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ON THE LOGARITHMIC DERIVATIVE OF THE ζ-DETERMINANT

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Abstract: A formula for the derivative of the logarithm of the ζ-determinant of the quotient of two elliptic pseudodifferential operators, acting between the fibers of a vector bundle over a n-dimensional closed manifold M, is presented in this article. Although this formula is not unknown, the hypotheses are relaxed in the version given here.

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1. Introduction

Given a trace class operator A acting on a Hilbert space, the Fredholm determinant of the operator $L = I - A$ is defined by

$$\det_1 L = \prod_{j=1}^{\infty}(1 - \lambda_j(A)), \tag{1.1}$$

where I is the identity operator and the numbers $\lambda_j(A)$ are the eigenvalues of A, repeated the times indicated by their corresponding multiplicities.

It is a very known fact the necessity of this concept in various areas of mathematics, as differential geometry (see [5]), and those of physics, for instance in

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the construction of quantum theories by means of functional integration (see [16], [6], [7], [2], etc.), in where the calculus of determinants of quotients of some elliptic differential operators recovers a special interest.

R. Forman studied some Fredholm determinant properties of L and the quotient of regularization of the determinants of two differential operators D_0 and D_1 by the Riemann ζ-function method, when $L = D_0 D_1^{-1} = I - A$ and A belongs to the trace class operators (see [5]). This type of determinant regularization procedure is called the ζ-determinant regularization and is denoted by Det_ζ.

In general, for an operator L acting on a Hilbert space H the notions of Fredholm determinant and the ζ-determinant have no sense. On the other hand, in several occasions, the interest is focalized on the quotient of the determinants of the operators instead of each determinant individually. On this line the works [6] and [7] fit in perfectly. It is shown in such papers that the quotient between the ζ-determinants of two elliptic operators $A+\epsilon A_1$ and A, defined on a compact differential manifold without boundary, is given by

$$\frac{\mathrm{Det}_\zeta(A+\epsilon A_1)}{\mathrm{Det}_\zeta(A)} = \exp\left\{\epsilon \left.\frac{d}{ds}\right|_{s=0} \left[s.\mathrm{Tr}\big(A^{-s-1}\, A_1\big)\right] + O(\epsilon^2)\right\}, \tag{1.2}$$

where A is pseudodifferential of positive order and A_1 is a differential operator with $\mathrm{order}(A_1) < \mathrm{order}(A)$.

Another version about the derivative of the logarithm of the ζ-determinant with respect to a parameter is presented in [5], where it is established that

$$\begin{aligned} \frac{d}{dt}\log \mathrm{Det}_\zeta L_{t\,B} &= \frac{d}{ds}\ \mathrm{Tr}\left[s.\left(\frac{d}{dt}L_{t\,B}\right).L_{t\,B}^{-s-1}\right]\Bigg|_{s=0} \\ &= \frac{d}{dt}\log \det{}_1\left(L_{t\,B}.L_{0\,B}^{-1}\right), \end{aligned} \tag{1.3}$$

for a quotient of elliptic differential operators belonging to a monoparametric family L_t, all with identical principal symbol (and, hence, with the same order) and the same elliptic boundary condition B for each member L_t of the family. For the veracity of this formula R. Forman requires the restrictive hypothesis that $\left(\frac{d}{dt}L_{t\,B}\right)L_{t\,B}^{-1}$ is a trace class operator for all t. It will be shown that this restriction can be removed. So, one aim of this paper is to extend this formula to the quotient of two classical elliptic pseudodifferential operators defined over a closed manifold.

The paper has three sections. Next part of the present section is devoted to expose the principal results. Same basic concepts, notation and definitions as the ζ-determinant regularization method and some differential properties of the Fredholm determinant are recalled in Section 2. In the last section the extended results to pseudodifferential operators are proved.

1.1. Main Results

Now we are in condition to present our principal statements. The main theorem refers to the logarithm of the derivative of the ζ-determinant of the quotient of two (classical) elliptic pseudodifferential operators acting between the fibers of a compact manifold without boundary.

Theorem 1.1.1. *Let Ω be an open subset of the complex plane and let $z(t) : [0,1] \longrightarrow \Omega$ be a differentiable curve. Over a compact, n-dimensional, differential manifold M without boundary define the z-analytic family $\{L_z\}_{z\in\Omega}$ of elliptic, invertible, pseudodifferential operators, having order $m > 0$. For simplicity, let it be denoted $L_t = L_z(t)$.*

It is supposed that all the operators of the family have the same principal symbol, which has a cone of minimum growth rays, that is, a cone of rays on $\mathbb{C}$ in which the principal symbol does not have any eigenvalue.

Then, for all $t \in [0,1]$ it is satisfied

$$\frac{d}{dt} \ln \mathrm{Det}_\zeta L_t = \left.\frac{d}{ds}\right|_{s=0} \mathrm{Tr}\left[s.\left(\frac{d}{dt}L_t\right).L_t^{-s-1}\right], \tag{1.4}$$

being the r.h.s. of this equality the "finite part" at $s = 0$ of the analytic extension of $\mathrm{Tr}\left[\left(\frac{d}{dt}L_t\right).L_t^{-s-1}\right]$.

Next, the corresponding integrated version will be enunciated. In order to deduce the first corollary it is enough to take the exponential function after integrating from 0 to t_o in equation (1.4) of the previous theorem.

Corollary 1.1.2. (Integrated Version) *Under the hypotheses of Theorem 1.1.1, it is true that*

$$\frac{\mathrm{Det}_\zeta L_{t_o}}{\mathrm{Det}_\zeta L_0} = \exp\left\{\int_0^{t_o} \left.\frac{d}{ds}\right|_{s=0} \left\{s.\mathrm{Tr}\left[\frac{d}{dt}(L_t).L_t^{-s-1}\right]\right\} dt\right\}.$$

Corollary 1.1.3. (Logarithmic Derivative Trace Class Case) *Under the hypotheses of Theorem 1.1.1, if besides $\left(\frac{d}{dt}L_t\right).L_t^{-1}$ is a trace class operator for all t, it is valid that*

$$\frac{d}{dt} \log \mathrm{Det}_\zeta L_t = \frac{d}{ds}\, \mathrm{Tr}\left[s.\left(\frac{d}{dt}L_t\right).L_t^{-s-1}\right]\Bigg|_{s=0}$$

$$= \frac{d}{dt} \log \det_1 \left(L_t . L_0^{-1} \right), \tag{1.5}$$

and also

$$\frac{\mathrm{Det}_\zeta L_{t_o}}{\mathrm{Det}_\zeta L_0} = \det_1 \left(L_{t_o} . L_0^{-1} \right).$$

Remark 1.1.4. It should be noted the proximity of the previous corollary to the one of the results established in [5] for the case of differential operators defined over a manifold with boundary.

2. Fredholm Determinant and ζ-Determinant Regularization Method

2.1. Basic Concepts, Technical Explanations and Notation

As usual, $\mathbb{N}$ will denote the set of the positive integers, $\mathbb{R}$ the set of real numbers and $\mathbb{C}$ the set of complex numbers. If $\omega \in \mathbb{C}$, its real and complex parts are denoted by $\mathrm{Re}(\omega)$ and $\mathrm{Im}(\omega)$, respectively. The greek letters $\alpha, \beta, ...$ are used for multi-indexes of numbers in $\mathbb{N}$; in this way

$$\begin{aligned}
&\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \quad , \quad \beta = (\beta_1, \beta_2, \ldots, \beta_n) \,, \\
&\alpha + \beta = (\alpha_1 + \beta_1, \alpha_2 + \beta_2, \ldots \alpha_n + \beta_n) \,, \\
&\alpha ! = \alpha_1 ! \alpha_2 ! \ldots \alpha_n ! \quad , \quad |\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n .
\end{aligned}$$

The letters x, y, ξ denote points in the Euclidean space $\mathbb{R}^n$. Then,

$$\begin{aligned}
&x = (x_1, x_2, \ldots, x_n) \quad , \quad y = (y_1, y_2, \ldots, y_n) \,, \\
&< x, y >= x_1 y_1 + x_2 y_2 + \cdots + x_n y_n \,, \\
&x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n} \,, \\
&\partial_x^\alpha = (\frac{\partial}{\partial x})^\alpha = (\frac{\partial}{\partial x_1})^{\alpha_1} (\frac{\partial}{\partial x_2})^{\alpha_2} \ldots (\frac{\partial}{\partial x_n})^{\alpha_n} .
\end{aligned}$$

Let M be a differential manifold equipped by a measure μ. The space of all the complex valued functions defined over M having derivatives of every order will be denoted by

$$\mathcal{C}^\infty(M) = \{ f : M \longrightarrow \mathbb{C} \, / \, f \text{ is infinitely differentiable} \}.$$

In general, H will be understood a Hilbert space and the set of all the linear and continuous operators $T : H \longrightarrow H$ will be denoted $\mathcal{L}(H)$. In particular, the

Hilbert space of the square integrable functions $f : M \longrightarrow \mathbb{C}$ will be denoted by $H = L^2(M)$.

The letters L, L_1, L_t, etc., indicate differential or pseudodifferential operators, and A, B, etc. boundary conditions. For the vector bundles over M it will be written (E, M, π_E) and (F, M, π_F).

A classical pseudodifferential operator L of order m defined from the $\mathcal{C}^\infty$ sections of the vector bundle (E, M, π_E) to the $\mathcal{C}^\infty$ sections of another vector bundle (F, M, π_F) is a linear operator that, for each local patch $(\mathcal{O}, \varphi)$ of M and for each local section f over $\mathcal{O}$, can be expressed

$$L\,f(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-i<\varphi(x),\xi>} \sigma(L)(\varphi(x), \xi)\, \widehat{f \circ \varphi^{-1}}(\xi)\, d\xi,$$

where $\widehat{g}(\xi)$ indicates the Fourier transform of the function g, and $\sigma(L)(y, \xi)$ is the so called (full) symbol de L and is a $\mathcal{C}^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ function satisfying

$$\left|\partial_y^\alpha \partial_\xi^\beta \sigma(L)(y, \xi)\right| \leq C_{\alpha\,\beta}(1 + |\xi|)^{m-|\beta|},$$

for all pair of multi-indexes α, β and some constant C only depending on them.

In the case in which the (full) symbol $\sigma(L)(x, \xi)$ of L admits an asymptotic expansion

$$\sum_{j \geq 0} a_{m-j}(x, \xi),$$

being $a_{m-j}(x, \xi)$ the $\mathcal{C}^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ functions which are homogeneous in $|\xi| \geq 1$ of degree $m - j$, we say that the operator L belongs to the class $I_h^m(M)$. The principal symbol of L, denoted by $\sigma_0(L)$, is the function $a_m(x, \xi)$ of the last asymptotic expansion of the symbol.

The composition of two operators L_1 and L_2 belonging to $I_h^{m_1}(M)$ and $I_h^{m_2}(M)$, respectively, is another classical pseudodifferential operator in the class $I_h^{m_1+m_2}(M)$. Its (full) symbol is given by the expression (cf. [3], [9]).

$$\sigma(L_1\, L_2) = \sigma(L_1) \circ \sigma(L_2) \sim \sum_{j=0}^{\infty} \sum_{|\alpha|=j} \frac{i^{|\alpha|}}{\alpha!} \left(\partial_\xi^\alpha p\right)\left(\partial_x^\alpha q\right),$$

with $p = \sigma(L_1)$ and $q = \sigma(L_2)$. In particular, for the principal symbol we have the simple relationship $\sigma_0(L_1\, L_2) = \sigma_0(L_1)\sigma_0(L_2)$.

A $k \times k$ matrix of pseudodifferential operators $L \in I_h^m(M)$ is called (uniformly) elliptic if its principal symbol $\sigma_0(L)$ satisfies

$$|\det\sigma_0(L)(x, \xi)| \geq C|\xi|^{mk}, \quad \text{for} \;\; |\xi| > N \;\; \text{and} \;\; C > 0.$$

2.2. Trace Class Operators and Fredholm Determinant

A compact operator A defined on a Hilbert space H is called to be a trace class operator if

$$\mathrm{Tr}(|A|) = \sum_{j=1}^{\infty} \mu_j(A) < \infty, \tag{2.1}$$

where $\mu_j(A)$, the singular values of A, are the eigenvalues of $|A| = \sqrt{A^*A}$. The set of the trace class operators on H form an ideal denoted $\mathcal{J}_1$. If I denotes the identity operator on H, the Fredholm determinant of $L = I - A$ was defined by (1.1) as

$$det_1\, L = \prod_{j=1}^{\infty} (1 - \lambda_j),$$

where $\{\lambda_j(A)\}_j$ denotes the proper values of A when A is a trace class operator. Of course, its trace is given by

$$\mathrm{Tr}(A) = \sum_{j=1}^{\infty} \lambda_j(A) < \infty.$$

The expression (2.1) defines a norm on $\mathcal{J}_1$, called the trace norm and denoted $\|A(z)\|_1 = \mathrm{Tr}(|A|)$.

Also, the integral representation of the Fredholm determinant given in [8] will be used

$$det_1(I - A) = \exp\left\{ - \int_{\gamma} \mathrm{Tr}\ \left[A\,(1 - zA)^{-1} \right]\ dz \right\}, \tag{2.2}$$

with $\gamma : [0,1] \longrightarrow \mathbb{C}$ a continuous path such that $\gamma(0) = 0$, $\gamma(1) = 1$ and that the operator $(1 - zA)^{-1}$ exists and is bounded for all z in γ.

2.2.1. Differentiability Properties of the Fredholm Determinant

In this paragraph some properties connected with the differentiability of the Fredholm determinants are recalled. The corresponding proofs are reproduced in the appendix and can be found in [1].

Lemma 2.2.1. *Let $A(z) : G \longrightarrow \mathcal{J}_1$ a holomorphic application over an open subset G of $\mathbb{C}$ taking values on the ideal $\mathcal{J}_1$ of the trace class operators equipped with the norm of $\mathcal{L}(H)$. Let us suppose that the trace norm $\|A(z)\|_1$ of $A(z)$ is bounded over each compact subset of G.*

Then, the funtion $det_1(I - A(z)) :\ G \longrightarrow \mathbb{C}$ is holomorphic.

Lemma 2.2.2. *Under the hypotheses of Lemma 2.2.1 we have:*
— *the derivative of $A(z)$ is a trace class operator for all $z \in G$;*
— *the funtion* $\mathrm{Tr}(A(z))$ *is holomorphic on G, and*
— $\frac{d}{dz}\left[\mathrm{Tr}(A(z))\right] = \mathrm{Tr}\left[\frac{d}{dz}A(z)\right]$.

Remark 2.2.3. Since $\mathcal{J}_1$ is not a closed subspace of $\mathcal{L}(H)$ in the norm of the bounded operators, the first statement is not evident at all.

Lemma 2.2.4. *Under the hypotheses of Lemma 2.2.1 it results*

$$\frac{d}{dz}\ln(\det{}_1(I - A(z)) = -\mathrm{Tr}\left[(I - A(z))^{-1}\frac{d}{dz}(A(z))\right].$$

Remark 2.2.5. Let us notice the very close connection between this last lemma and the formula (2.2) given in [8].

2.3. ζ-Determinant

Let L be an endomorphism on a vectorial space of finite dimension. If $\lambda_1, \lambda_2, \ldots, \lambda_k$ are the eigenvalues of L repeated the times indicated by their multiplicities, the determinant of L is defined by $\det_1 L \det L = \prod_{j=1}^{k}\lambda_j$.

So much,

$$\ln\det L = \sum_{j=1}^{k}\ln\lambda_j = \frac{d}{ds}\left[-\sum_{j=1}^{k}\lambda_j^{-s}\right]\Bigg|_{s=0} = -\frac{d}{ds}\left[\mathrm{Tr}\big(L^{-s}\big)\right]\big|_{s=0},$$

for a suitable determination of the logarithm. From here it results

$$\det L = \exp\left\{-\frac{d}{ds}\Big|_{s=0}\left[\mathrm{Tr}\big(L^{-s}\big)\right]\right\}. \tag{2.3}$$

Let us treat the case of a classical elliptic pseudodifferential operator L of order $m > 0$ defined over the Hilbert space $L^2(M)$, if M is a compact manifold without boundary. Since L is an unbounded operator, it is clear that the product of the eigenvalues is divergent. In order to establish for this case a similar expression to (2.3) that allows to obtain a finite quantity as a function of these eigenvalues, it is necessary to define the generalized Riemann ζ-function, associated to the operator L. To that end it is imperative to recall the precise notion of complex powers of L. Given a complex number s, one of the ways to define the operator L^{-s} is ([12], [13] and [15])

$$\left.\begin{aligned} L^{-s} &= \frac{i}{2\pi}\int_\Gamma \lambda^{-s}(L-\lambda I)^{-1}\,d\lambda, \quad \text{if} \quad \mathrm{Re}(s) > 0\,,\\ L^{-s} &= L^k . L^{-(k+s)}, \quad \text{if} \quad -k < \mathrm{Re}(s) \le -(k-1) \le 0, \end{aligned}\right\} \tag{2.4}$$

with $k \geq 1$ an integer number and $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ the path on the complex plane, where for some angle θ each path is defined by

$$\begin{aligned}
&\Gamma_1 = \{te^{i\theta}\}, \quad \text{varying } \ t \text{ from } \infty \text{ to } \epsilon > 0, \\
&\Gamma_2 = \{|\lambda| = \epsilon\} \ \text{ clockwise oriented, and} \\
&\Gamma_3 = \{te^{i\theta}\}, \quad \text{varying } \ t \text{ from } \epsilon \text{ to } \infty,
\end{aligned} \tag{2.5}$$

assuming that there exists a cone of directions around the ray $\arg \lambda = \theta$ in such a way that no eigenvalue of L belongs to the cone. In [12], [13] and [14] it was proved that the function $\mathrm{Tr}\big(L^{-s}\big)$ is holomorphic in a half-plane and that admits a meromorphic extension to the whole complex s-plane, being analytic at $s = 0$. Then, the generalized Riemann ζ-function, associated to L is defined by

$$\zeta(L, s) = \mathrm{Tr}\big(L^{-s}\big).$$

Note its similitude with the numerical Riemann ζ-function. In this way formula (2.3) gives the definition of the regularized determinant of the operator L by means of the generalized Riemann ζ-function and that, in what follows, it will be denoted $\mathrm{Det}_\zeta L$. Therefore,

$$\mathrm{Det}_\zeta L = \exp\left\{ -\frac{d}{ds}\bigg|_{s=0} \zeta(L, s) \right\}. \tag{2.6}$$

3. Proofs

Proof of Theorem 1.1.1 Under the hypotheses the complex powers of L_t are given by ([12], [13] and [15])

$$\begin{aligned}
L_t^{-s} &= \frac{i}{2\pi} \int_\Gamma \lambda^{-s} (L_t - \lambda)^{-1}\, d\lambda, \qquad \text{if } \ \mathrm{Re}(s) > 0\,, \\
L_t^{-s} &= L_t^k . L_t^{-(k+s)}, \qquad \text{if } \ -k < \mathrm{Re}(s) \leq -(k-1) \leq 0,
\end{aligned}$$

where $k \geq 1$ is an integer and Γ is the curve described in (2.4).

Let $k > \frac{n}{m}$ be an integer and $s \in \mathbb{C}$ such that $\mathrm{Re}(s) \geq k$. According to [12], [13], [14], [15] and [17], L_t^{-s} is a trace class operator and its kernel is continuous on the diagonal of M. Since the complex powers depend analytically on the parameter s ([10], [12]), from Lemma 2.2.2 it follows for $\mathrm{Re}(s) > k$ that

$$\frac{d}{dt} \mathrm{Tr}\big(L_t^{-s}\big) = \frac{d}{dt} \mathrm{Tr}\big[L_t^{k-s}\, L_t^{-k}\big]$$

$$
\begin{aligned}
&= \frac{d}{dt}\mathrm{Tr}\left[\frac{i}{2\pi}\int_\Gamma \lambda^{k-s}(L_t-\lambda)^{-1}\,L_t^{-k}\,d\lambda\right] \\
&= \mathrm{Tr}\left\{\frac{i}{2\pi}\int_\Gamma \lambda^{k-s}\left[-(L_t-\lambda)^{-1}\,\frac{d}{dt}(L_t)\,(L_t-\lambda)^{-1}\,L_t^{-k}\right.\right. \\
&\quad \left.\left.+\,(L_t-\lambda)^{-1}\,\left(\sum_{j=1}^{k} L_t^{-j+1}\frac{d}{dt}(L_t^{-1})L_t^{-k+j}\right)\right]d\lambda\right\} \\
&= -\frac{i}{2\pi}\int_\Gamma \lambda^{k-s}\mathrm{Tr}\left[(L_t-\lambda)^{-1}\,\frac{d}{dt}(L_t)\,(L_t-\lambda)^{-1}\,L_t^{-k}\,d\lambda\right] \\
&\quad +\frac{i}{2\pi}\sum_{j=1}^{k}\int_\Gamma \lambda^{k-s}\mathrm{Tr}\left[-(L_t-\lambda)^{-1}\,L_t^{-j+1}\,L_t^{-1}\frac{d}{dt}(L_t)\,L_t^{-1}L_t^{-k+j}\right]d\lambda.
\end{aligned}
$$

By the cyclic property of the trace it can be written as

$$
\begin{aligned}
\frac{d}{dt}\mathrm{Tr}\big(L_t^{-s}\big) &= -\frac{i}{2\pi}\int_\Gamma \lambda^{k-s}\mathrm{Tr}\left((L_t-\lambda)^{-2}\frac{d}{dt}(L_t)\,L_t^{-k}\right)\,d\lambda \\
&\qquad -\frac{i}{2\pi}\sum_{j=1}^{k}\int_\Gamma \lambda^{k-s}\mathrm{Tr}\left[(L_t-\lambda)^{-1}\,\frac{d}{dt}(L_t)\,L_t^{-k-1}\right]d\lambda \\
&= \mathrm{Tr}\left[-\frac{i}{2\pi}\int_\Gamma \lambda^{k-s}(L_t-\lambda)^{-2}\,d\lambda\frac{d}{dt}(L_t)\,L_t^{-k}\right]-k\mathrm{Tr}\left[L_t^{k-s}\,\frac{d}{dt}(L_t)\,L_t^{-k-1}\right].
\end{aligned}
$$

Integrating by parts and taking into account that $\mathrm{Re}(s)>k$, we have

$$
\begin{aligned}
\frac{d}{dt}\mathrm{Tr}\big(L_t^{-s}\big) &= \mathrm{Tr}\left[(k-s)\frac{d}{dt}(L_t)L_t^{-s-1}-k\frac{d}{dt}(L_t)L_t^{-s-1}\right] \\
&= \mathrm{Tr}\left[-s\frac{d}{dt}(L_t)L_t^{-s-1}\right]=(-s).\mathrm{Tr}\left[\frac{d}{dt}(L_t).L_t^{-s-1}\right]. \qquad (3.1)
\end{aligned}
$$

As a function of s the r.h.s. of (3.1) has a meromorphic extension to the whole complex plane ([12], [13], [14], [15] and [17]) with only simple poles possibly localized at $s=n-\frac{j}{m}$, for $j=1,2,\dots$. In particular, at $s=0$ such extension is analytical.

Eventually, in virtue of definition of ζ-determinant given by formula (2.6) and expression (3.1), it is clear that

$$
\frac{d}{dt}\ln\mathrm{Det}_\zeta L_t=\frac{d}{dt}\left\{-\frac{d}{ds}\bigg|_{s=0}\mathrm{Tr}\big(L_t^{-s}\big)\right\}
$$

$$= -\frac{d}{ds}\bigg|_{s=0} \left\{ \frac{d}{dt} \mathrm{Tr}\big(L_t^{-s}\big) \right\} = \frac{d}{ds}\bigg|_{s=0} \left\{ s.\mathrm{Tr}\left[\frac{d}{dt}(L_t).L_t^{-s-1} \right] \right\}. \quad (3.2)$$

Proof of Corollary 1.1.3. In fact, from the integral representation (2.2) of the Fredholm determinant det_1 it results that

$$\frac{d}{ds}\bigg|_{s=0} \left\{ s.\mathrm{Tr}\left[\frac{d}{dt}(L_t).L_t^{-s-1} \right] \right\} = \mathrm{Tr}\left[\frac{d}{dt}(L_t).L_t^{-1} \right] = \frac{d}{dt} \ln \det{}_1 \left(L_t.L_0^{-1} \right).$$

The conclusion follows straightforward after integrating the last equality from 0 to t_o and taking the exponential function. □

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Appendix

In this appendix we recall the proofs of the technical lemmas exposed in Section 2 related to differentiability properties of the Trace class operators and the Fredholm determinant. These proofs were borrowed from [1].

Lemma 2.2.1. *Let $A(z) : G \longrightarrow \mathcal{J}_1$ a holomorphic application over an open subset G of $\mathbb{C}$ taking values on the ideal $\mathcal{J}_1$ of the trace class operators equipped with the norm of $\mathcal{L}(H)$. Let us suppose that the trace norm $\|A(z)\|_1$ of $A(z)$ is bounded over each compact subset of G.*

Then, the funtion $det_1(I - A(z)) : \; G \longrightarrow \mathbb{C}$ is holomorphic.

Proof. Let be $\{\Phi_j\}_1^\infty$ an orthogonal basis of H and for each $n \geq 1$, let P_n the orthogonal projection onto the subspace spanned by $\{\Phi_j\}_{j=1}^n$.

Let us define $A_n(z) = P_n A(z) P_n$. Since, for each fixed $z \in G$, $A_n(z) \to A(z)$ for $n \to \infty$ in $\mathcal{J}_1$-norm,

$$\det{}_1(I - A(z)) = \lim_{n\to\infty} \det{}_1(I - A_n(z)).$$

because $\det_1$ is continuous in this norm.

For $A(z)$ is holomorphic on G,

$$\det{}_1(I - A_n(z)) = \det(\delta_{jk} - (A(z)\Phi_k, \Phi_j))_{j,k=1,\dots,n}$$

is holomorphic on G and $\det_1(I - A(z))$ is a measurable function.

Then, for each n we have:

$$\det{}_1(I - A_n(z)) = \frac{1}{2\pi i} \int\limits_{|w-z|=r} \frac{\det_1(I - A_n(w))}{w - z}\, dw, \tag{4.1}$$

where the path $\{|w - z| = r\} \subset G$, is nonclockwise oriented. If we denote by $\lambda_j(A_n(z))$ and $z_j(A_n(z))$ the eigenvalues and the singular values of the operator $A_n(z)$ respectively, we have

$$|\det{}_1(I - A_n(z))| = \prod_{j=1}^n |1 - \lambda_j(A_n(z))| \leq \prod_{j=1}^n 1 + |\lambda_j(A_n(z))|$$
$$\leq \prod_{j=1}^n 1 + z_j(A_n(z)) \leq \prod_{j=1}^n e^{z_j(A_n(z))} = e^{\sum_{j=1}^n z_j(A_n(z))} = e^{\|A_n(z)\|_1} \leq e^{\|A(z)\|_1},$$

which is bounded by hypothesis for $z \in K$, being K any compact subset of G.

Finally, by applying Lebesgue Dominated Convergence Theorem we have from (4.1) the integral representation

$$\det{}_1(I - A(z)) = \frac{1}{2\pi i} \int\limits_{|w-z|=r} \frac{\det_1(I - A(w))}{w - z}\, dw,$$

which implies that $\det_1(I - A(z))$ is holomorphic in G. □

Lemma 2.2.2. *Under the hypotheses of Lemma 2.2.1 we have:*

— *the derivative of $A(z)$ is a trace class operator for all $z \in G$;*

— *the funtion $\mathrm{Tr}(A(z))$ is holomorphic on G, and*

— $\frac{d}{dz}\left[\mathrm{Tr}(A(z))\right] = \mathrm{Tr}\left[\frac{d}{dz}A(z)\right]$.

Proof. We will prove (a) by showing that the series

$$\sum_{j=1}^{\infty} < \partial_z A(z)\phi_j, \phi_j >$$

is absolutely convergent for all $z \in G$ and any orthonormal basis $\{\phi_j\}_1^\infty$ of H. By hypothesis, the functions $a_j(z) =< \partial_z A(z)\phi_j, \phi_j > \ G \longrightarrow \mathbf{C}$ are holomorphic. Then the sequence $S_n(z) = \sum_{j=1}^{n} a_j(z)$ of holomorphic functions in G tends to $\mathrm{Tr}(A(z))$ and is uniformly bounded in compact sets of G, because the hypothesis and the following inequality

$$|S_n(z)| \leq \sum_{j=1}^{n} |a_j(z)| \leq \sum_{j=1}^{\infty} |a_j(z)| = \|A(z)\|_1.$$

For the path $\gamma = \{|w - z| = r\} \subset G$, it is valid the integral representation

$$S_n(z) = \frac{1}{2\pi i} \int_\gamma \frac{S_n(w)}{w - z}\, dw,$$

and applying the Lebesgue Dominated Convergence Theorem, we get:

$$\mathrm{Tr}(A(z)) = \frac{1}{2\pi i} \int_\gamma \frac{\mathrm{Tr}(A(w))}{w - z}\, dw.$$

This shows that the function $\mathrm{Tr}(A(z))$ is holomorphic in G and then

$$\begin{aligned} \partial_z \left[\mathrm{Tr}(A(z))\right] &= \lim_{n\to\infty} \partial_z(S_n(z)) = \lim_{n\to\infty} \sum_{j=1}^{n} \partial_z < A(z)\phi_j, \phi_j > \\ &= \lim_{n\to\infty} \sum_{j=1}^{n} < \partial_z A(z)\phi_j, \phi_j >= \sum_{j=1}^{\infty} < \partial_z A(z)\phi_j, \phi_j >, \quad (4.2) \end{aligned}$$

independently of the choice of the orthonormal basis $\{\phi_j\}_1^\infty$. In particular this formula is independent of any rearrangement of the basis and the series is absolutely convergent. So $\partial_z A(z)$ is trace class and the equality (4.2) can be write as

$$\partial_z \left[\mathrm{Tr}(A(z))\right] = \mathrm{Tr}\left[\partial_z A(z)\right].$$

□

Lemma 2.2.4. *Under the hypotheses of Lemma 2.2.1 it results*

$$\frac{d}{dz} \ln(\det{}_1(I - A(z)) = -\mathrm{Tr}\left[(I - A(z))^{-1} \frac{d}{dz}(A(z))\right].$$

Proof. Arguing as in Lemma 2.2.2 for the function $\ln\det_1(I-A(z))$, being z such that $\det_1(I-A(z)) \neq 0$, we have:

$$\begin{aligned} \ln[\det_1(I-A(z))] &= \lim_{n\to\infty} \ln[\det_1(I-A_n(z))] \\ \text{and} \quad \partial_z \ln[\det_1(I-A(z))] &= \lim_{n\to\infty} \partial_z \ln[\det_1(I-A_n(z))]. \end{aligned}$$

For the finite dimension matrices $A_n(z)$ of Lemma A.1 is valid that:

$$\begin{aligned} \partial_z \ln[\det_1(I-A_n(z))] &= \partial_z \mathrm{Tr}[\ln(I-A_n(z))] \\ &= \mathrm{Tr}\left[(I-A_n(z))^{-1}\partial_z(I-A_n(z))\right] \\ &= -\mathrm{Tr}\left[(I-A_n(z))^{-1}\partial_z A_n(z)\right], \end{aligned}$$

and moreover $(I-A_n(z))^{-1} \to (I-A(z))^{-1}$ in $\mathcal{J}_1$.

Then by the continuity of the functional Tr in the ideal $\mathcal{J}_1$, we get

$$\begin{aligned} \partial_z \ln[\det_1(I-A(z))] &= \lim_{n\to\infty} -\mathrm{Tr}\left[(I-A_n(z))^{-1}\partial_z A_n(z)\right] \\ &= -\mathrm{Tr}\left[(I-A(z))^{-1}\partial_z A(z)\right]. \quad \square \end{aligned}$$