= \frac{G(0)F(0)^2}{1+k}\left(2F(0)t + (k-1)\,t^2\right), \quad k = \frac{G'(0)}{F'(0)}. \end{aligned}$$

The linear function α_0 is illustrated graphically on Figure 2, while the quadratic function α_0 on Figure 3.

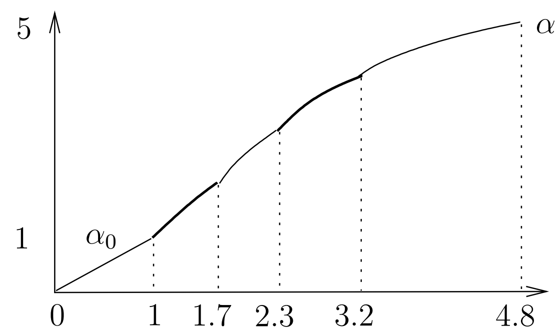

Figure 2: Linear example for the conjugacy equation $\alpha(e^{t/2}) = 1 + \alpha(t)$

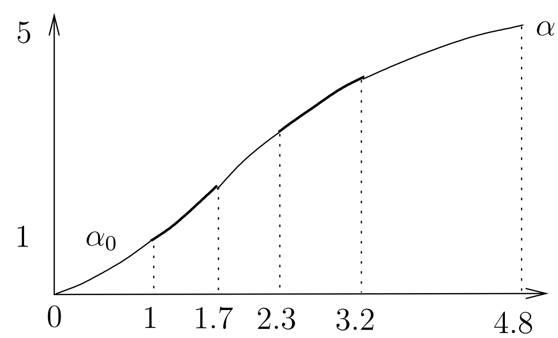

Figure 3: Quadratic example for the conjugacy equation $\alpha(e^{t/2}) = 1 + \alpha(t)$

7. Historical Background

Functional equation (1.1) and the Abel functional equation are studied in monographs of M. Kuczma and his colleagues [2], [3].

In the monograph of O. Borůvka [1], the Abel functional equation

$$\alpha f(x) = \alpha(x) + 1 \tag{7.1}$$

has a significant place in the theory of linear second-order differential transformations. In equation (7.1), the function α is called a *phase* and the function f is called a *central dispersion*. The *first phase* α is a continuous function on $(-\infty, \infty)$ which is constructed from a basis u, v of solutions for an oscillatory differential equation

$$y'' = q(t)y, \quad t \in (-\infty, \infty).$$

It satisfies except at the zeros of v (see [1], [3], [4]) the defining equation

$$\tan \alpha(t) = \frac{u(t)}{v(t)}.$$

The *central dispersion* f of the first kind for $y'' = q(t)y$ is defined by $f(t_1) = t_2 > t_1$ where some solution y satisfies $y(t_1) = y(t_2) = 0$ and $y(t)$ is nonzero between t_1 and t_2.

The problem of determining the differential equation $y'' = q(t)y$ from a knowledge of its central dispersion is studied in [1]. It leads to consideration of the Abel functional equation for the central dispersion with α an unknown phase function. In [1], O. Borůvka gives a construction due to E. Barvínek for infinitely many phase functions $\alpha \in C_3$ which are solutions of the Abel functional equation $\alpha(f) = a(t) + \pi \operatorname{sgn} \alpha'$.

References

[1] O. Borůvka, *Linear Differential Transformations of the Second Order*, The English University Press, London (1971).

[2] M. Kuczma, *Functional Equations in a Single Variable*, PWN, Warszawa (1968).

[3] M. Kuczma, B. Choczewski, R. Ger, *Iterative Functional Equations*, Cambridge University Press, Cambridge (1990).

[4] J. Laitochová, Algebraic structure of phase functions, *Acta Math. Hungarica*, **73**, No. 4 (1996), 301-318.

[5] F. Neuman, *Funkcionální Rovnice*, SNTL, Praha (1986).