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NO-ARBITRAGE IN L^p-PROBABILITY SPACES

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Abstract: We prove the fundamental theorem of asset pricing in finite discrete time and for finitely many stocks using the topology generated by the order convergence in L^p-spaces, $p \geq 1$.

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1. Introduction

In this paper we fix integers $N \geq 0$, $m \geq 1$, and work with indices $n \in \{0, 1, \ldots, N\}$, $k \in \{1, 2, \ldots, m\}$. We assume that the value of a financial asset k at time n is given by a random variable $S_n^{(k)}$, defined on a probability space $(\Omega, \mathcal{F}, P)$ and adapted to a filtration $\{\mathcal{F}_n\}$ (by convention, $\mathcal{F}_0$ is the trivial σ-algebra). A *trading strategy* θ consists of the number of shares $\theta_n^{(k)}$ of asset k you hold after time n (and strictly before time $n+1$), for all n and k. We assume that $\theta_n^{(k)} \in L^p(\Omega, \mathcal{F}_n, P; \mathbb{R})$ for some finite $p \geq 1$, and $S_n^{(k)} \in L^\infty(\Omega, \mathcal{F}_n, P; \mathbb{R})$ for all n and k. Since the time horizon ends at N, we have $\theta_N^{(k)} = 0$ for all k. The *payment stream* generated by θ is the vector $\delta^\theta = (\delta_0^\theta, \delta_1^\theta, \ldots, \delta_N^\theta) \in L^p(\Omega, \mathcal{F}_N, P; \mathbb{R}^{N+1})$, given by

$$\delta_n^\theta = < \theta_{n-1}^{(\cdot)} - \theta_n^{(\cdot)}, S_n^{(\cdot)} >_{\mathbb{R}^m}$$

for all n, where $\theta_{-1}^{(\cdot)} = 0$ and $< \cdot, \cdot >_{\mathbb{R}^m}$ is the scalar product in $\mathbb{R}^m$.

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The *no arbitrage condition*, or the absence of arbitrage opportunities, is becoming increasingly important since the Black, Scholes and Merton's seminal papers in the 1970's; with the above notations and definitions, it reads

$$\mathbb{M} \cap L^p_+(\Omega, \mathcal{F}_N, P; \mathbb{R}^{N+1}) = \{\mathbf{0}\}, \tag{NA}$$

where $\mathbb{M}$ is the set of all payment streams δ^θ generated by trading strategies, the subscript "+" refers to positive elements in $L^p(\Omega, \mathcal{F}_N, P; \mathbb{R}^{N+1})$, and $\mathbf{0}$ is the null payment stream. Alternatively, we may say that the trading strategy θ provides an arbitrage opportunity if $\delta^\theta \geq 0$ and $\delta^\theta \neq 0$ with positive probability.

The *fundamental theorem of asset pricing*, in finite discrete time and for finitely many stocks, gives functional and martingale characterizations of the above no arbitrage condition. Namely, we have the following theorem.

Theorem. *The no-arbitrage condition (NA) is equivalent to the existence of a strictly positive continuous linear functional Q on*

$$L^p(\Omega, \mathcal{F}_N, P; \mathbb{R}^{N+1}),$$

which equals zero for all elements of $\mathbb{M}$. When Q is restricted to the componentwise $\{\mathcal{F}_n\}$-adapted vectors in $L^p(\Omega, \mathcal{F}_N, P; \mathbb{R}^{N+1})$, condition (NA) is equivalent to the existence of a unique strictly positive vector $\phi = (\phi_0, \phi_1, \ldots, \phi_N) \in L^q(\Omega, \mathcal{F}_N, P; \mathbb{R}^{N+1})$ such that the weighted value process $\{\phi_n S_n^{(k)}; n = 0, 1, \ldots, N\}$ of asset k is a $\{\mathcal{F}_n\}$-martingale, where $1/p + 1/q = 1$, with the usual convention: $q = +\infty$ if $p = 1$.

The proof of the above theorem in the case $p = 1$, based on deep functional machinery, was obtained in [2] and, in the case $p = 2$, based on Hilbert space techniques, was obtained in [4]. Nevertheless, we point out that any restriction on strategies, such as $p = 1$ or 2, automatically becomes a restriction on the integrability properties of the martingale $\{\phi_n S_n^{(k)}\}$. From a practitioner's point of view, the latter means less specification of the model in its cumulants (e.g. kurtosis, skewness). From a theoretical point of view, the above restrictions diminish the natural domain of the Markov resolvent that generates the "deflators" $(\phi_0, \phi_1, \ldots, \phi_N)$, hence the spot rate process may be undefined (see [3]).

It is our aim to present a unitary proof of the fundamental theorem of asset pricing *for any finite value* $p \geq 1$. The idea is to consider a strictly stronger topology than the norm topology on L^p-spaces, generated by their intrinsic latticial structure. More precisely, in the sequel X denotes one of the spaces $L^p(\Omega, \mathcal{F}_N, P; \mathbb{R}^{N+1})$ or $L^p(\Omega, \mathcal{F}_n, P; \mathbb{R}^m)$. As X is an Archimedean complete vector lattice with respect to the usual operations, we can consider the order convergence on X, in short (o)-*convergence*, defined as follows: $f_i \to f$ as

$i \to +\infty$ iff there is a sequence g_i decreasing towards 0 in X, and such that $|f_i - f| \leq g_i$ a.e. for all i. Equivalently, (o)-convergence means a.e. convergence together with the following: there exists $g \in X$ such that $|f_i(\omega)| \leq g(\omega)$ a.e. for all i (if p is finite), or $|f_i(\omega)| \leq c$ a.e. for all i, where c is a constant (if $p = +\infty$). In addition, (o)-convergence implies norm convergence (by Lebesgue's Dominated Convergence Theorem), the converse is not true (consider for instance non-uniformly integrable sequences), and norm continuous linear functionals coincide with (o)-continuous linear functionals on X (by monotonic convergence and Riesz-Fischer Completeness Theorems). For proofs of these statements and other details on (o)-convergence, one may consult [5].

Our proof of the fundamental theorem of asset pricing for any $p \geq 1$ consists of three steps, as follows:

Step 1. Is to prove the equivalence of condition (NA) and the existence of a strictly positive continuous linear functional Q on $L^p(\Omega, \mathcal{F}_N, P; \mathbb{R}^{N+1})$ that equals 0 on $\mathbb{M}$.

Indeed, an inspection of Step 2 of the proof in Theorem 1 in [1] shows that the linear functional Q is (o)-continuous iff Yan-Schaefer's separating linear functionals Q_ε therein are all (o)-continuous (see [4], Theorem 3.1) and Schachermayer's Lemma (Step 1 in the proof of Theorem 1 in [1]) holds true for (o)-limits. The first part is obvious, and for the second we modify the a.e.-sequential proof given in [4] with (o)-convergent sequences, for the fact that, if (NA) holds, then $L^p_+(\Omega, \mathcal{F}_N, P; \mathbb{R}^{N+1})$ and the closure in the (o)-convergence topology of the set $\mathbb{M} - L^p_+(\Omega, \mathcal{F}_N, P; \mathbb{R}^{N+1})$ have in common the null payment stream only.

Step 2. Is a reduction argument, that is, instead of Q we can consider $(N+1)$ functionals Q_n on $L^p(\Omega, \mathcal{F}_n, P; \mathbb{R})$. More precisely, for $f \in L^p(\Omega, \mathcal{F}_N, P; \mathbb{R}^{N+1})$, define $Q_n(f) := Q(I_n(f))$, where $I_n(f) = (0, \ldots, 0, f, \ 0, \ldots, 0)$ with f on the n^{th}-position. Remark that the general form of (o)-continuous linear functionals on $L^p(\Omega, \mathcal{F}_N, P; \mathbb{R}^{N+1})$ is $Q(f) = \sum_{n=0}^N Q_n(f_n)$, where $f = (f_0, \ldots, f_N)$, see [5]. More, Q is strictly positive (resp. (o)-continuous) iff *all* Q_n are strictly positive (resp. (o)-continuous), because the operators I_n are strictly positive and (o)-continuous. As such, by Riesz' representation theorem, each Q_n can be represented as $Q_n(f) := \int f\phi_n \ dP$ for a strictly positive $\phi_n \in L^q(\Omega, \mathcal{F}_N, P; \mathbb{R})$.

Step 3. Is to prove the equivalence of $Q(\delta^\theta) = 0$ and the martingale property of $\{\phi_n S_n^{(k)}; n = 0, 1, \ldots, N\}$.

For one implication fix $F_{n-1} \in \mathcal{F}_{n-1}$ and consider $\theta_{n-1}^{(k)} := \mathbf{1}_{F_{n-1}}$, the indicator function of F_{n-1}, for any k; all other $\theta_\cdot^{(\cdot)}$ are 0. We have that

$$\delta_{n-1}^\theta = -\mathbf{1}_{F_{n-1}} S_{n-1}^{(k)} \text{ and } \delta_n^\theta = \mathbf{1}_{F_n} S_n^{(k)}$$

are the only non-zero components of δ^θ. Hence $Q_{n-1}(\delta^\theta_{n-1}) + Q_n(\delta^\theta_n) = 0$, that is,

$$\int_{F_{n-1}} \phi_{n-1} S^{(k)}_{n-1} \; dP = \int_{F_{n-1}} \phi_n S^{(k)}_n \; dP,$$

or $\phi_{n-1} S^{(k)}_{n-1} = E[\phi_n S^{(k)}_n | \mathcal{F}_n]$ for $n = k, k+1, \ldots, N$, and the latter is the required martingale property.

Conversely, integrating by parts, we obtain:

$$Q(\delta^\theta) = \sum_{n=0}^{N} Q_n(\delta^\theta_n) = \int \sum_{n=0}^{N} \phi_n < \theta^{(\cdot)}_{n-1} - \theta^{(\cdot)}_n, S^{(\cdot)}_n >_{\mathbb{R}^m} \; dP$$

$$= \int \sum_{j=1}^{N} < \theta^{(\cdot)}_{j-1}, \phi_j S^{(\cdot)}_j - \phi_{j-1} S^{(\cdot)}_{j-1} >_{\mathbb{R}^m} \; dP = 0,$$

and the proof is now complete.

Remark. The above proof works for other spaces with differential structure, isomorphic to some L^p's (as ordered vector spaces), such as the spaces of: absolutely continuous, continuous on a compact, Lebesgue, Orlicz, test, or of bounded p-variation functions. On the other hand, for a proof of the fundamental theorem in the space of Lebesgue measurable functions, (o)-convergence is not appropriate, for it is equivalent to a.e. convergence *only*, and hence leading to trivial dual; the same happens with the spaces $L^p(\Omega, \mathcal{F}, P)$, for $0 \leq p < 1$, on some infinite set Ω and atomless P.

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