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AN OPTIMIZATION MODEL FOR THE TWO-DIMENSIONAL PACKING PROBLEM AND ITS AUGMENTED LAGRANGIAN METHOD

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Abstract: This paper formulates a two-dimensional packing problem as a nonlinear programming problem and establishes the first-order optimality conditions for the NLP problem. The augmented Lagrangian method is applied to solve this NLP problem and the computational experiments show the effectiveness of this method.

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Key Words: packing problem, augmented Lagrangian method, first-order optimality conditions

1. Introduction

Given a set $J = \{1, \cdots, n\}$ of n rectangular items each having width l_j and height $h_j (j \in J)$, we consider the special two-dimensional packing problem (2DP) of orthogonally allocating all the items, without overlapping, to a square box by minimizing the size of the box. We assume that the items have fixed orientation, i.e., they cannot be rotated. No further constraint is imposed to

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the packing.

Problem 2DP has many industrial applications, especially in cutting (wood and glass industries) and packing (transportation and warehousing). Numerous applications of cutting and packing can be found in a survey paper by Lodi et al [5]. Many publications dealing with the two-dimensional packing problem are concentrating on the development of approximation algorithms, for instance, see Baker et al [1], Baker and Schwarz [2], Coffman et al [3], Golan [4] and Steinberg [10].

In this paper, we try to find the exact solution to the two-dimensional packing problem using a nonlinear optimization approach, which is different from the exact approach proposed by Silvauo Martello et al [8]. In Section 2 we formulate 2DP as a general model, in which constraints include empty-intersections between any two different set-valued mappings. In Section 3, we give the general model a nonlinear programming representation and prove the first-order optimality conditions. In Section 4, we resolve this problem by the augmented Lagrangian method and show the exact solutions to three small-sized problems.

2. A General Mathematical Model

Let $(x,y)=(x_1,\cdots,x_n,y_1,\cdots,y_n)\in R^{2n}$ be a decision variable, where (x_i,y_i) is the bottom left-hand corner of item i if the bottom left-hand corner of the square box containing all these items taken as a position reference. Define set-valued mappings

$$S_i(x_i,y_i)=\{(u,w)\in R^2|\quad x_i<u<x_i+l_i, y_i<w<y_i+h_i\}.$$

Considering the set of all different item pairs defined by $H=\{(i,j)|i\in J, j\in J, i<j\}$, we obtain $|H|=n(n-1)/2$, the non-overlapping condition for any two different items i and j can be expressed as

$$S_i(x_i,y_i)\cap S_j(x_j,y_j)=\emptyset\,,\quad (i,j)\in H.$$

Obviously the two-dimensional packing problem can be equivalently formulated as the following optimization model.

Model General

$$\begin{array}{ll}\text{minimize} & v\\ \text{subject to} & S_i(x_i,y_i)\cap S_j(x_j,y_j)=\emptyset\,,\quad (i,j)\in H,\end{array}$$

$$0 \le x_i \le v - l_i, \quad 0 \le y_i \le v - h_i, \quad i \in J.$$

This model is a non-smooth mathematical programming problem because the constraints $S_i(x_i, y_i) \cap S_j(x_j, y_j) = \emptyset$, $(i,j) \in H$, are not usual function constraints. In the next section we study how to describe these empty intersections of set-valued mappings as usual mathematical formulations.

3. A Smooth Optimization Representation

The reformulation of Model General in this section comes from the fact that two rectangles (whose edges are parallel to x-axis and y-axis respectively) intersect if and only if both their projections to x-axis and projections to y-axis intersect. Let $P^X(S)$ and $P^Y(S)$ denote the projections of set S to x-axis and y-axis respectively, then $P^X(S_i(x_i, y_i)) = (x_i, x_i + l_i)$, $\quad P^Y(S_i(x_i, y_i)) = (y_i, y_i + h_i)$. It is easy to check

$$P^X(S_i(x_i, y_i)) \cap P^X(S_j(x_j, y_j)) \neq \emptyset \Longleftrightarrow |x_i - x_j + \frac{l_i - l_j}{2}| - \frac{l_i + l_j}{2} \le 0$$

and

$$P^Y(S_i(x_i, y_i)) \cap P^Y(S_j(x_j, y_j)) \neq \emptyset \Longleftrightarrow |y_i - y_j + \frac{h_i - h_j}{2}| - \frac{h_i + h_j}{2} \le 0.$$

Therefore, Model General can be equivalently expressed as the following model.

Model Absolute

$$\begin{aligned}
\text{minimize} \quad & v \\
\text{subject to} \quad & \max\{|x_i - x_j + \frac{l_i - l_j}{2}| - \frac{l_i + l_j}{2}, \\
& \quad |y_i - y_j + \frac{h_i - h_j}{2}| - \frac{h_i + h_j}{2}\} \ge 0, \ (i,j) \in H, \\
& 0 \le x_i \le v - l_i, \quad 0 \le y_i \le v - h_i, \quad i \in J.
\end{aligned}$$

Model Absolute is a non-convex non-smooth optimization problem, because the feasible set is non-convex as it contains a set of anti-convex constraints.

We can convert Model Absolute into a smooth optimization problem by introducing artificial variables w^X, w^Y, w^A. Let $z = (x, y, w^X, w^Y, w^A, v)$, $M = n(n-1)/2$, define $f_0 : \mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}^M \times \mathbf{R}^M \times \mathbf{R}^M \times \mathbf{R} \longrightarrow \mathbf{R}$ and $F : \mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}^M \times \mathbf{R}^M \times \mathbf{R}^M \times \mathbf{R} \longrightarrow \mathbf{R}^M \times \mathbf{R}^M \times \mathbf{R}^M \times \mathbf{R}^n$ as follows

$$f_0(z) = v, \quad F(z) = (F_1(z)^\top, F_2(z)^\top, F_3(z)^\top, F_4(z)^\top, F_5(z)^\top)^\top,$$

with

$$\begin{aligned}
&F_1(z) = ((x_1 - x_2 + \tfrac{l_1-l_2}{2})^2 - w_{1,2}^{X2}, \cdots, (x_{n-1} - x_n + \tfrac{l_{n-1}-l_n}{2})^2 - w_{n-1,n}^{X2})^\top,\\
&F_2(z) = ((y_1 - y_2 + \tfrac{h_1-h_2}{2})^2 - w_{1,2}^{Y2}, \cdots, (y_{n-1} - y_n + \tfrac{h_{n-1}-h_n}{2})^2 - w_{n-1,n}^{Y2})^\top,\\
&F_3(z) = ((w_{1,2}^A - w_{1,2}^X + \tfrac{l_1+l_2}{2})(w_{1,2}^A - w_{1,2}^Y + \tfrac{h_1+h_2}{2}), \cdots,\\
&\qquad (w_{n-1,n}^A - w_{n-1,n}^X + \tfrac{l_{n-1}+l_n}{2})(w_{n-1,n}^A - w_{n-1,n}^Y + \tfrac{h_{n-1}+h_n}{2}))^\top,\\
&F_4(z) = (v - x_1 - l_1, \cdots, v - x_n - l_n)^\top.\\
&F_5(z) = (v - y_1 - h_1, \cdots, v - y_n - h_n)^\top.
\end{aligned}$$

Let $\Omega = \{z \in Z | F(z) \in D\}$, where $D = \{0_M\} \times \{0_M\} \times \{0_M\} \times \mathbf{R}_+^n \times \mathbf{R}_+^n$ and

$$Z = \left\{ z \in \mathbf{R}^{2n+3M+1} \;\middle|\; \begin{array}{l} x \geq 0_n,\, y \geq 0_n \\ w^A \geq 0_M,\, w^X \geq 0_M \\ w^Y \geq 0_M,\, v \in \mathbf{R} \end{array} \right\}.$$

Then Model Absolute can be equivalently formulated as a smooth optimization problem as below.

Model NLP

$$\begin{array}{ll} \text{minimize} & f_0(z) \\ \text{subject to} & F(z) \in D, \\ & z \in Z. \end{array}$$

Let us introduce some notations for the purpose of deriving the first-order optimality conditions for Model NLP:

$$W^X = \mathrm{diag}_{(i,j)\in H}(w_{i,j}^X), \quad W^Y = \mathrm{diag}_{(i,j)\in H}(w_{i,j}^Y),$$

$$A = \mathrm{diag}_{(i,j)\in H}(\alpha_{i,j}), \quad B = \mathrm{diag}_{(i,j)\in H}(\beta_{i,j}), \quad C = \mathrm{diag}_{(i,j)\in H}(\gamma_{i,j}),$$

$$D_1 =$$

$$\begin{pmatrix}
d_{1,2}(x) & d_{1,3}(x) & \cdots & d_{1,n}(x) & 0 & \cdots & 0 & \cdots & 0 \\
-d_{1,2}(x) & 0 & \cdots & 0 & d_{2,3}(x) & \cdots & d_{2,n}(x) & \cdots & 0 \\
0 & -d_{1,3}(x) & \cdots & 0 & -d_{2,3}(x) & \cdots & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & d_{n-1,n}(x) \\
0 & 0 & \cdots & -d_{1,n}(x) & 0 & \cdots & -d_{2,n}(x) & \cdots & -d_{n-1,n}(x)
\end{pmatrix},$$

$$D_2 =$$

$$\begin{pmatrix} c_{1,2}(y) & c_{1,3}(y) & \cdots & c_{1,n}(y) & 0 & \cdots & 0 & \cdots & 0 \\ -c_{1,2}(y) & 0 & \cdots & 0 & c_{2,3}(y) & \cdots & c_{2,n}(y) & \cdots & 0 \\ 0 & -c_{1,3}(y) & \cdots & 0 & -c_{2,3}(y) & \cdots & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & c_{n-1,n}(y) \\ 0 & 0 & \cdots & -c_{1,n}(y) & 0 & \cdots & -c_{2,n}(y) & \cdots & -c_{n-1,n}(y) \end{pmatrix},$$

where $d_{i,j}(x) = 2(x_i - x_j) + l_i - l_j$, $c_{i,j}(y) = 2(y_i - y_j) + h_i - h_j$, $\alpha_{i,j} = -(w^A_{i,j} - w^Y_{i,j} + (h_i + h_j)/2)$, $\beta_{i,j} = -(w^A_{i,j} - w^X_{i,j} + (l_i + l_j)/2)$, $\gamma_{i,j} = 2w^A_{i,j} - w^X_{i,j} + (l_i + l_j)/2 - w^Y_{i,j} + (h_i + h_j)/2$. Then $\nabla F_i(z)^\top, i = 1, \cdots, 5$ can be expressed as

$$\nabla F_1(z)^\top = \begin{pmatrix} D_1 \\ 0_{n\times M} \\ -2W^X \\ 0_{M\times M} \\ 0_{M\times M} \\ 0_{1\times M} \end{pmatrix}, \nabla F_2(z)^\top = \begin{pmatrix} 0_{n\times M} \\ D_2 \\ 0_{M\times M} \\ -2W^Y \\ 0_{M\times M} \\ 0_{1\times M} \end{pmatrix}, \nabla F_3(z)^\top \begin{pmatrix} 0_{n\times M} \\ 0_{n\times M} \\ A \\ B \\ C \\ 0_{1\times M} \end{pmatrix},$$

$$\nabla F_4(z)^\top = \begin{pmatrix} -I_{n\times n} \\ 0_{n\times n} \\ 0_{M\times n} \\ 0_{M\times n} \\ 0_{M\times n} \\ 1_{1\times n} \end{pmatrix}, \nabla F_3(z)^\top = \begin{pmatrix} 0_{n\times n} \\ -I_{n\times n} \\ 0_{M\times n} \\ 0_{M\times n} \\ 0_{M\times n} \\ 1_{1\times n} \end{pmatrix},$$

and the transpose of the Jacobian of F is

$$\nabla F(z)^\top = (\nabla F_1(z)^\top, \nabla F_2(z)^\top, \nabla F_3(z)^\top, \nabla F_4(z)^\top, \nabla F_5(z)^\top).$$

Now we give the formulas for calculating the normal cones $N_Z(\bar{z})$ and $N_D(F(\bar{z}))$ so that the formula $N_\Omega(\bar{z})$ can be derived for $\bar{z} \in \Omega$. Let

$$Z_x = \{x \in \mathbf{R}^n | x \geq 0_n\}, \quad Z_y = \{y \in \mathbf{R}^n | y \geq 0_n\},$$

with $Z^i_x = [0, \infty)$ and $Z^i_y = [0, \infty)$. Then

$$N_Z(\bar{z}) = N_{Z_x}(\bar{x}) \times N_{Z_y}(\bar{y}) \times N_{\mathbf{R}^M}(\bar{w}^X) \times N_{\mathbf{R}^M}(\bar{w}^Y) \times N_{\mathbf{R}^M_+}(\bar{w}^A) \times N_{\mathbf{R}}(\bar{v}),$$

where $N_{Z_u}(\bar{u}) = N_{Z^1_u}(\bar{u_1}) \times \cdots \times N_{Z^n_u}(\bar{u_n}), u = x, y$, with

$$N_{Z^i_u}(\bar{u}_i) = \begin{cases} \mathbf{R}_-, & \bar{u}_i = 0, \\ \{0\}, & \bar{u}_i > 0, \end{cases}$$

$N_{\mathbf{R}_+^{\mathbf{M}}}(\bar{w}^H) = N_{\mathbf{R}_+}(\bar{w}^H_{1,2}) \times \cdots \times N_{\mathbf{R}_+}(\bar{w}^H_{n-1,n}), H = X, Y, A$, with

$$N_{\mathbf{R}_+}(\bar{w}^H_{i,j}) = \begin{cases} \mathbf{R}_-, & \bar{w}^H_{i,j} = 0\,, \\ \{0\}, & \bar{w}^H_{i,j} > 0\,, \end{cases}$$

$N_{\mathbf{R}}(\bar{v}) = \{0\}$.

The normal cone of D at $F(\bar{z})$, $N_D(F(\bar{z}))$ is

$$N_D(F(\bar{z})) = N_{\{0_M\}}(F_1(\bar{z})) \times N_{\{0_M\}}(F_2(\bar{z})) \times N_{\{0_M\}}(F_3(\bar{z})) \\ \times N_{\mathbf{R}_+^n}(F_4(\bar{z})) \times N_{\mathbf{R}_+^n}(F_5(\bar{z}))\,,$$

where

$$N_{\{0_M\}}(F_1(\bar{z})) = N_{\{0_M\}}(F_2(\bar{z})) = N_{\{0_M\}}(F_3(\bar{z})) = \mathbf{R}^M,$$
$$N_{\mathbf{R}_+^n}(F_\alpha(\bar{z})) = N_{\mathbf{R}_+}(F_\alpha^1(\bar{z})) \times \cdots \times N_{\mathbf{R}_+}(F_\alpha^n(\bar{z})), \quad \alpha = 4, 5,$$

with

$$N_{\mathbf{R}_+}(F_\alpha^i(\bar{z})) = \begin{cases} \mathbf{R}_-, & F_\alpha^i(\bar{z}) = 0; \\ \{0\}, & F_\alpha^i(\bar{z}) > 0. \end{cases}$$

Lemma 1. (Normal Cone of Ω at $\bar{z}$) *Let $\bar{z} \in \Omega$, satisfy $\bar{w}^X_{i,j} \neq 0, \bar{w}^Y_{i,j} \neq 0, \bar{w}^X_{i,j} - (l_i + l_j)/2 \neq \bar{w}^Y_{i,j} - (h_i + h_j)/2, (i, j) \in H$ and $\bar{w}^A > 0$. Then the normal cone of Ω at $\bar{z}$ can be expressed as $N_\Omega(\bar{z}) = \{\bigtriangledown F(\bar{z})^\top p + q | p \in N_D(F(\bar{z})), q \in N_Z(\bar{z})\}$.*

Proof. According to Theorem 6.14 in Rockafellar and Wets [9], we only need to check the validity of the following constraint qualification

$$- \bigtriangledown F(\bar{z})^\top p \in N_Z(\bar{z}),\ p \in N_D(F(\bar{z})) \Rightarrow p = 0. \tag{1}$$

Let $p = (p^{1T}, \cdots, p^{5T})^T \in N_D(F(\bar{z}))$, then $- \bigtriangledown F(\bar{z})^\top p \in N_Z(\bar{z})$ can be written as $\bigtriangledown F_1(\bar{z})^\top(-p^1) + \cdots + \bigtriangledown F_5(\bar{z})^\top(-p^5)$, or in the matrix form:

$$\begin{pmatrix} D_1 & 0_{n\times M} & 0_{n\times M} & -I_{n\times n} & 0_{n\times n} \\ 0_{n\times M} & D_2 & 0_{n\times M} & 0_{n\times n} & -I_{n\times n} \\ -2W^X & 0_{M\times M} & A & 0_{M\times n} & 0_{M\times n} \\ 0_{M\times M} & -2W^Y & B & 0_{M\times n} & 0_{M\times n} \\ 0_{M\times M} & 0_{M\times M} & C & 0_{M\times n} & 0_{M\times n} \\ 0_{1\times M} & 0_{1\times M} & 0_{1\times M} & 1_{1\times n} & 1_{1\times n} \end{pmatrix} \begin{pmatrix} -p^1 \\ -p^2 \\ -p^3 \\ -p^4 \\ -p^5 \end{pmatrix} \in N_Z(\bar{z})\,,$$

that is

$$\begin{array}{rcl} -D_1 p^1 + p^4 & \in & N_{Z_x}(\bar{x}), \\ -D_2 p^2 + p^5 & \in & N_{Z_y}(\bar{y}), \\ 2W^X p^1 - Ap^3 & \in & N_{\mathbf{R}^M}(\bar{w}^X), \qquad (2) \\ 2W^Y p^2 - Bp^3 & \in & N_{\mathbf{R}^M}(\bar{w}^Y), \qquad (3) \\ -Cp^3 & \in & N_{\mathbf{R}_+^M}(\bar{w}^A), \\ \sum_{i=1}^n (-p_i^4 - p_i^5) & \in & N_{\mathbf{R}}(\bar{v}). \end{array}$$

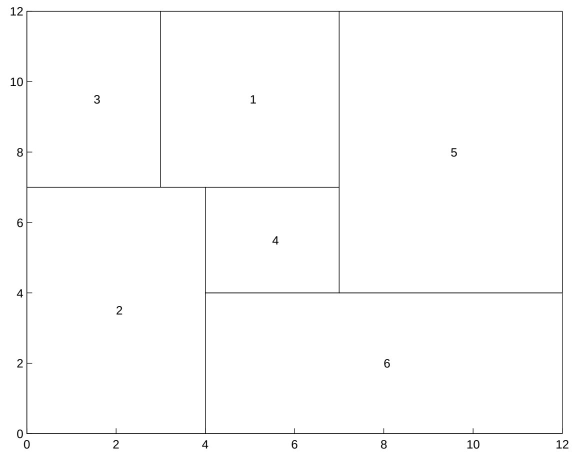

Figure 1:

In view of $p^4 \in N_{\mathbf{R}^{\mathbf{n}}_+}(F_4(\bar{z}))$ and $p^5 \in N_{\mathbf{R}^{\mathbf{n}}_+}(F_5(\bar{z}))$ we know that $p^4 \leq 0_n$ and $p^5 \leq 0_n$ while we obtain $\sum_{i=1}^{n}(-p_i^4 - p_i^5) = 0$ from $N_{\mathbf{R}}(\bar{v}) = \{0\}$, therefore we must have $p^4 = 0_n, p^5 = 0_n$. Since $\bar{w}^A > 0$, we have that $N_{\mathbf{R}^{\mathbf{M}}_+}(\bar{w}^A) = \{0_M\}$, thus $Cp^3 = 0_M$. We can say C is nonsingular (otherwise, there is at least one $\gamma_{i,j} = 0$, it implies that $\bar{w}_{i,j}^X - (l_i + l_j)/2 = \bar{w}_{i,j}^Y - (h_i + h_j)/2$, which is contrary to the assumption). Therefore we obtain $p^3 = 0_M$. In view of (2) and (3), we know $W^X p^1 = 0_M$, and $W^Y p^2 = 0_M$ (since $N_{\mathbf{R}^{\mathbf{M}}_+}(\bar{w}^X) = N_{\mathbf{R}^{\mathbf{M}}_+}(\bar{w}^Y) = \{0_M\}$). While W^X and W^Y are nonsingular, we have $p^1 = p^2 = 0_M$. By now we have verified the correctness of (1) and the formula for calculating $N_\Omega(\bar{z})$ can be obtained by Theorem 6.14 of Rockafellar and Wets [9]. □

Theorem 1. (First-Order Optimality Conditions for Model NLP) *Let $\bar{z} \in \Omega$ be a local minimizer to Model NLP, satisfying $\bar{w}_{i,j}^X \neq 0, \bar{w}_{i,j}^Y \neq 0, \bar{w}_{i,j}^X - (l_i + l_j)/2 \neq \bar{w}_{i,j}^Y - (h_i + h_j)/2, (i,j) \in H$ and $\bar{w}^A > 0$. then there exists a vector $p \in N_D(F(\bar{z}))$ such that $-[\nabla f_0(\bar{z}) + \bigtriangledown F(\bar{z})^\top p] \in N_Z(\bar{z})$.*

Proof. It follows from Theorem 6.12 of Rockafellar and Wets [9] that $-\nabla f_0(\bar{z}) \in N_\Omega(\bar{z})$. Thus the conclusion is obvious from Lemma 1. □

3. A Numerical Algorithm

In the Model NLP, the constraint $z \in Z$ is a simple box constraint, it can be written as usual inequality constraints. Therefore we can simply prescribe the

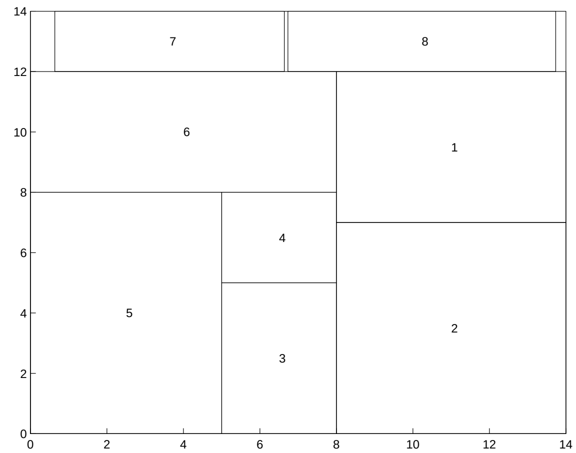

Figure 2:

Model NLP as

$$\begin{array}{lll} \text{minimize} & f_0(z) & \\ \text{subject to} & c_i(z) = 0, & i = 1, \cdots, 3M, \\ & c_i(z) \geq 0, & i = 3M+1, \cdots, 6M+4n. \end{array}$$

We suggest solving this constrained optimization problem by the augmented Lagrangian method (ALM) based on solving a sequence of unconstrained optimization problem involving parameter $\sigma > 0$ and Lagrange multiplier vector λ:

$$\begin{aligned} \min P(z, \lambda, \sigma) &= f_0(z) + \sum_{i=1}^{3M} [-\lambda_i c_i(z) + \tfrac{1}{2}\sigma_i c_i^2(z)] \\ &\quad + \sum_{i=3M+1}^{4n+6M} \begin{cases} -\lambda_i c_i(z) + \frac{1}{2}\sigma_i c_i^2(z), & c_i(z) < \frac{\lambda_i}{\sigma_i}; \\ -\frac{1}{2}\lambda_i^2/\sigma_i, & \text{otherwise.} \end{cases} \end{aligned}$$

Algorithm ALM:

Step 1. Given $z_1 \in \mathbf{R}^{2n+3M+1}, \lambda^{(1)} \in \mathbf{R}^{6M+4n}$ and $\lambda_i^{(1)} > 0 (i = 3M + 1, \cdots, 6M + 4n)$; $\sigma_i^{(1)} > 0, i = 1, \cdots, 4n + 6M; \varepsilon \geq 0, k := 1$.

Step 2. Solve min $P(z, \lambda^{(k)}, \sigma^{(k)})$ for z_{k+1}; if $\| c^{(-)}(z_{k+1}) \|_\infty \leq \varepsilon$ (where $c_i^{(-)}(z_k) = \min\{0, c_i(z_k)\}$), then stop.

Step 3. For $i = 1, \cdots, 6M + 4n$, let

$$\sigma_i^{(k+1)} = \begin{cases} \sigma_i^{(k)}, & |c_i^{(-)}(z_{k+1})| \leq 1/4 |c_i^{(-)}(z_k)|; \\ \max[10\sigma_i^{(k)}, k^2], & \text{otherwise.} \end{cases}$$

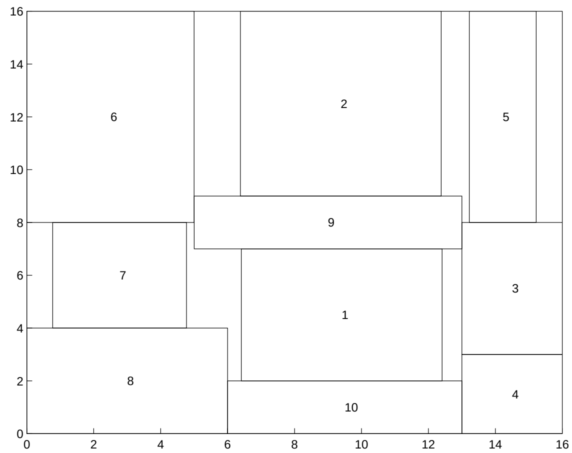

Figure 3:

Step 4. Compute

$$\lambda_i^{(k+1)} = \lambda_i^{(k)} - \sigma_i^{(k)} c_i(z_{k+1}),\ i = 1, \cdots, 3M,$$
$$\lambda_i^{(k+1)} = \max\{\lambda_i^{(k)} - \sigma_i^{(k)} c_i(z_{k+1}), 0\},\ i = 3M+1, \cdots, 6M+4n.$$

$k := k + 1$; go to *Step 2.*

This algorithm was coded in *C++* and run on a *Pentium III, 1000 MHz* on a series of instances from the literature. The numerical experiments show that the algorithm can find exact solutions to small-sized problems with less than 10 items. We select three of them to show the results. Figure 1 - Figure 3 are the geometrical patterns for rectangular items corresponding to the solutions obtained by the algorithm involving up to 6, 8 and 10 items respectively, which show the very good behavior of the proposed augmented Lagrangian method.

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