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A NOTE ON YOKOI'S D-INVARIANTS

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Abstract: In this paper, upper bounds for fundamental unit of real quadratic fields are obtained in the cases norm of fundamental unit ∓ 1.

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Throughout this paper, we denote by $\mathbf{N}$, the set of positive rational integers, and $\mathbf{N}_0 = \mathbf{N} \cup \{0\}$. For a square-free $D \in \mathbf{N}$, the real quadratic field is $\mathbf{Q}(\sqrt{D})$ and its fundamental unit is $\varepsilon_D = (t + u\sqrt{D})/2 > 1$. The norm map from $\mathbf{Q}(\sqrt{D})$ to $\mathbf{Q}$ will be denoted by N.

Let p be prime congruent to $1 \mod 4$ and $\varepsilon_p = (t_p + u_p\sqrt{p})/2 > 1$ be the fundamental unit of the real quadratic field $\mathbf{Q}(\sqrt{p})$. In [5], Yokoi showed that $\varepsilon_p < 2p$. Mollin and Williams gave a complete generalized form of Yokoi's p-invariants for arbitrary real quadrqtic field $\mathbf{Q}(\sqrt{D})$ and $\varepsilon_D < 8D/\sigma^2$ such that

$$\sigma = \begin{cases} 2, & d \equiv 1 \pmod{4}, \\ 1, & d \equiv 2, 3 \pmod{4}, \end{cases}$$

for $n_D \neq 0$ (n_D is defined in [3]) is proved.

In this paper by using the invariant values which are defined by Yokoi in [7], the upper bounds for ε_D are obtained. Morever, we prove that when $n \neq 0$ (n is defined in [6, 7]), then the Artin-Ankey-Chowla conjecture holds.

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Theorem 1. *Let $D > 3$. If $n \neq 0$ then:*
i) $\varepsilon_D < 2D$ *for* $N(\varepsilon_D) = 1$,
ii) $\varepsilon_D < D$ *for* $N(\varepsilon_D) = -1$.

We begin by citing three known results as Lemma 1, Lemma 2 and Lemma 3. The letters $N, D, \varepsilon_D, t, u$ will always keep the meaning explained above. For a real number x, $[x]$ means as usual the greatest integer $\leq x$.

For a square-free $D \in \mathbf{N}$, Yokoi defined the following sets in [7]. If $N(\varepsilon_D) = 1$, then

$$\begin{cases} V_D = \{v \mid 0 \leq v < u^2,\ v^2 \equiv 4 \pmod{u^2}\}, \\ (V,W)_D = \{(v,w) \mid v \in V_D,\ v^2 - 4 = wu^2\}, \end{cases}$$

and if $N(\varepsilon_D) = -1$, then

$$\begin{cases} V_D = \{v \mid 0 \leq v < u^2,\ v^2 \equiv -4 \pmod{u^2}\}, \\ (V,W)_D = \{(v,w) \mid v \in V_D,\ v^2 + 4 = wu^2\}. \end{cases}$$

Lemma 1. (see Yokoi [7]) *For any square-free $D \in \mathbf{N}$, there are uniquely determined $n \in \mathbf{N}_0$ and (v,w) in $(V,W)_D$ such that $t = u^2 n + v$ and $D = u^2 n^2 + 2vn + w$. Additionally if $u > 2$, then $0 \leq w < v < u^2$ and $n = [t/u^2]$.*

Lemma 2. (see Yokoi [4]) *Let $\varepsilon_D = (t + u\sqrt{D})/2 > 1$ be the fundamental unit of a real quadratic field $\mathbf{Q}(\sqrt{D})$. Then, in the case $N(\varepsilon_D) = 1$, it holds $t > \varepsilon_D > u\sqrt{D}$ and in the case $N(\varepsilon_D) = -1$, it holds $t < \varepsilon_D < u\sqrt{D}$.*

Lemma 3. (see Karaali and İşcan [2]) *If $N(\varepsilon_D) = 1$, then $\varepsilon_D < 2u\sqrt{D}$.*

Proof of Theorem 1. i) Let $N(\varepsilon_D) = 1$. In the case $u = 1$ or $u = 2$, using $\varepsilon_D < 2u\sqrt{D}$ we have $\varepsilon_D < 2D$. In the case $u > 2$, using the equation $t = u^2 n + v$ we have

$$\frac{t}{u^2} - n = \frac{v}{u^2} < 1$$

and therefore $t/u^2 > 1$ for $n \neq 0$ is obtained. Also $t^2 - Du^2 = 4$ for $N(\varepsilon_D) = 1$. Hence

$$\frac{t^2}{u^2} = D + \frac{4}{u^2}$$

is obtained. Thus

$$2D > t\,\frac{t}{u^2} > D.$$

Using Lemma 2, we obtain $\varepsilon_D < 2D$.

ii) Let $N(\varepsilon_D) = -1$. If $u = 1$ or $u = 2$, using $\varepsilon_D < u\sqrt{D}$ we have $\varepsilon_D < D$. If $u > 2$, then we know that $t/u^2 > 1$ for $n \neq 0$. Since $t^2 - Du^2 = -4$ for $N(\varepsilon_D) = -1$, we have

$$\frac{t^2}{u^2} = D - \frac{4}{u^2}$$

and so

$$D > t\,\frac{t}{u^2} > D - \frac{1}{2}.$$

Therefore $t < D$. On the other hand $t^2 - Du^2 = -4$ implies $t^2 + 4 = Du^2$. Since $t < D$ we obtain $Du^2 < D + 4$ and so $Du^2 < D^2$, which implies $u\sqrt{D} < D$. In this case, $\varepsilon_D < u\sqrt{D} < D$. □

Using Theorem 1, we obtain the following theorem.

Theorem 2. *Let If $n \neq 0$ for a square-free D, then $u \not\equiv 0 \pmod{D}$.*

Proof. Let $N(\varepsilon_D) = 1$. If $u \equiv 0 \pmod{D}$, then $D \mid u$ and $D^2 < u^2 < t$ from Theorem 1. Hence $D^2 < 2D$. Therefore $D < 2$, a contradiction. Therefore $u \not\equiv 0 \pmod{D}$.

Let $N(\varepsilon_D) = -1$. If $u \equiv 0 \pmod{D}$, then $D \mid u$. Hence $u \geq D$. But $u^2 < D$ from Theorem 1, this is a contradiction. Therefore

$$u \not\equiv 0 \,(\mathrm{mod}\ D).$$ □

In particular, the Artin-Ankeny-Chowla conjecture holds if $n \neq 0$, i.e. $u \not\equiv 0 \pmod{p}$ when $p \equiv 1 \pmod{4}$ is prime.

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