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RIABOUCHINSKY FLOW THROUGH POROUS MEDIA

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Abstract: Exact solutions to the two-dimensional, viscous fluid flow through porous media, as governed by the Darcy-Lapwood-Brinkman model, are obtained for flows where the streamfunction is linear in one of the space variables. A solution algorithm is developed in this work to handle the type of flow considered, and is capable of treating similar flows without resorting to the Riabouchinsky assumptions.

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1. Introduction

The steady flow of an incompressible, viscous fluid through porous media is governed by the conservation of mass and conservation of linear momentum principles. In the absence of sources and sinks, conservation of mass takes the following form of velocity continuity:

$$\nabla \cdot \mathbf{v} = 0, \tag{1}$$

where $\mathbf{v}$ is the macroscopic velocity vector.

Conservation of linear momentum takes different forms depending on the porous medium microstructure, viscous shear effects and the presence of macroscopic boundaries, curvilinearity of the flow path and the effects of inertia. When viscous shear and macroscopic inertial effects are significant, fluid flow

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through a porous medium may be described by the Darcy-Lapwood-Brinkman equation (referred to hereafter as the DLB equation). This equation has the form, [3]:

$$(\mathbf{v} \cdot \nabla)\mathbf{v} = -\frac{\nabla p}{\rho} + \nu \nabla^2 \mathbf{v} - \frac{\nu}{k}\mathbf{v}, \qquad (2)$$

where p is the pressure, k is the permeability, $\nu = \frac{\mu}{\rho}$ is the dynamic viscosity coefficient, μ is the kinematic viscosity coefficient, ρ is the fluid density, and $\nabla^2 \equiv \partial_{xx} + \partial_{yy}$ is the Laplacian operator.

The DLB equation is capable of handling the presence of a macroscopic boundary, on which the no-slip condition is imposed, through the viscous shear term.

Comparison of the structure of equation (2) with that of the Navier-Stokes equations that govern the flow of a viscous fluid in free-space shows the presence in equation (2) of the damping term, $\frac{\nu}{k}\mathbf{v}$, which represents the Darcy resistance to motion that is exerted by the porous structure on the traversing fluid. However, the presence of this damping term does not alter the nonlinearity exhibited in the Navier-Stokes equations Nonlinearity renders the superposition principle inapplicable, and introduces a degree of difficulty in obtaining analytical solutions to the Navier-Stokes equations, and to the DLB equation.

In the case of the Navier-Stokes equations, analytical solutions have been obtained for parallel, laminar flows for which the Navier-Stokes equations can be linearised. The source of nonlinearity is the convective inertial terms which, as Taylor [11] noted, vanish for two-dimensional flows when the vorticity is a function of the Stokes streamfunction. Taylor [11] obtained an exact solution to the Navier-Stokes equations by taking the vorticity to be proportional to the streamfunction of the flow. His solution represents a double infinite array of vortices decaying exponentially with time.

An extension of Taylor's approach was accomplished by Kovasznay [7], where he linearised the Navier-Stokes equations by taking the vorticity to be proportional to the streamfunction perturbed by a uniform stream. The two-dimensional solution that Kovasznay obtained represents the flow downstream of a two-dimensional grid. Two solutions representing the reverse flow over a flat plate with suction and blowing were obtained by Lin and Tobak, [8], who extended Kovasznay's approach. Various other authors have obtained exact solutions to the Navier-Stokes and other equations for special types of flow (cf. [2, 6, 9, 12] and the review in [13]).

For the case of flow through porous media, as governed by the DLB model, exact solutions are rare. However, some of the methods that are applicable to the Navier-Stokes equations may be readily used for the DLB model. An extension of the method used by Kovasznay [7] has been employed by Hamdan and Ford [5] to solve the DLB model and analyze the effect of permeability on the resulting flow. The resulting solution represents the flow behind a two-dimensional porous grid.

In this work, we analyze the nonlinear Darcy-Lapwood-Brinkman equation in an attempt to find possible two-dimensional solutions corresponding to a particular form of the streamfunction. The choice of the streamfunction is one that is linear with respect to one of the independent variables. The corresponding flow governed by the Navier-Stokes equations it is referred to as the Riabouchinsky flow, [10], and has received considerable attention in the literature due to its application in boundary layer analysis (cf. [1] and the references therein).

In the case of Riabouchinsky flow in free-space, the two-dimensional Navier-Stokes equations, written as a fourth order partial differential equation in terms of the streamfunction, may be replaced by two coupled fourth order ordinary differential equations in two unknown functions of a single variable. Solutions to the coupled set are then obtained based on the knowledge of particular integrals of one of the equations Different types of flow may then be studied with the knowledge of one of the functions.

Riabouchinsky [10] assumed one of the functions to be zero and studied the resulting flow, which represents a plane stagnation flow in which the flow is separated in the two-dimensional symmetrical regions by a vertical or a horizontal plate. The approach used to solve the resulting coupled set of ordinary differential equations has some limitations among which is its dependence on the knowledge of the particular solutions of one of the two equations In addition, the Riabouchinsky solutions involve a number of arbitrary constants the determination of which may involve making restrictive assumptions on the flow. For the case of flow through porous media, a disadvantage of the Riabouchinsky approach is the difficulty of its implementation in obtaining solutions to the DLB model.

In order to overcome the above limitations, we offer a modest modification of the Riabouchinsky approach. The resulting methodology is capable of handling a wider class of flow problems, including Brinkman-type flow through porous media.

2. Problem Formulation

We consider the two-dimensional, steady flow of an incompressible fluid through a porous structure. The flow is governed by the equation of continuity and the

Darcy-Lapwood-Brinkman model, equations (1) and (2), respectively. These equations may be cast in the following vorticity-streamfunction form

$$\nabla^2 \psi = -\xi \,, \tag{3}$$

$$\psi_y \, \xi_x - \psi_x \, \xi_y = \nu \nabla^2 \xi - \frac{\nu}{k} \xi \,, \tag{4}$$

where ψ is the streamfunction, and ξ is the vorticity, and subscript notation denotes partial differentiation.

The streamfunction and vorticity are defined in terms of the tangential and normal velocity components, u and v, respectively, as

$$\psi_y = u \,, \tag{5}$$

$$\psi_x = -v \,, \tag{6}$$

$$\xi = v_x - u_y \,. \tag{7}$$

An equation for the pressure, $p(x, y)$, is derived as follows. Equation (2) is written in the following components' form:

$$\rho \left\{ \left[\frac{q^2}{2} \right]_x - v[v_x - u_y] \right\} + p_x = \mu \nabla^2 u - \frac{\mu}{k} u \,, \tag{8}$$

$$\rho \left\{ \left[\frac{q^2}{2} \right]_y + u[v_x - u_y] \right\} + p_y = \mu \nabla^2 v - \frac{\mu}{k} v \,, \tag{9}$$

where $q^2 = u^2 + v^2$ is the square of the speed.

Letting $h(x, y)$ be a generalized pressure function defined as:

$$h = -\frac{1}{2} \rho q^2 + p \,, \tag{10}$$

then with the help of equations (7) and (10), we write equations (8) and (9), respectively, as:

$$h_x - \rho\nu\xi = \mu\nabla^2 u - \frac{\mu}{k}u\,, \tag{11}$$

$$h_y + \rho u\,\xi = \mu\nabla^2 v - \frac{\mu}{k}v\,. \tag{12}$$

Equations (11) and (12) can also be written in terms of the streamfunction, respectively, as:

$$h_x - \rho\psi_x(\nabla^2\psi) = \mu\nabla^2\psi_y - \frac{\mu}{k}\psi_y\,, \tag{13}$$

$$h_y - \rho\psi_y(\nabla^2\psi) = -\mu\nabla^2\psi_x + \frac{\mu}{k}\psi_x\,. \tag{14}$$

Thus far, in the vorticity-streamfunction formulation, the equations to be solved are equations (3) and (4) for the unknowns $\psi(x,y)$ and $\xi(x,y)$. The velocity components may then be obtained from equations (5) and (6), while the pressure can be obtained from equations (11) and (12), or from equations (13) and (14).

A compatibility condition for the DLB equation can be obtained by substituting equation (3) into equation (4). This yields:

$$\psi_x(\nabla^2\psi)_y - \psi_y(\nabla^2\psi)_x = \frac{\nu}{k}\nabla^2\psi - \nu\nabla^4\psi\,, \tag{15}$$

where $\nabla^4 \equiv \partial_{xxxx} + 2\partial_{xxyy} + \partial_{yyyy}$, and subscript notation denotes partial differentiation.

It is thus required to solve equation (15) for $\psi(x,y)$. Velocity components may then be obtained from equations (5) and (6); vorticity from equation (3); and pressure from equations (11) and (12) or (13) and (14).

By contrast, the Navier-Stokes equations take the following vorticity-streamfunction form, [4]:

$$\nabla^2\psi = -\xi\,, \tag{16}$$

$$\psi_y\xi_x - \psi_x\xi_y = \nu\nabla^2\xi\,. \tag{17}$$

Compatibility condition is of the form:

$$\psi_x(\nabla^2\psi)_y - \psi_y(\nabla^2\psi)_x = -\nu\nabla^4\,,\psi \tag{18}$$

and the pressure equations take the form:

$$h_x - \rho\psi_x(\nabla^2\psi) = \mu\nabla^2\psi_y\,, \tag{19}$$

$$h_y - \rho\psi_y(\nabla^2\psi) = -\mu\nabla^2\psi_x\,. \tag{20}$$

Velocity components' relationships with the Navier-Stokes streamfunction and vorticity are given by equations of the same form as equations (5), (6) and (7).

3. Solution Methodology

In this section, we discuss the solution methodology for the DLB model. For the sake of comparison, we will also solve the flow problem governed by the Navier-Stokes equations In the following subsections the Riabouchinsky approach is used to solve the Navier-Stokes flow equations, a modification to the Riabouchinsky approach of solving the Navier-Stokes equations is presented, and the modified approach to the solution of the DLB model is applied.

3.1. The Riabouchinsky Approach

In order to obtain solutions to the Navier-Stokes equations, it is assumed that the streamfunction is linear with respect to one of the independent variables, say y, taken in the form:

$$\psi(x,y) = yf(x) + g(x)\,, \tag{21}$$

where $f(x)$ and $g(x)$ are arbitrary functions of x.

We are thus required to find $f(x)$ and $g(x)$ such that given by equation (21) satisfies equation (18). The equations that the functions $f(x)$ and $g(x)$ must satisfy are obtained by substituting equation (21) in (18), and equating the coefficients of similar powers of y. This yields the following set of coupled fourth order ordinary differential equations:

$$\nu f^{(iv)} + f'f'' - ff''' = 0\,, \tag{22}$$

$$\nu g^{(iv)} + g'f'' - gf''' = 0\,. \tag{23}$$

In the absence of a general solution to equations (22) and (23), the functions $f(x)$ and $g(x)$ can be determined in accordance with the following algorithm (referred to here as the Riabouchinsky approach).

(a) A particular solution satisfying equation (22) is found for $f(x)$.

(b) The solution for $f(x)$, found in step (a), is then substituted in equation (23) and a general solution is found for $g(x)$.

(c) Once $f(x)$ and $g(x)$ are found, the streamfunction $\psi(x,y)$ can be calculated using equation (21). The velocity components and vorticity can be found from equations (5), (6), and (7), and the pressure from equations (19) and (20).

The above algorithm is detailed in what follows. Equation (22) admits the three particular solutions, [1]:

$$f_1(x) = -\frac{6\nu}{x}\,, \quad x \neq 0\,, \tag{24}$$

$$f_2(x) = \nu a[1 + b\ \exp(ax)]\,, \tag{25}$$

$$f_3(x) = cx + d\,, \tag{26}$$

where a, b, c, and d are constants.

Substituting equations (24), (25), and (26), in turn, in equation (23), and solving for $g(x)$ we obtain, respectively, the following cases:

Case 1. When $f(x) = f_1(x) = -\frac{6\nu}{x}; x \neq 0$, then

$$g_1(x) = a_0 + a_1 x^{-1} + a_2 x^{-2} + a_3 x^3 \tag{27}$$

and the streamfunction takes the form:

$$\psi(x,y) = yf_1(x) + g_1(x) = -6\nu\frac{y}{x} + a_0 + a_1 x^{-1} + a_2 x^{-2} + a_3 x^3\,. \tag{28}$$

Case 2. When $f(x) = f_2(x) = \nu a[1 + b\exp(ax)]$, then $g_2(x) = b_0 + b_1 \exp(x) + b_2 \int \exp(x)\,dx \int \exp[b\ \exp(x) - x]dx$

$$+b_3 \int \exp(x)dx \int \exp[b\ \exp(x) - x]dx \int \exp[-b\ \exp(x)]dx\,. \tag{29}$$

The corresponding streamfunction takes the form:

$$\begin{aligned}\psi(x,y) = yf_2(x) + g_2(x) &= \nu a[1 + b\ \exp(ax)]y \\ &+ b_0 + b_1\ \exp(x) + b_2 \int \exp(x)dx \int \exp[b\exp(x) - x]dx \\ &+ b_3 \int \exp(x)dx \int \exp[b\exp(x) - x]dx \int \exp[-b\exp(x)]dx\,. \end{aligned} \tag{30}$$

Case 3. When $f(x) = f_3(x) = cx + d$, then

$$g_3(x) = \int\int\int c_1\ \exp\left[\frac{1}{\nu}\left\{\frac{1}{2}\,c_2\,x^2 + c_3 x\right\}\right] dxdxdx\,, \tag{31}$$

and the streamfunction takes the form:

$$\psi(x,y) = yf_3(x) + g_3(x) = (cx+d)y + \int\int\int c_1 \exp\left[\frac{1}{\nu}\left\{\frac{1}{2}c_2\,x^2 + c_3 x\right\}\right] dx dx dx\,. \quad (32)$$

The above solutions involve a number of arbitrary constants, the determination of which may involve restrictive assumptions on the flow. Furthermore, evaluation of the resulting integrals is rather involved. This hinders the evaluation of $g(x)$ and the flow variables.

In the absence of a systematic procedure for determining the arbitrary constants, Riabouchinsky assumed that $g(x) = 0$ and obtained the special form of the streamfunction: $\psi(x,y) = yf(x)$. This amounts to assigning the value of zero to each of the arbitrary constants appearing in $g_1(x), g_2(x)$ and $g_3(x)$.

3.2. A Modified Approach

In an attempt to overcome some of the drawbacks of the above algorithm, Hamdan [4] proposed a modified algorithm to solve equations (22) and (23). The algorithm and its application to equations (22) and (23) are summarized as follows.

(a) Assume the form of the function $g(x)$.

(b) Substitute the assumed form of $g(x)$ in equation (23) (this produces an ordinary differential equation for $f(x)$).

(c) Solve the equation that arises in step (b) for $f(x)$ (solution obtained in this step will involve some arbitrary constants, this solution must satisfy equation (22)).

(d) Substitute $f(x)$ that is obtained in step (c) in equation (22) and determine the arbitrary constants.

Utility of this algorithm is illustrated by applying the procedure to equations (22) and (23). We will assume a form of $g(x)$ that will render the forms of $f(x)$ that are viscosity dependent, namely Case 1 and Case 2 of Subsection 3.1.

Case 1. Let $g(x) = \alpha x^3$, where α is a known parameter.

Using this form of $g(x)$ in equation (23), we obtain the following equation for $f(x)$:

$$x^2 f'' - 2f = 0\,, \quad (33)$$

which has the general solution:

$$f(x) = c_1 x^{-1} + c_2 x^2\,, \quad (34)$$

where c_1 and c_2 are arbitrary constants that can be determined by substituting equation (34) in (22). This yields $c_2 = 0$ and $c_1 = -6\nu$. Hence, $f(x) = -6\nu x^{-1}; x \neq 0$.

The following corresponding streamfunction, vorticity and velocity components are computed from equations (21), (7), (5), and (6), respectively:

$$\psi(x,y) = -6\nu \frac{y}{x} + \alpha x^3\,, \tag{35}$$

$$\xi(x,y) = 12\nu \frac{y}{x^3} - 6\alpha x\,, \tag{36}$$

$$u(x,y) = -6\frac{\nu}{x}\,, \tag{37}$$

$$\nu(x,y) = 3\alpha x^2 - 6\nu \frac{y}{x^2}\,. \tag{38}$$

It is clear that the solutions obtained do not contain arbitrary constants. However, depending on the choice of α, different flow patterns may be obtained. For example, the corresponding Riabouchinsky solution that was obtained by setting $g(x) = 0$ in equation (21) can be obtained here by setting $\alpha = 0$ in equation (35).

Case 2. Let $g(x) = \gamma + \alpha \exp(\beta, x)$, where α, β and γ are known parameters.

Using this form of $g(x)$ in equation (23), we obtain the following equation for $f(x)$:

$$f'' - \beta^2 f = -\beta^3 \nu\,. \tag{39}$$

Equation (39) has the general solution:

$$f(x) = c_1 \exp(\beta x) + c_2 \exp(-\beta x) + \beta\nu\,, \tag{40}$$

where c_1 and c_2 are arbitrary constants that can be determined by substituting equation (40) in (22). This yields $c_2 = 0$, and $c_1 \neq 0$ is an arbitrary constant. Hence, $f(x) = c_1 \exp(\beta x) + \beta\nu$. This function is the same as $f_2(x)$, that appeared in the Riabouchinsky approach in Case 2 of the previous subsection, provided that $c_1 = ab\nu$ and $\beta = a$. It should be noted that the constant c_1 can be determined provided that one condition is posed on the streamfunction of the flow, or it can be assigned different values, thus leading to various flow patterns.

The following corresponding streamfunction, vorticity and velocity components are computed from equations (21), (7), (5), and (6), respectively:

$$\psi(x,y) = y + \beta\nu y + (\alpha + c_1 y)\exp(\beta x)\,, \tag{41}$$

$$\xi(x,y) = -\beta^2(\alpha + c_1 y)\exp(\beta x)\,, \tag{42}$$

$$u(x,y) = \beta\nu + c_1 \exp(\beta x)\,, \tag{43}$$

$$v(x,y) = -\beta(\alpha + c_1 y)\exp(\beta x)\,. \tag{44}$$

3.3. Solution to the Darcy-Lapwood-Brinkman Model

It is required to solve the DarcyLapwood-Brinkman model as expressed in terms of the streamfunction equation (15). We assume that the streamfunction is linear with respect to one of the independent variables, say y, and takes the form given by equation (21).

We are thus required to find $f(x)$ and $g(x)$ such that $\psi(x,y)$ given by equation (21) satisfies equation (15). The equations that the functions $f(x)$ and $g(x)$ must satisfy are obtained by substituting equation (21) in (15), and equating the coefficients of similar powers of y. This yields the following set of coupled fourth order ordinary differential equations:

$$\nu f^{(iv)} - \frac{\nu}{k} f'' + f' f'' - f f''' = 0\,, \tag{45}$$

$$\nu g^{(iv)} - \frac{\nu}{k} g'' + g' f'' - g f''' = 0\,. \tag{46}$$

In order to follow the Riabouchinsky approach, and use the algorithm presented earlier, it is necessary to have at hand a particular solution to equation (45). A particular solution is not readily available, however. Thus, we attempt to find $f(x)$ and $g(x)$ using the modified algorithm, discussed in the last subsection. We implement the modified algorithm as follows.

Let $g(x) = \gamma + \alpha\exp(\beta x)$, where α, β, and γ are known parameters.

Using this form of $g(x)$ in equation (46), we obtain the following equation for $f(x)$:

$$f'' - \beta^2 f = -\beta^3\nu + \frac{\nu}{k}\beta\,. \tag{47}$$

Equation (47) has the general solution:

$$f(x) = c_1\exp(\beta x) + c_2\exp(-\beta x) + \beta\nu - \frac{\nu}{\beta k}\,. \tag{48}$$

where c_1 and c_2 are arbitrary constants that can be determined by substituting equation (48) in (45), equating the coefficients of $\exp(\beta x)$, and equating the coefficients of $\exp(-\beta x)$. This process yields the following results:

Case (i). $c_2 = 0$ and $c_1 \neq 0$ is arbitrary. Using these in equation (48) yields:

$$f(x) = c_1 \exp(\beta x) + \beta\nu - \frac{\nu}{\beta k}. \tag{49}$$

The streamfunction thus takes the form:

$$\psi(x, y) = \left\{ c_1 \exp(\beta x) + \beta\nu - \frac{\nu}{\beta k} \right\} y + \{\gamma + \alpha \exp(\beta x)\}. \tag{50}$$

The following corresponding vorticity and velocity components are computed from equations (7), (5), and (6), respectively:

$$\xi(x, y) = -\beta^2 (\alpha + c_1 y) \exp(\beta x), \tag{51}$$

$$u(x, y) = \beta\nu - \frac{\nu}{\beta k} + c_1 \exp(\beta x), \tag{52}$$

$$v(x, y) = -\beta(\alpha + c_1 y) \exp(\beta x). \tag{53}$$

Case (ii). $c_2 \neq 0$ and $c_1 \neq 0$ are arbitrary. This case leads to a connection between the parameter β and the permeability k, given by: $\beta^2 = \frac{1}{k}$. Using these in equation (48) yields:

$$f(x) = c_1 \exp(\beta x) + c_2 \exp(-\beta x) = \alpha_1 \exp(1/\sqrt{k}) + \alpha_2 \exp(-1/\sqrt{k}), \tag{54}$$

where $a_1 = c_1, a_2 = c_2$ if the positive square root of β^2 is used, and $a_1 = c_2, a_2 = c_1$ if the negative square root of β^2 is used in equation (54).

The streamfunction thus takes the following form, written in terms of β:

$$\psi(x, y) = \{c_1 \exp(\beta x) + c_2 \exp(-\beta x)\} y + \{\gamma + \alpha \exp(\beta x)\}. \tag{55}$$

The following corresponding vorticity and velocity components are computed from equations (7), (5), and (6), respectively:

$$\xi(x, y) = -\beta^2 \{(\alpha + c_1 y) \exp(\beta x) + c_2 y \exp(-\beta x)\}, \tag{56}$$

$$u(x, y) = c_1 \exp(\beta x) + c_2 \exp(-\beta x), \tag{57}$$

$$v(x, y) = -\beta(\alpha + c_1 y) \exp(\beta x) + c_2 \beta y \exp(-\beta x). \tag{58}$$

The above solution of the Darcy-Lapwood-Brinkman model provides a first step in the analysis of flow patterns that result for different values of the parameters involved. The effect of the permeability can be studied by comparing the DLB solution to the solution obtained for the Navier-Stokes equations, for the same $g(x)$.

It should be emphasized here that our ability to obtain a solution to the model equations, for a given form of $g(x)$, is predicated upon our ability to solve the resulting ordinary differential equation for $f(x)$ and obtain values of, or conditions on the arbitrary constants and the parameters involved.

At this point we contend that the non-trivial, non-constant form of $g(x)$ that results in determinate values of the arbitrary constants is the linear combination of a constant function and an exponential function of x. An attempt has been made to use a power function for $g(x)$ of the forms: $g(x) = \alpha x^{\lambda}$; a non-linear polynomial form; and a periodic trigonometric function. However, these have not rendered determinate values for the arbitrary constants involved.

4. Conclusions

In this work, we offered a modification to the solution steps that have been typically followed in the solution of a special type of flow problems (one with the streamfunction being linear in one of the independent variables, of the form: $\psi(x, y) = yf(x) + g(x)$).

The typical approach gives rise to a number of arbitrary constants the determination of which may be restrictive. Furthermore, the form of the solution obtained is rather complicated. In an attempt to simplify the approach, Riabouchinsky assigned $g(x)$ to zero, thus rendering a value of zero to each arbitrary constant. Our current modification avoids these drawbacks, and offers a means of implicitly determining the form of the solution without restrictive assumptions to determine the arbitrary constants. The number of arbitrary constants is considerably less than in the typical approach.

Using the modified approach, we obtained the same form of solution that has been obtained for the linearised Navier-Stokes equations using the typical approach (without the extensive number of arbitrary constants). In addition, the modified approach has been implemented in this work to solve a linearised form of the Darcy-Lapwood-Brinkman model, which governs flow through porous media. It should be noted that the Riabouchinsky approach would have been restrictive in this case due to the absence of a particular solution to one of the coupled ordinary differential equations.

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