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# BOUNDARY TARGET IN PARABOLIC SYSTEMS APPROACH AND SIMULATIONS

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**Abstract:** In this paper we examine the boundary target control problem, so we establish a sequence of controls which converge to the optimal control, also we give an important numerical approach that leads to the computation of a such control. This method, based on an extension of the internal regional controllability, is successfully tested through computer simulations leading to some conjectures.

**AMS Subject Classification:** 26A33
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## 1. Introduction

Many real problems in the control and observation of distributed systems can be reformulated as problem of analysis for infinite dimensional systems. Among the most important is controllability which has been widely developed (Curtain

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and Pritchard [3], Curtain and Zwart [4] and references therein).

For a given distributed system, the concept of regional controllability introduced by El Jai et al [7] is defined as the possibility to reach a desired state only on a subregion $\omega$ of the system domain $\Omega$. This concept was extended by Zerrik et al [14], to boundary subregion. This leads to the so called regional boundary controllability. Many theoretical results have been obtained for this controllability. This study finds its application in various real world problems, this is the case for example of a biological reactor (Figure 1) in which the concentration regulation of a substratum at the bottom of the reactor Jacob [5], and to give an application of the controllability concept not in the whole domain but controllable on the boundary of the evolution domain Zerrik et al [14].

This work is a natural consequence of the results obtained in Zerrik et al [14], and our interest here is focused on the development of an approach that leads to numerical implementation for the computation of the control which steers the system from an initial state to a given regional boundary state, with minimum energy. This is the purpose of this paper and the content is organized as follows. Section 2 is devoted to the presentation of the system under consideration and characterization of approximate regional boundary controllability in term of actuators. In the Section 3 we consider the regional boundary target control problem and we establish a sequence of control which converge to the optimal control. The method is based on an extension of the internal regional controllability. In the last section the obtained approach is successfully tested through computer simulations.

## 2. Regional Boundary Controllability

### 2.1. Boundary Controllability

Let $\Omega$ be a regular bounded open set of $\mathbb{R}^n$ with boundary $\partial\Omega$, let $\Gamma$ be a nonempty subset of $\partial\Omega$. For a given time $T > 0$ let $Q = \Omega\times]0,T[$ , $\Sigma = \partial\Omega\times]0,T[$, consider the system described by the equation

$$\begin{cases} \dfrac{\partial y}{\partial t}(x,t) = Ay(x,t) + Bu(t) & Q\,, \\ \dfrac{\partial y}{\partial \nu_A}(\xi,t) = 0 & \Sigma\,, \\ y(x,0) = y_0(x) & \Omega\,, \end{cases} \tag{2.1}$$

where we apply the following points:

— $A$ is a second order linear differential operator with compact resolvent and generates a strongly continuous semi-group $(S(t))_{t\geq 0}$ on the Hilbert state space $L^2(\Omega)$. $A^*$ is considered for the adjoint operator of $A$, $\frac{\partial y}{\partial \nu_A}$ denotes the conormal with respect to $A$. $B \in \mathcal{L}(I\!R^m, L^2(\Omega))$, $y_0 \in L^2(\Omega)$ and $u \in \widetilde{U} = \{u \in L^2(0,T,\ I\!R^m)/y_u(T) \in H^1(\Omega)\}$, where $y_u(.)$ is the solution of (2.1) when it is excited by the control $u$.

We denote by $U$ the completion of the set $\widetilde{U}$ with respect to the norm $\| u \|_U = \| u \|_{L^2(0,T;I\!R^m)}$ .

— $\gamma_0 : H^1(\Omega) \longrightarrow L^2(\partial\Omega)$, denotes the trace operator of order zero. It is linear continuous , $\gamma_0^*$ denotes the adjoint operator.

— If $\Gamma \subset \partial\Omega$, we consider $\chi_\Gamma : L^2(\partial\Omega) \longrightarrow L^2(\Gamma)$ the restriction to $\Gamma$ and $\chi_\Gamma^*$ is considered for its adjoint.

— If $\omega \subset \Omega$, we consider also $\chi_\omega : H^1(\Omega) \longrightarrow H^1(\omega)$ the restriction to $\omega$.

— $H : U \longrightarrow H^1(\Omega)$, the controllability operator defined by $Hu = \int_0^T S(T-s)Bu(s)ds$.

We recall some definitions related to regional controllability described by El Jai et al [7]).

Let $\omega \subset \Omega$ be of positive Lebesgue measure and $\mathcal{G}$ be any subset of $H^1(\Omega)$. The system (2.1) is said to be $\omega-$exactly (resp. $\omega$- approximately) controllable to $\mathcal{G}$ if for all $z_d \in \mathcal{G}$ (given $\varepsilon > 0$) there exists a control $u \in U$ such that $\chi_\omega y_u(T) = z_d$ (resp. $\|\chi_\omega y_u(T) - z_d\|_{H^1(\omega)} \leq \varepsilon$).

The problem of regional boundary controllability consists in steering the system (2.1) from an initial state to a desired state only given in the boundary subregion $\Gamma$.

Let $E$ be any subset of $H^{\frac{1}{2}}(\Gamma)$, then we have the following definition.

**Definition 2.1.** 1. The system (2.1) is said to be exactly regionally boundary controllable to $E$ on $\Gamma$ if

$$E \subset \chi_\Gamma \gamma_0 S(T)y_0 + \mathrm{Im}\,(\chi_\Gamma \gamma_0 H).$$

2. The system (2.1) is said to be approximately regionally boundary controllable to $E$ on $\Gamma$ if

$$E \subset \chi_\Gamma \gamma_0 S(T)y_0 + \overline{\mathrm{Im}\,(\chi_\Gamma \gamma_0 H)}.$$

In what follows, we shall say that a system is $\mathcal{B}$-controllable on $\Gamma$ ($\mathcal{B}$ for the boundary).

It is clear that:

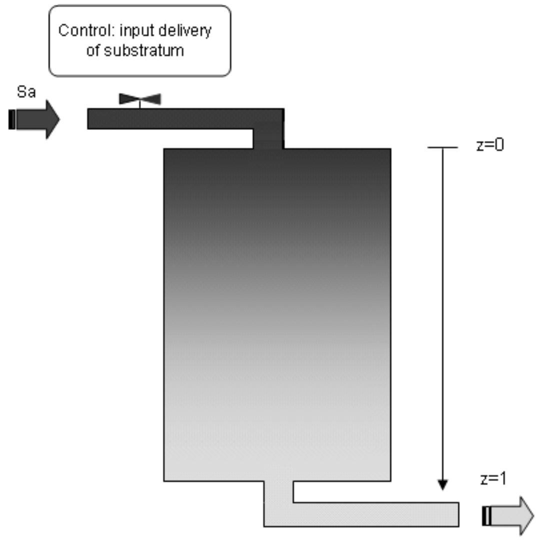


Figure 1: Regulation of the concentration of the substratum at the bottom of the reactor

1. The above definitions mean that we are only interested by the transfer of the system (2.1) to a desired state on the subregion $\Gamma \subset \partial\Omega$ and the control $u$ implicitly depends on $\Gamma$.

2. The above definitions do not allow for pointwise or boundary controls since, for such systems the operator $B$ is not bounded. However the extension can be carried out in similar manner if one takes regular controls such that $y_u(T) \in H^1(\Omega)$ as in El Jai and Pritchard [6].

### 2.2. Boundary Controllability and Actuator

Regional boundary controllability analysis can be done from a purely theoretical viewpoint – see Zerrik et al [7]. But the study may be also become concrete, in some sense, by using the structure of actuators, the actuators form an important link between a system and its environment. They have an active role and are used to excite the system. Their structure depends on the geometry, the location of the support, and the spatial distribution of the action.

The purpose of this section is to give characterization for actuators (number and location) in order that the system to be regionally approximately boundary controllable.

Consider the system (2.1) excited by $p$ zone actuators $(D_i, f_i)_{1\leq i\leq p}$, where

$D_i \subset \Omega$ and $f_i \in L^2(D_i)$ .

$$\begin{cases} \dfrac{\partial y}{\partial t}(x,t) = Ay(x,t) + \displaystyle\sum_{i=1}^{p} \chi_{D_i} f_i u_{i(t)}\,, & Q\,, \\ \dfrac{\partial y}{\partial \nu_A}(\xi,t) = 0\,, & \Sigma\,, \\ y(x,0) = 0\,, & \Omega\,. \end{cases} \tag{2.1}$$

**Definition 2.2.** A sequence of actuators $(D_i, f_i)_{1\le i\le p}$ is said to be $\Gamma$-strategic if the excited system is approximately $\mathcal{B}$-controllable on $\Gamma$.

Assume that there exists a complete set of eigenfunctions $(\varphi_{m_j})_{m\in I, j=1\ldots r_m}$ of $A$ in $H^1(\Omega)$ orthonormal in $L^2(\Omega)$ and associated with the eigenvalues $\lambda_m$ of multiplicities $r_m$ and $r = \sup\limits_{m\in I} r_m$ is finite. For $x = (x_1, ..., x_n) \in \Omega$ and $m = (m_1, ..., m_n) \in I$. Let $\overline{x} = (x_1, ..., x_{n-1})$ and $\overline{m} = (m_1, ..., m_{n-1})$. Suppose that the functions $\psi_{\overline{m}_j}(\overline{x}) = \chi_\Gamma \gamma_0\, \varphi_{m_j}(x)$, $m \in I$, form a complete set in $L^2(\Gamma)$. Then we have the following result.

**Proposition 2.3.** *The sequence of actuators $(D_i, f_i)_{1\le i\le p}$ is $\Gamma$-strategic if and only if:*

*1. $p \ge r$, and* *2.* rank $G_m = r_m$,

*where*

$$(G_m)_{ij} = \begin{cases} <\varphi_{m_j}, f_i>_{L^2(D_i)}\,, & \textit{in the zone case}\ , \\ \varphi_{m_j}(b_i)\,, & \textit{in the pointwise case}\ , \end{cases}$$

*for $1 \le i \le p$, $1 \le j \le r_m$.*

For the proof, we refer to Zerrik et al [13].

**Remark 2.4.** By infinitesimally deforming the domain, the multiplicity of the eigenvalues can be reduced to one in El Jai and El Yaccoubi [8]. Consequently, the $\mathcal{B}$-controllability on $\Gamma$ can be guaranteed by employing only one actuator.

**Example.** Consider a two-dimensional system defined on $\Omega = ]0, a[\times]0, d[$ and excited by a pointwise actuator located in $b = (\alpha, \beta) \in \Omega$

$$\begin{cases} \dfrac{\partial y}{\partial t}(x_1,x_2,t) = \dfrac{\partial^2 y}{\partial x_1^2}(x_1,x_2,t) + \dfrac{\partial^2 y}{\partial x_2^2}(x_1,x_2,t) + \delta((x_1,x_2)-b)u(t)\,, & Q\,, \\ \dfrac{\partial y}{\partial \nu}(\xi_1,\xi_2,t) = 0\,, & \Sigma\,, \\ y(x_1,x_2,0) = 0\,, & \Omega\,. \end{cases} \tag{2.2}$$

The problem here is to analyze the controllability on $\Gamma = ]0, a[\times\{0\}$. The eigenfunctions associated to the system (2.2) are of the form

$$\varphi_{ij}(x_1, x_2) = \frac{2a_{ij}}{\sqrt{ad}} \cos\left(i\pi \frac{x_1}{a}\right) \cos\left(j\pi \frac{x_2}{d}\right) \quad \text{with} \quad a_{ij} = (1 - \lambda_{ij})^{-\frac{1}{2}} .$$

In this case we have

$$I = I\!N^2, \quad \overline{(i,j)} = i, \quad \overline{(x_1, x_2)} = x_1 .$$

The functions $\psi_i(x_1) = \left(\frac{2}{a}\right)^{\frac{1}{2}} \cos\left(i\pi \frac{x_1}{a}\right), \quad i \in I\!N$ form a complete set in $L^2(\Gamma)$.

The actuator is not $\Gamma$-strategic if there exists $k, l \in I\!N^*$ such that $2k\frac{\alpha}{a}$ or $2l\frac{\beta}{d}$ is odd.

Various interesting results concerning the choice of actuators structure in specific situations are given in Zerrik et al [13].

## 3. Regional Boundary Target Control Problem

In this section, we explore the possibility of finding a minimum energy control which ensures the transfer of the system (2.1) to a desired $y_d$ on the boundary subregion $\Gamma$.

Let us suppose that the solution $y_u(.)$ of (2.1) is such that $y_u(T) \in H^1(\Omega)$ and consider the problem

$$\left\{ \begin{array}{lcl} \min\ J(u) & = & \displaystyle\int_0^T ||u(t)||^2_{I\!R^m}\ dt , \\ u \in U_{ad} & = & \{u \in U \ |\ \chi_\Gamma \gamma_0 y_u(T) = y_d\} . \end{array} \right. \tag{3.1}$$

If the system (2.1) is approximately $\mathcal{B}$-controllable on $\Gamma$, using the pseudo-inverse method it was proved in Zerrik et al [14], that the problem (3.1) has a unique solution $u^\star$.

### 3.1. Theoretical Approach

We shall develop an approach that leads to a sequence of controls $(u^\star_p)_p$ which converge to $u^\star$ the solution of (3.1).

Let $p > 0$ integer, $F_p = \bigsqcup_{z\in\Gamma} B(z, \frac{1}{p})$ and $\omega_p = F_p \cap \Omega$, where $B(z, \frac{1}{p})$ is the open ball of radius $\frac{1}{p}$ and center z.

Consider the problem

$$\begin{cases} \min \ J(u) = \displaystyle\int_0^T \ ||u(t)||^2_{I\!R^m} \ dt \,, \\ u \in U^p_{ad} = \{u \in U \ \mid \ \chi_{\omega_p} y_u(T) = \chi_{\omega_p} z_d\} \,, \end{cases} \tag{3.2}$$

where $z_d \in H^1(\Omega)$, is the solution of the equation

$$\begin{cases} \Delta z = 0 \,, & \Omega \,, \\ z = \bar{y}_d \,, & \partial\Omega \,, \end{cases} \tag{3.3}$$

where $\Delta$ is the Laplace operator and $\bar{y}_d$ is the extension of $y_d$ in $H^{\frac{1}{2}}(\partial\Omega)$. The problem of reaching $y_d$ on $\Gamma$ may be solved by reaching $z_d$ on $\omega_p$.

We note $R_{\omega_p} = (\chi_{\omega} H)(\chi_{\omega} H)^\star$, $\tilde{z}_p = \chi_{\omega}(z_d - S(T)y_0)$, then we have the following result.

**Proposition 3.1.** *If the system (2.1) is exactly controllable on $\omega_p$, and $\tilde{z}_p \in \mathrm{Im}\,(R_{\omega_p})$, then the problem (3.2) has a unique solution given by*

$$u^\star_p = (\chi_{\omega_p} H)^* R^{-1}_{\omega_p} \tilde{z}_p,$$

*and the sequence $u^\star_p$ converges strongly to $u^\star$ solution of the problem (3.1).*

*Proof.* 1. The problem (3.2) has only one solution proved in El Jai et al [7].

2. We show that the sequence $(U^p_{ad})_p$ converges to $U_{ad}$ in Kuratowski's sense (see [1, 2]), indeed:

The sequence $(U^p_{ad})_p$ is strictly increasing then it converges to $\displaystyle\bigcup_{p>0} U^p_{ad} \subset U_{ad}$.

Now, if $u \in U_{ad}$ we have $\chi_\Gamma \gamma_0 H u = y_d$ then $\chi_\Gamma \gamma_0 H u = \chi_\Gamma \gamma_0 z_d$, as the sequence $(\omega_p)_p$ converges to $\Gamma$ in Kuratowski's sense then $u$ steers the system to $y_d$ on $\displaystyle\lim_{p \longrightarrow +\infty} \omega_p = \Gamma$ then $u \in \displaystyle\bigcup_{p>0} U^p_{ad}$.

3. Consider the problem

$$\begin{cases} \min \ J(v) = \displaystyle\int_0^T \ ||v(t)||^2_{I\!R^m} \ dt \,, \\ v \in V_{ad} = \{v \in U \ \mid \ y_v(T) = z_d\,\} \,, \end{cases} \tag{3.4}$$

we have $V_{ad} \subset U^p_{ad}$ and $\displaystyle\min_{u \in U^p_{ad}} \|u\|^2 \leq \min_{v \in V_{ad}} \|v\|^2$ then $\|u^\star_p\| \leq \|v^\star\|$, where $\|\ \ \|$ denotes the norm in $L^2(0,T;I\!R^m)$, $u^\star_p$ is the unique solution of (3.2) and $v^\star$ is the unique solution of (3.4). One deduce that the sequence $(u^\star_p)_p$ is bounded in

$U$. As $U$ is a reflexive space, there exists a subsequence $(u^\star_{p_k})_k$ which converges weakly to $u^\star$ in $U$.

Suppose that $(u^\star_p)_p$ has at least two adherent point $u^\star_1$ and $u^\star_2$, then there exists two subsequences $(u^\star_{p_{k_1}})_{k_1}$ and $(u^\star_{p_{k_2}})_{k_2}$ of $(u^\star_p)_p$ such that $u^\star_{p_{k_1}}$ converges weakly to $u^\star_1$ and $u^\star_{p_{k_2}}$ converges weakly to $u^\star_2$.

For all $v \in U_{ad}$, there exists $v_p \in U^p_{ad}$ which converges to $v$ and we have $J(u^\star_p) \leq J(v_p)$ which gives $J(u^\star_{p_{k_1}}) \leq J(v_{p_{k_1}})$ thus $\underline{\lim}\ J(u^\star_{p_{k_1}}) \leq \underline{\lim}\ J(v_{p_{k_1}})$ and $J(u^\star_1) \leq \underline{\lim}\ J(u^\star_{p_{k_1}}) \leq \underline{\lim}\ J(v_{p_{k_1}})$.

Therefore $\quad J(u^\star_1) \leq J(v)$, $\forall v \in U_{ad}$.

Hence $u^\star_1$ is solution of the problem (3.1) and by similar techniques, $u^\star_2$ is also solution of (3.1).

Consequently $u^\star_1 = u^\star_2 = u^\star$.

The sequence $(u^\star_p)_p$ is bounded in a reflexive space and has one adherent point $u^\star$ for the weak topology then $u^\star_p$ converges weakly to $u^\star$.

Let $l = \lim\limits_{p \longrightarrow +\infty} \|u^\star_p\|$, we shall prove that $l = \|u^\star\|$.

For all $v \in U_{ad}$, there exists $v_p \in U^p_{ad}$ which converges to $v$ and we have $\|u^\star\| \leq \|u^\star_p\| \leq \|v_p\|$ then $\|u^\star\| \leq \lim\limits_{p \longrightarrow +\infty} \|u^\star_p\| \leq \lim\limits_{p \longrightarrow +\infty} \|v_p\|$ and so $\|u^\star\| \leq l \leq \|v\|$ for all $v \in U_{ad}$ therefore $\|u^\star\| \leq l \leq \min\limits_{v \in U_{ad}} \|v\|$ which shows that $l = \|u^\star\|$. Hence, $u^\star_p$ converges strongly to $u^\star$ solution of the problem (3.1). □

### 3.2. Numerical Approach

We have seen that $u^\star_p$ approximate $u^\star$ the optimal control solution of the problem (3.1). In this subsection we explore a numerical approach devoted to the calculation of $u^\star_p$ which is an extension of the internal regional controllability in connection with the Hilbert uniqueness method El Jai et al [7] and Lions [9]. The following set will be used in future constructions:

$$\bar{G} = \left\{ g \in H^1(\Omega) \text{ such that } g = 0 \quad \text{on } \Omega \backslash \omega_p \right\} .$$

Let us consider the system (2.1) with a pointwise internal actuator and is given by

$$\begin{cases} \dfrac{\partial y}{\partial t}(x,t) = Ay(x,t) + \delta(x-b)u(t)\,, & Q\,, \\ \dfrac{\partial y}{\partial \nu_A}(\xi,t) = 0 & \Sigma\,, \\ y(x,0) = y_0(x)\,, & \Omega\,, \end{cases} \tag{3.5}$$

where $b$ denotes the actuator support in $\Omega$ and $u(t) \in I\!R$. We assume that the solution of (3.5) is such that $y_u(T) \in H^1(\Omega)$. The construction method is based on the three following steps.

*Step 1.* For $\varphi_0 \in \bar{G}$, consider the system

$$\begin{cases} \dfrac{\partial \varphi_p}{\partial t}\,(x,t) = -A^* \varphi_p(x,t)\,, & Q\,, \\ \dfrac{\partial \varphi_p}{\partial \nu_A}\,(\xi,t) = 0\,, & \Sigma\,, \\ \varphi_p(x,T) = \varphi_0(x)\,, & \Omega\,, \end{cases} \tag{3.6}$$

which has a unique solution $\varphi_p \in L^2(0,T,H^1(\Omega)) \cap C^0(0,T,L^2(\Omega))$ (Lions and Magenes [10]).

The mapping

$$\varphi_0 \in \bar{G} \to \|\varphi_0\|_{\bar{G}} = \left[\int_0^T \varphi_p^2(b,t)dt\right]^{\frac{1}{2}} \tag{3.7}$$

defines a semi-norm on $\bar{G}$. If the system (3.5) is $\omega_p$-approximately controllable the mapping (3.7) defines a norm on $\bar{G}$, Zerrik et al [14]. Then we denote also by $\bar{G}$ the completion of $\bar{G}$.

*Step 2.* Now if we consider the system

$$\begin{cases} \dfrac{\partial \Psi}{\partial t}(x,t) = A\Psi(x,t) + \varphi_p(b,t)\delta(x-b)\,, & Q\,, \\ \dfrac{\partial \Psi}{\partial \nu_A}(\xi,t) = 0\,, & \Sigma\,, \\ \Psi(x,0) = y_0(x)\,, & \Omega\,. \end{cases} \tag{3.8}$$

Then the knowledge of $\varphi_0 \in \bar{G}$ gives $\varphi_p$ from equation (3.6) and thus equation (3.8) gives $\Psi$. The above system (3.8) can be decomposed considering an autonomous system and an homogeneous initial condition one

$$\begin{cases} \dfrac{\partial \Psi_1}{\partial t}\,(x,t) = A\Psi_1(x,t)\,, & Q\,, \\ \dfrac{\partial \Psi_1}{\partial \nu_A}\,(\xi,t) = 0\,, & \Sigma\,, \\ \Psi_1(x,0) = y_0(x)\,, & \Omega\,, \end{cases} \tag{3.9}$$

and

$$\begin{cases} \dfrac{\partial \Psi_2}{\partial t}(x,t) = A\Psi_2(x,t) + \varphi_p(b,t)\delta(x-b)\,, & Q\,, \\ \dfrac{\partial \Psi_2}{\partial \nu_A}(\xi,t) = 0\,, & \Sigma\,, \\ \Psi_2(x,0) = 0\,, & \Omega\,. \end{cases} \tag{3.10}$$

*Step 3.* Let $\wedge$ be the operator

$$\begin{array}{rccc} \wedge : & \bar{G} & \longrightarrow & \bar{G}^*, \\ & \varphi_0 & \longrightarrow & P(\Psi_2(T)), \end{array}$$

where $P = \chi^*_{\omega_p}\chi_{\omega_p}$, $\Psi_2(T)$ is the solution of (3.10) and $\bar{G}^*$ is the dual of $\bar{G}$. Then the regional control problem on $\omega_p$ is transformed by El Jai et al [7] into solving the equation.

$$\wedge\, \varphi_0 = P(\, z_d - \Psi_1(T)) \tag{3.11}$$

and we have the following result.

**Proposition 3.2.** *If the system (3.5) is $\omega_p$-approximately controllable, the equation (3.11) has a unique solution $\varphi_0 \in \bar{G}$ and a control steering the system to the desired state $y_d$ on $\Gamma$ is given by*

$$u_p^\star(t) = \varphi_p(b,t)\,. \tag{3.12}$$

*Proof.* For the proof we refer the reader to Zerrik et al [14]. □

The numerical approach of (3.11) is realized very easily when one can calculate the eigenfunctions of the system. The idea, is to calculate the components $\wedge_{ij}$ of $\wedge$, in a suitable basis $(\varphi_i)$. The problem will be approximated by the solution of the finite linear system

$$\sum_{j=1}^{N} \wedge_{ij}\varphi_{0,j} = z_i, \quad i = 1, N \tag{3.13}$$

where $N$ is the order of approximation and $z_i's$ are the components of $P(z_d - \Psi_1(T))$ in the considered basis.

Assume that $(\varphi_i)$ is the set of eigenfunctions of the operator $A^*$ associated to the eigenvalues $\lambda_i$, then we have

$$< \wedge\varphi_0, \varphi_0 >= \int_0^T (\varphi_p(b,t))^2\, dt\,,$$

and with

$$\varphi_p(b,t) = \sum_{j=1}^{\infty} e^{\lambda_j(T-t)} < \varphi_0, \varphi_j >_{L^2(\omega)} \varphi_j(b)\,,$$

we have

$$< \wedge\varphi_0, \varphi_0 >$$

$$= \sum_{i,j=1}^{\infty} < \varphi_0, \varphi_i >_{L^2(\omega)} < \varphi_0, \varphi_j >_{L^2(\omega)} \varphi_i(b)\varphi_j(b)\frac{(e^{(\lambda_i+\lambda_j)T} - 1)}{(\lambda_i + \lambda_j)} .$$

Finally, the components of $\wedge$ are given by

$$\wedge_{ij} = \varphi_i(b)\varphi_j(b)\frac{e^{(\lambda_i+\lambda_j)T} - 1}{(\lambda_i + \lambda_j)} .$$

In the case of many pointwise actuators $(b_q)_{q=1,p}$ the components of $\wedge$ are given by

$$\wedge_{ij} = \sum_{q=1}^{p} \varphi_i(b_q)\varphi_j(b_q)\frac{(e^{(\lambda_i+\lambda_j)T} - 1)}{(\lambda_i + \lambda_j)} .$$

**Remark.** 1. The obtained control ensures the transfer of the system to $z_d$ on $\omega_p$ with minimum energy (Zerrik et al [14]) and since $\chi_\Gamma \gamma_0 z_d = y_d$, the obtained control also ensures the regional boundary transfer on $\Gamma$, this control is optimal for $z_d$ but non optimal for $y_d$.

2. For the cases of zone and boundary actuators, the same developments lead to similar formula.

3. When it is difficult to find a convenient basis, the problem turns up naturally to find $\varphi_0$ which is the solution of the following problem

$$\left\{ \begin{array}{l} \min \ \|\chi_\Gamma \gamma_0 y_{u^\star}(T) - y_d\|_{L^2(\Gamma)} \, , \\ \varphi_0 \in \bar{G} \ . \end{array} \right. \tag{3.14}$$

The resolution of this problem can easily be achieved by the any minimization algorithm.

## 4. Simulation

In this section we develop numerical example which illustrate the efficiency of the previous approach. The results are related to the choice of the subregion and the desired state to be reached.

Consider the two-dimensional diffusion system described on $\Omega = ]0,1[\times ]0,1[$ by

$$\left\{ \begin{array}{ll} \dfrac{\partial y}{\partial t}(x_1,x_2,t) = 0,01 \left[\dfrac{\partial^2 y}{\partial x_1^2}(x_1,x_2,t) + \dfrac{\partial^2 y}{\partial x_2^2}(x_1,x_2,t)\right] & \\ \qquad\qquad + \delta(x_1 - b_1, x_2 - b_2)u(t) \, , & Q \, , \\ \dfrac{\partial y}{\partial \nu}(\xi_1,\xi_2,t) = 0 \, , & \Sigma \, , \\ y(x_1,x_2,0) = 0 \, , & \Omega \, . \end{array} \right. \tag{4.1}$$

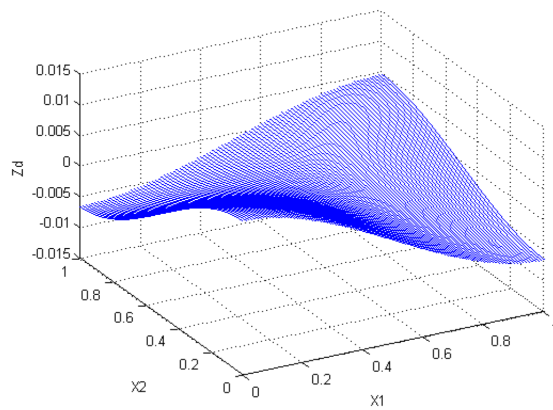


Figure 2: Desired state on $\omega$

For the simulations, we assume that:

— The actuator is located in $(b_1, b_2) = (0.13, 0.19)$.

— The considered boundary subregion is $\Gamma = \{0\} \times [0, 1]$.

— The desired state $y_d(\xi) = 0.1\left(\frac{\xi^3}{3} - \frac{\xi^2}{2} + 0.1\right)$ and is to be reached on $\Gamma$.

— The internal subregion $\omega = ]0, 0.2[\times]0, 1[$ and the final time $T = 2$.

— The internal state $z_d(x_1, x_2) = \left(\frac{x_1^3}{3} - \frac{x_1^2}{2} + 0.1\right)\left(\frac{x_2^3}{3} - \frac{x_2^2}{2} + 0.1\right)$ is an extension on $\omega$ of $y_d$.

Using the previous approach, we obtain the following Figure 2, Figure 3 and Figure 4.

Figure 4 shows how the final reached state is very close to the desired state on the boundary subregion $\Gamma$.

The regional boundary desired state $y_d$ is reached with error

$$\|\chi_\Gamma \gamma_0 y_{u^*}(T) - y_d\|^2_{L^2(\Gamma)} = 3.795 \times 10^{-9}$$

and with transfer cost $||u^\star||^2_{L^2(0,T)} = 36.40$.

## 5. Conclusion

This work provide an interesting tool to achieve regional boundary target for a parabolic system excited by actuator. Also it permits us to avoid “bad” actua-

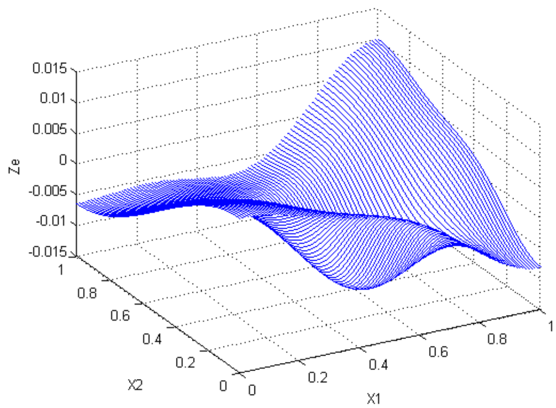


Figure 3: Reached state on $\omega$

tors location. The dual concept of observability which concerns the state reconstruction on a given part of the boundary in connection with sensors structure was also studied by Zerrik and Badraoui [12] and is based on similar techniques. Various questions are still open. This is the case for example, the problem of finding the optimal actuators location considered by D. Ucinski, [11], ensuring such an objective. Also the problem of constrained control as well as the case of systems with more complicated dynamics which are of great interest.

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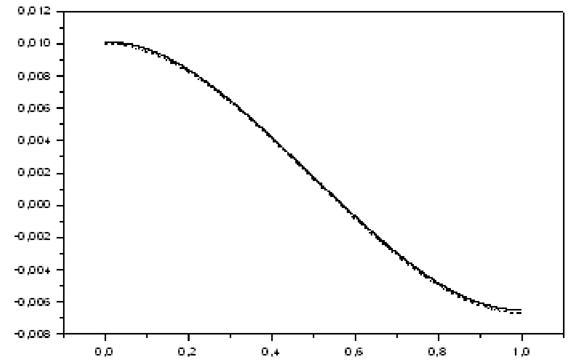


Figure 4: Desired (dashed line) and reached (continuous line) state on $\Gamma$

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