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EXACT AND ASYMPTOTIC SOLUTIONS OF THE ELASTIC WAVE PROPAGATION PROBLEM IN A ROD

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Abstract: An exact solution to the problem of elastic wave propagation in a bar of finite length with variable Young's modulus is presented. The problem is reduced to a Bessel equation using the Laplace transform and asymptotic form of the Bessel function is used to obtain the solution in the final form.

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1. Introduction

The study of elastic waves in rods and bars is of importance in various engineering applications. The idealized model of a homogeneous and isotropic elastic bar presents the simplest case in which time-harmonic and transient waves have been considered under a variety of boundary conditions [1]. In many practical situations, the bar either has impurities or inhomogeneities causing change in the Young's modulus. In some cases corrosion may also cause variation in the elastic properties of the bar. In such situations, the method of choice has been mostly perturbation method; see for example Dutta [4] and Vasudeva [7]. Recently, Bhattacharya and Bera [3] considered transient waves in a rod having

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small variations in the Young's modulus and dimensions of the rod. They used the Laplace transform to obtain a perturbed second order ordinary differential equation and presented a solution using the Adomian method [2]. However, as the problem is in the transformed domain, term by term inversion may not be possible until the transformed function $F(s) \to 0$ as $s \to \infty$ for any function $f(t)$ [5]. This poses some concerns about validity of such a procedure that involves an infinite series of transformed functions.

In this note, we present an alternate method to solve the Laplace transformed problem based upon reduction of the transformed equation in an equation of the Bessel type. An exact solution to this equation can then be found which, using the asymptotic form of the Bessel function involved does not pose any violation to the limiting behavior of the transformed function. This could then be extended to include random variations in the elastic properties of the material.

2. Formulation of the Problem

Let us consider an elastic rectangular bar of length l and cross section area a, constant density ρ a variable Young's modulus $E(x)$. The variation, for the time being need not to be assumed to be small. We assume that one end of the bar is subjected to a displacement while the other end is kept fixed. Assuming that the initial displacement and the velocity is zero, the governing equation and the initial and boundary conditions can be stated as [3]

$$\frac{\partial}{\partial x}\left[aE\frac{\partial u}{\partial x}\right] = \rho a \frac{\partial^2 u}{\partial t^2}, \tag{1}$$

$$u(x,0) = \frac{\partial u(x,0)}{\partial t} = 0, \tag{2}$$

$$u(x,t) = \begin{cases} f(t), & \text{at } x = l \\ 0, & \text{at } \quad x = 0 \end{cases} \tag{3}$$

where $f(t)$ is a function of t. Let us write the Young's modulus as sum of a constant E_0, corresponding its value in the background medium and a variation $E(x)$, which measures its variation form the background value. Thus

$$E = E_0\left(1 + \frac{E_1(x)}{E_0}\right). \tag{4}$$

We shall assume that $E_1(x)$ is a linear function of x given by $E_1(x) = \alpha E_0 x$, where α is a constant.

Putting value of E in (1), and taking Laplace transform in t, denoting the Laplace transform of $u(t)$ by $U(s), s$ being the Laplace transform parameter, we get, after some re-arrangement

$$(1+\alpha x)\,\frac{d^2U}{dx^2}+\alpha\frac{dU}{dx}-\frac{s^2}{c^2}U=0\,, \tag{5}$$

where $c=\sqrt{E_0/\rho}$ is the longitudinal wave velocity for the bar with constant elastic parameters.

3. Solution in Exact Form

We find an exact form of the solution of (5). If we put $z=(1+\alpha x)$, we obtain from (5)

$$z^2\frac{d^2U}{dx^2}+z\frac{dU}{dz}-z\frac{s^2}{c^2\alpha^2}U=0\,. \tag{6}$$

Eq. (6) can be transformed into a modified Bessel equation having the general solution in terms of the modified Bessel functions of the first and second kind [6]

$$U(z,s)=C_1I_0(\frac{2s}{\alpha c}\sqrt{z})+C_2K_0(\frac{2s}{\alpha c}\sqrt{z})\,. \tag{7}$$

The boundary conditions at $x=0$ and $x=l$ will now become at $z=l$ and $z=1+\alpha l$, respectively. These lead to

$$C_1I_0(\frac{2s}{\alpha c})+C_2\left(\frac{2s}{\alpha c}\right) \quad = \quad 0, \tag{8}$$

$$C_1I_0(\frac{2s}{\alpha c})\sqrt{1+\alpha l}+C_2K_0(\frac{2s}{\alpha c}\sqrt{1+\alpha l}) \quad = \quad F(s). \tag{9}$$

The system of equations (8), (9) can be solved to obtain

$$C_1=-K_0(\frac{2s}{\alpha c})\frac{F(s)}{\triangle}\,,\quad C_2=I_0(\frac{2s}{\alpha c})\frac{F(s)}{\triangle}, \tag{10}$$

where $L=\sqrt{1+\alpha l}$ and

$$\triangle=I_0(\frac{2s}{\alpha c})K_0(\frac{2s}{\alpha c}L)-I_0(\frac{2s}{\alpha c}L)K_0(\frac{2s}{\alpha c})\,. \tag{11}$$

The solution $U(z,s)$ in the transformed domain now vanishes as $s\to\infty$.

4. Asymptotic Solution

If we assume that the variation E_1 in the Young's modulus is small, we have $\alpha \ll 1$, and so we can use the asymptotic approximations of the modified Bessel functions [5],

$$I_0(x) \sim \frac{e^x}{\sqrt{2\pi x}}\,, \tag{12}$$

and

$$K_0(x) \sim e^{-x}\sqrt{\frac{\pi}{2x}}\,, \tag{13}$$

to obtain

$$\triangle \sim -\frac{\alpha c}{4L^{1/2}s}\left\{e^{\frac{2s}{\alpha c}(L-1)} - e^{-\frac{2s}{\alpha c}(L-1)}\right\}. \tag{14}$$

The asymptotic form of the solution for $\alpha \ll 1$ can now be obtained using (7) as

$$U(s,z) = \frac{L^{1/2}}{z^{1/4}}F(s)\left\{\frac{e^{\frac{2s}{\alpha c}(z^{1/2}-1)} - e^{-\frac{2s}{\alpha c}(z^{1/2}-1)}}{e^{\frac{2s}{\alpha c}(L-1)} - e^{-\frac{2s}{\alpha c}(L-1)}}\right\}. \tag{15}$$

The denominator can be expanded as the Binomial series and thus (15) becomes

$$\begin{aligned} U(s,z) = {} & \frac{L^{1/2}}{z^{1/4}}F(s) \\ & \times \sum_{n=0}^{\infty} e^{-\frac{2s}{\alpha c}(L-z^{1/2}+2n(L-1))} - e^{-\frac{2s}{\alpha c}(L+z^{1/2}-2+2n(L-1))}. \end{aligned} \tag{16}$$

Finally, the inverse Laplace transform gives, setting $N = L + 2n(L-1)$

$$\begin{aligned} u(z,t) = {} & \frac{L^{1/2}}{z^{1/4}}\sum_{n=0}^{\infty}\left[f\left\{t - \frac{2}{\alpha c}(N - z^{1/2})\right\} H\left\{t - \frac{2}{\alpha c}(N - z^{1/2})\right\}\right. \\ & \left. - f\left\{t - \frac{2}{\alpha c}(N - 2 + z^{1/2}\right\} H\left\{t - \frac{2}{\alpha c}(N - 2 + z^{1/2}\right\}\right], \end{aligned} \tag{17}$$

where H denotes the unit step function.

5. Conclusion

The solution of a rectangular elastic bar of finite length with variable Young's modulus has been presented under more general boundary conditions without assuming a series of perturbation or Adomian type. The present approach does not involve any power series in the transform parameter s and so term by term Laplace inversion does not pose any problem. Moreover, this approach enables one to get the exact solution in a closed form. The results in [3] dealing with determinate variation in the Young's modulus can be recovered as a special case of our solution.

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