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SMOOTH CURVES COVERING TWO CURVES OF POSITIVE GENUS

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Abstract: Fix integers $z \ge 2$, $q_i \ge 0$, $1 \le i \le z$, $a_i \ge 2$, $1 \le i \le z$, g_j, $1 \le j \le z-1$. Set $k_z := a_z$ and define by decreasing induction the integers k_j, $1 \le j \le z-1$ by the formula $k_j := a_j k_{j+1}$. Fix smooth and connected projective curves Y_i, $1 \le i \le z$, such that $p_a(Y_i) = q_i$ and there is $M_j \in \mathrm{Pic}^{a_j}(Y_j)$, $1 \le j \le z$, spanned by its global sections. Set $x_j := h^0(Y_j, M_j)$ and $g := g_z$. Assume $1 + a_1a_2 + a_1q_1 + a_2q_2 - a_1 - a_2 + 1 - \lfloor x_1/2 \rfloor \cdot \lfloor x_2/2 \rfloor \le g_1 \le 1 + a_1a_2 + a_1q_1 + a - 2q_2 - a_1 - a_2$ and $1 + a_ja_{j+1} + a_jg_j + a_{j+1}q_{j+1} - a_j - a_{j+1} + 1 - \lfloor x_j/2 \rfloor \cdot \lfloor x_{j+1}/2 \rfloor \le g_1 \le 1 + a_ja_{j+1} + a_jg_j + a_{j+1}q_{j+1} - a_j - a_{j+1}$ for $1 \le j \le z-1$. Here we prove the existence of a smooth and connected genus g_z projective curve X and degree k_i morphisms $f_i : X \to Y_i$, $1 \le i \le z$, such that $p_a(X) = g$, $p_a(Y_i) = q_i$, and all morphisms $(f_j, f_z) : X \to Y_j \times Y_z$, $1 \le i \le z-1$, are birational onto its image.

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1. The Statement

Here we study the existence of smooth and connected projective curves X equipped with several morphisms $f_i : X \to Y_i$ with Y_i a prescribed curve of genus > 0. Here we prove the following result.

Theorem 1. *Fix integers $z \ge 2$, $q_i \ge 0$, $1 \le i \le z$, $a_i \ge 2$, $1 \le i \le z$, g_j, $1 \le j \le z-1$. Set $k_z := a_z$ and define by decreasing induction the*

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integers k_j, $1 \le j \le z-1$ by the formula $k_j := a_j k_{j+1}$. Fix smooth and connected projective curves Y_i, $1 \le i \le z$, such that $p_a(Y_i) = q_i$ and there is $M_j \in \mathrm{Pic}^{a_j}(Y_j)$, $1 \le j \le z$, spanned by its global sections. Set $x_j := h^0(Y_j, M_j)$ and $g := g_z$. Assume $1 + a_1a_2 + a_1q_1 + a_2q_2 - a_1 - a_2 + 1 - \lfloor x_1/2 \rfloor \cdot \lfloor x_2/2 \rfloor \le g_1 \le 1 + a_1a_2 + a_1q_1 + a - 2q_2 - a_1 - a_2$ and $1 + a_ja_{j+1} + a_jg_j + a_{j+1}q_{j+1} - a_j - a_{j+1} + 1 - \lfloor x_j/2 \rfloor \cdot \lfloor x_{j+1}/2 \rfloor \le g_1 \le 1 + a_ja_{j+1} + a_jg_j + a_{j+1}q_{j+1} - a_j - a_{j+1}$ for $1 \le j \le z-1$. Then there exist a smooth and connected genus g projective curve X and degree k_i morphisms $f_i : X \to Y_i$, $1 \le i \le z$, such that $p_a(X) = g$, $p_a(Y_i) = q_i$, and all morphisms $(f_j, f_z) : X \to Y_j \times Y_z$, $1 \le i \le z-1$, are birational onto its image.

Remark 1. Take the set-up of the statement of Theorem 1. Notice that we may always take $a_i = q_i + x_i - 1$ and $h^1(Y_i, M_i) = 0$ for all i.

We work over an algebraically closed field $\mathbb{K}$ with $\mathrm{char}(\mathbb{K}) = 0$.

2. The Proof

We will first prove the case $z = 2$ of Theorem 1. The general case will easily follow by induction on z. To prove the case $z = 2$ we will use the following set-up.

Remark 2. Set $S := Y_1 \times Y_2$. Hence S is a smooth and connected projective surface. Let $\pi_i : S \to Y_i$, $i = 1, 2$, denote the projection. By Künneth formula we have $h^1(S, \mathcal{O}_S) = q_1 + q_2$. For all $M_i \in \mathrm{Pic}(Y_i)$, $i = 1, 2$, set $\mathcal{O}_S(M_1, M_2) := \pi_i^*(M_1) \otimes \pi_2^*(M_2) \in \mathrm{Pic}(S)$. In this way we obtain an inclusion of abelian groups $\mathfrak{s} : \mathrm{Pic}(Y_1) \oplus \mathrm{Pic}(Y_2) \to \mathrm{Pic}(S)$. Y_1 and Y_2 are called Picard-indipendent if the inclusion $\mathfrak{s}$ is surjective (see [4]). This is often (but not always) the case. The pair of integers $(\deg(M_1), \deg(M_2))$ is called the bidegree of the line bundle $\mathcal{O}_S(M_1, M_2)$. Fixing any $P_i \in Y_i$ and using the integers $D \cdot \pi_i^{-1}(P_i)$ we see that every Weil divisor of S has a bidegree, even if Y_1 and Y_2 are not Picard-independent. We have $(\mathcal{O}_S(M_1, M_2) \cdot \mathcal{O}_S(N_1, N_2)) = a_1b_2 + a_2b_1$ if $\deg(M_i) = a_i$ and $\deg(N_i) = b_i$, $i = 1, 2$. We have $h^0(S, \mathcal{O}_S(M_1, M_2)) = h^0(Y_1, M_1) \cdot h^0(Y_2, M_2)$, $h^1(S, \mathcal{O}_S(M_1, M_2)) = h^0(Y_1, M_1) \cdot h^1(Y_2, M_2) + h^1(Y_1, M_1) \cdot h^0(Y_2, M_2)$ and $h^2(S, \mathcal{O}_S(M_1, M_2)) = h^1(Y_1, M_1) \cdot h^1(Y_2, M_2)$ (Künneth formula). We have $\omega_S \cong \mathcal{O}_S(\omega_{Y_1}, \omega_{Y_2})$. Hence $\omega_C \cong \mathcal{O}_S(\omega_{Y_1} \otimes M_1, \omega_{Y_2} \otimes M_2)$ for any effective divisor $C \in |\mathcal{O}_S(M_1, M_2)|$ (adjunction formula). Thus $p_a(C) = 1 + a_1a_2 + a_1q_1 + a_2q_2 - a_1 - a_2$ if $\ge (M_i) = a_i$. For any $P \in S$ let $2P$ denote the first infinitesimal neighborhood of P in S, i.e. the closed zero-dimensional subscheme of S with $(\mathcal{I}_P)^2$ as its ideal sheaf. Hence $(2P)_{red} = \{P\}$ and length$(2P) = 3$. Similarly, for any scheme A and any $P \in A_{reg}$, let $\{2P, A\}$ denote the first infinitesimal neighborhood of P in A, i.e. the closed zero-dimensional subscheme of A with $(\mathcal{I}_{P,A})^2$ as its ideal sheaf. Notice that $\{2P, A\}_{red} = \{P\}$ and length$(\{2P, A\}) = \dim_P(A) + 1$.

Remark 3. Fix integers $a_i \geq 2$, $i = 1, 2$, such that there are $M_i \in \mathrm{Pic}^{a_i}(Y_i)$, which are spanned by their global sections. Hence the linear system $\Gamma := |\mathcal{O}_S(M_1, M_2)|$ has no base points. Hence a general $T \in \Gamma$ is smooth (Bertini's Theorem). There is $T' \in \Gamma$ formed by the union of a_2 fibers of π_1 and a_1 fibers of π_2. Thus T' is reduced and connected. Hence T is connected. Thus T is irreducible. Since each M_i is spanned, every irreducible component of T' moves and hence it is a nef divisor. Hence T is a nef divisor. Notice that $h^1(S, \mathcal{O}_S(M_1, M_2)) = x_1 h^1(Y_2, M_2) + x_2 h^1(Y_1, M_1)$. Set $x_i := h^0(Y_i, M - i)$. Thus $\dim(\Gamma) = x_1 x_2 - 1$. Fix an integer $b \geq 0$ and a general $B \subset S$ such that $\sharp(B) = b$. Set $Z := \bigcup_{P \in B} 2P$. Let C be a general member of the linear system $|\mathcal{I}_Z(M_1, M_2)|$.

Claim A. *Assume* $b \leq \lfloor x_1/2 \rfloor \cdot \lfloor x_2/2 \rfloor$. *Then* $h^1(S, \mathcal{I}_Z(M_1, M_2)) = x_1 h^1(Y_2, M_2) + x_2 h^1(Y_1, M_1)$.

Proof of Claim A. For any zero-dimensional subscheme $W \subset S$ the exact sequence

$$0 \to \mathcal{I}_W(M_1, M_2) \to \mathcal{O}_S(M_1, M_2) \to \mathcal{O}_W(M_1, M_2) \to 0 \qquad (1)$$

gives

$$h^1(S, \mathcal{I}_W(M_1, M_2)) \geq h^1(S, \mathcal{O}_S(M_1, M_2)) = x_1 h^1(Y_2, M_2) + x_2 h^1(Y_1, M_1).$$

Hence to prove Claim A it is sufficient to check the inequality

$$h^1(S, \mathcal{I}_Z(M_1, M_2)) \leq h^1(S, \mathcal{O}_S(M_1, M_2)).$$

Fix $\lfloor x_i/2 \rfloor$ general points $Q_{j,i}$, $1 \leq j \leq \lfloor x_i/2 \rfloor$, of Y_i, $i = 1, 2$. Set $Q[h, k] := (Q_{h,1}, Q_{k,2}) \in S$. Let $\tilde{Q}[h, k]$ denote the closed zero-dimensional subscheme of S which is the product of the scheme $\{2Q_{h,1}, Y_1\} \subset Y_1$ and the scheme $\{2Q_{h,2}, Y_2\} \subset A_2$. Set $Z_i := \cup_{j=1}^{\lfloor x_i/2 \rfloor} \{2Q_{j,i}, A_i\}$. Set $\tilde{Q} := Z_1 \times Z_2 = \bigcup_{h,k} Q[h, k] \subset S$. Since $\mathrm{char}(\mathbb{K}) = 0$ and the points $Q_{j,i}$ are general in Y_i, we have $h^1(Y_i, \mathcal{I}_{Z_i} \otimes M_i) = h^1(Y_i, M_i)$ (see [6]). Hence $h^1(S, \mathcal{I}_{\tilde{Q}}(M_1, M_2)) = h^1(S, \mathcal{O}_S(M_1, M_2))$. Notice that $\sharp(\bigcup_{h,k} Q[h, k]) = \lfloor x_1/2 \rfloor \cdot \lfloor x_2/2 \rfloor$. Take any $A \subseteq \bigcup_{h,k} Q[h, k]$ with $\sharp(A) = a$ and set $\tilde{A} := \bigcup_{P \in A} 2P$. Since $\tilde{Q}$ is zero-dimensional and $\tilde{A} \subseteq \tilde{Q}$, we have $h^1(S, \mathcal{I}_{\tilde{Z}}(M_1, M_2)) \leq h^1(S, \mathcal{I}_{\tilde{Q}}(M_1, M_2)) = h^1(S, \mathcal{O}_S(M_1, M_2))$. Hence Claim A is true for $\tilde{Z}$. By semicontinuity we get Claim A for the general Z. $\square$

Claim B. *Assume $b \le \lfloor x_1/2 \rfloor \cdot \lfloor x_2/2 \rfloor$. Then every singularity of C is an ordinary node.*

Proof of Claim B. Take $Q_{j,i} \in Y_i$, $i = 1, 2$, $1 \le j \le \lfloor x_i/2 \rfloor$, and $Q[h,k] := (Q_{h,1}, Q_{k,2}) \in S$ as in the proof of Claim A. Fix any $A \subset \bigcup_{h,k} Q[h,k]$ such that $\sharp(A) = b$ and set $\tilde{A} := \bigcup_{P \in A} 2P$. We saw in the proof of Claim A that $h^1(S, \mathcal{I}_{\tilde{A}}(M_1, M_2)) = h^1(S, \mathcal{O}_S(M_1, M_2))$, i.e. $h^1(S, \mathcal{I}_{\tilde{A}}(M_1, M_2)) = h^1(S, \mathcal{I}_Z(M_1, M_2))$ (the minimal possible value). Take any reduced divisor $E_i \in |\mathcal{O}_{Y_i}(M_i(-\sum_j Q_{j,i}))$ such that the support of each E_i does not contain any of the points $Q_{j,i}$, $1 \le j \le \lfloor x_i/2 \rfloor$. Set $E := \pi_1(D_1 \cup \bigcup_j Q_{j,1}) \cup \pi_2(D_2 \cup \bigcup_j Q_{j,2})$. Hence $E \in |\mathcal{I}_{\tilde{A}}(M_1, M_2)|$ and E has only ordinary nodes. Since $h^1(S, \mathcal{I}_{\tilde{A}}(M_1, M_2)) = h^1(S, \mathcal{O}_S(M_1, M_2))$, we may apply semicontinuity and get that C has only ordinary nodes, i.e. Claim B is true. □

Claim C. *Assume $0 \le b \le \lfloor x_1/2 \rfloor \cdot \lfloor x_2/2 \rfloor$. Then $C = C'$.*

Proof of Claim C. Assume $C \ne C'$ and call E the closure of $C \backslash C'$ in S. Since we assumed $b \le \lfloor x_1/2 \rfloor \cdot \lfloor x_2/2 \rfloor$, $h^0(S, \mathcal{O}_S(E)) \ge 2$ and $h^0(S, \mathcal{I}_Z(C')) \ge 2$ then there is the following elementary proof. By assumption for a general $P \in S$ there is $E_1 \in |E|$ and $C'' \in |\mathcal{I}_Z(C')|$ such that $P \in E_1$ and $P \in C''$. Furthermore, the set of all such curves E_1 (resp. C'') has codimension one in $|E|$ (resp. $|\mathcal{I}_Z(C')|$). Hence $h^0(S, \mathcal{I}_{Z \cup 2P}(C)) \ge h^0(S, \mathcal{I}_Z(C)) - 2$. Since $b' := b+1 \le \lfloor x_1/2 \rfloor \cdot \lfloor x_2/2 \rfloor$ and $B \cup \{P\}$ is general in S, the contradiction comes from Claim A. Making a weaker assumption on b we may give the following proof which heavily use our characteristic zero assumption. Since $\mathcal{O}_S(M_1, M_2)$ is spanned and $E \ne \emptyset$, we have $h^0(S, \mathcal{O}_S(M_1, M_2)(-E)) < h^0(S, \mathcal{O}_S(M_1, M_2))$. Since $h^0(S, \mathcal{I}_Z(M_1, M_2)(-E)) = h^0(S, \mathcal{I}_Z(M_1, M_2))$, the scheme Z does not impose $3b$ independent conditions to $H^0(S, \mathcal{O}_S(M_1, M_2)(-C))$. By a lemma of Terracini (see [2]) this implies that the base locus of the linear system $|\mathcal{I}_Z(M_1, M_2)|$ contains a curve with multiplicity two, contradicting the fact that C is nodal and hence it has no multiple component. □

Claim D. *Assume $b + 1 \le \lfloor x_1/2 \rfloor \cdot \lfloor x_2/2 \rfloor$. Fix $O \in Y_1$ and let $\bar{Z}$ be the residual scheme of Z with respect to the Cartier divisor $\pi_1^{-1}(O)$. Then $h^1(S, \mathcal{O}_{\bar{Z}}(M_1(-O), M_2) = h^1(S, \mathcal{O}_S(M_1(-O), M_2))$.*

Proof of Claim D. Copy the proof of Claim A. We only observe that $\bar{Z} = Z$ if $O \notin \pi_1(B)$, while $\bar{Z} \subset Z$ and length$(\bar{Z}) = 3b - 2$ if $O \in \pi_1(B)$. □

Claim E. *Fix $O \in Y_2$ and let $\tilde{Z}$ be the residual scheme of Z with respect to the Cartier divisor $\pi_2^{-1}(O)$. Then $h^1(S, \mathcal{O}_{\tilde{Z}}(M_1, M_2(-O)) = h^1(S, \mathcal{O}_S(M_1, M_2(-O)))$.*

Proof of Claim E. Exchange Y_1 and Y_2 in the proof of Claim D. □

Claim F. *Assume $b+1 \le \lfloor x_1/2 \rfloor \cdot \lfloor x_2/2 \rfloor$. Then B is the set-theoretic base point of $|\mathcal{I}_Z(M_1, M_2)|$.*

Proof of Claim F. By Claims D (resp. E) for any $O \in Y_1$ (resp. $O \in Y_2$) the restriction map $H^0(S, \mathcal{I}_Z(M_1, M_2)) \to H^0(\pi_1^{-1}(O), \mathcal{I}_{Z \cap \pi_1^{-1}(O)}(M_2))$ (resp. $H^0(S, \mathcal{I}_Z(M_1, M_2)) \to H^0(\pi_2^{-1}(O), \mathcal{I}_{Z \cap \pi_2^{-1}(O)}(M_1))$ is surjective. Hence the set-theoretic base locus of $|\mathcal{I}_Z(M_1, M_2)|$ is contained in the set $\pi_1^{-1}(\pi_1(B)) \cap \pi_2^{-1}(\pi_2(B))$. Since $\pi_1|B$ and $\pi_2|B$ are injective, we have $\pi_1^{-1}(\pi_1(B)) \cap \pi_2^{-1}(\pi_2(B)) = B$, proving Claim F. □

Claim G. *Assume* $b + 1 \leq \lfloor x_1/2 \rfloor \cdot \lfloor x_2/2 \rfloor$. *Then* C *is integral, nodal and* $\mathrm{Sing}(C) = B$.

Proof of Claim G. By Claim F and Bertini's Theorem C is smooth outside B. By Claim C the curve C is nodal. Now we take again the set-up of Claim B. Now we will prove that C is irreducible. Call C' the union of all irreducible components of C containing at least one point of B. Notice that $B \subset C'$ because $B \subset C$. Since the symmetric product of b copies of S is irreducible and we may vary B in an open subset of it, a monodromy argument gives that one of the following three cases must occur:

(a) C' is irreducible.

(b) C' has exactly b irreducible components, each of them containing exactly one point of B. Furthermore, all irreducible components, say $C_1, \dots, C_b$, of C' are numerically equivalent and $h^0(S, \mathcal{O}_S(C_i)) = h^0(S, \mathcal{O}_S(C_j))$ for all i, j.

(c) C' has two irreducible components, say C_1 and C_2, and $B \subseteq (C_1)_{reg} \cap (C_2)_{reg}$. Furthermore, C_1 and C_2 are numerically equivalent and $h^0(S, \mathcal{O}_S(C_1)) = h^0(S, \mathcal{O}_S(C_2))$.

If $b = 0$, then we are in case (a) and $C = C'$ is smooth by Bertini's Theorem. If $b = 1$, then cases (a) and (b) are the same. First assume that we are in case (b) or in case (c) and fix any irreducible component D of C'. Let (d_1, d_2) be the bidegree of D. Hence in case (b) C' has bidegree (bd_1, bd_2), while in case (c) C' has bidegree $(2d_1, 2d_2)$. Since C' is reduced and $Z \subset C'$, in case (c) we get $d_1 > 0$ and $d_2 > 0$. Every effective divisor of bidegree $(x, 0)$ (resp. $(0, x)$ is of the form $\pi_1^{-1}(\tilde{D})$ (resp. $\pi_2^{-1}(\tilde{D})$) for some effective degree x divisor $\tilde{D}$ of Y_1 (resp. Y_2). Since in case (b) D has at least one singular point, but it is reduced, while Y_1 and Y_2 are smooth, in case (b) we have $d_1 > 0$ and $d_2 > 0$. Thus case (b) cannot occur if $b > \min\{a_1, a_2\}$. Every effective divisor of S with bidegree $(1, 1)$ and at least one singular point is reducible. Hence we see that in case (b) either $d_1 \geq 2$ or $d_2 \geq 2$. Thus case (b) cannot occur if $b > \max\{a_1/2, a_2/2\}$. Now we fix B and hence Z, but we move C (and hence C') in an open subset of $|\mathcal{I}_Z(M_1, M_2)|$. We assume $b \geq 2$ and that we are in case (b). Since $|\mathcal{I}_Z(M_1, M_2)|$ is a projective space, C varies in a rational family. Since B is fixed, each C_i varies in a rational family. Furthermore,

$h^0(S, \mathcal{O}_{2P_i}(C_i)) > 0$, where $P_i := B \cap C_i$. Hence the contradiction comes as in the proof of Claim C by the quoted lemma of Terracini (see [2]), unless $h^0(S, \mathcal{I}_{2P_i}(C_i)) = h^0(S, \mathcal{O}_S(C_i) + 3 \geq 4$ for all i, i.e. $h^0(S, \mathcal{O}_S(M_1, M_2)) = \sum_{i=1}^{b} h^0(S, \mathcal{O}_S(C_i)$ and that each $\mathcal{O}_S(C_i)$ is spanned. By [3], Corollary 5.2, we also have that all pairs of linear systems $(\mathcal{O}_S(C_i), H^0(S, \mathcal{O}_S(C_i)))$, $1 \leq i \leq b$, are obtained from the pair $(\mathbf{P}^1, H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(z))$, $z := h^0(S, \mathcal{O}_S(C_1))$, taking the composition of it with a rational map ψ from S into $\mathbf{P}^1$ (the same map for all $i \in \{1, \dots, b\}$). Since all C_i, $1 \leq i \leq b$, are numerically equivalent and spanned, the existence of ψ implies that they are rationally (and hence linearly) equivalent. Hence $\mathcal{O}_S(C_i) \cong \mathcal{O}_S(C_1)$ for all i. By Claim C we also get $\mathcal{O}_S(M_1, M_2) \cong \mathcal{O}_S(bC_1)$ and hence $a_j \equiv 0 \pmod{b}$ for $j = 1, 2$, and each C_i has bidegree $(a_1/b, a_2/b)$. Since $\mathcal{O}_S(C_1)$ is spanned, ψ is a morphism. There is no morphism $\tau : S \to \mathbf{P}^1$ such that $\mathcal{O}_S(M_1, M_2) \cong \tau^*(\mathcal{O}_{\mathbf{P}^1}(t))$ for some $t \in \mathbb{Z}$, because $\mathcal{O}_S(M_1, M_2)$ is ample. Hence case (b) cannot occur. Now assume that we are in case (c). We saw that C_1 and C_2 must be smooth and $C_1 \cap C_2 = B$ (scheme-theoretically). Hence $C_1 \cdot C_2 = b$. Since C_1 and C_2 are numerically equivalent, we get $(C_1 + C_2) \cdot (C_1 + C_2) = 4b$. i.e. $a_1 a_2 = 2b$. Since $x_i \leq a_i$, we get $2b \geq x_1 x_2$, contradiction. □

Remark 4. Claim D of Remark 3 does not hold if $x_2 = x_2 = 2$ and $b = 1$. Indeed, take a general $(P_1, P_2) \in S$. Hence $|M_i(-P_i)| = \{D_i\}$ and hence $|\mathcal{I}_Z(M_1, M_2)| = \{\pi_1^{-1}(D_1 + P_1) \cup \pi_2^{-1}(D_2 + P_2)\}$ is formed by a reducible divisor.

Example 1. Here we assume $q_1 \geq 2$, $q_2 \geq 2$, $k_1 = k_2 = 2$ and that Y_1 and Y_2 hyperelliptic. In the construction of Remark 3 we have to take as M_i the ${g^1}_2$ on Y_i. We saw in Remark 3 that we have to take C smooth. Hence $g = 1 + k_1 k_2 + k_1 q_1 + k_2 q_2 - k_1 - k_2 = 2q_1 + 2q_2 + 1$ and that, conversely, such a quintuple (X, Y_1, Y_2, f_1, f_2) with X of genus $2q_2 + 2q_1 + 1$ exists for all hyperelliptic curves Y_1, Y_2. Notice that $M_1|X$ and $M_2|X$ are ${g^1}_4$'s on X and that $h^0(X, (M_1 \otimes M_2)|X) = 3$. By Castelnuovo-Severi inequality (see [5] or [1], p. 21) we see that these two g_4^1's are the same, i.e. that that the two involutions of X associated to the double coverings f_1 and f_2 commute. Hence X is of this type (for some q_1 and q_2) if and only if it is a Galois covering of $\mathbf{P}^1$ with $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ as its Galois group.

Proof of Theorem 1. First assume $z = 2$. Hence $a_1 = k_1$ and $a_2 = k_2$. Apply Remark 3. Now assume $z \geq 3$. Call X' a solution for the curves $Y_1, \dots, Y_{z-1}$ and apply the case $z = 2$ of the proof to the curves X' and Y_z. Use that (calling $u_j : X' \to Y_j$, $1 \leq j \leq z-1$, the given map) we have $h^0(X', u_j^*(M_j)) \geq h^0(Y_j, M_j) = x_j$. □

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