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ON A NUMERICAL SOLUTION TO THE DIRICHLET PROBLEM

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Abstract: It is proved in this paper that any solution to the Dirichlet boundary value problem for the homogeneous equation $(I - \Delta)\, u = 0$ in a bounded domain $\Omega \subset \mathbb{R}^n$ (I and Δ being the identity operator and the Laplacian, respectively) is represented in the form of the volume potential with a density supported in an arbitrarily thin boundary layer exterior to $\partial\Omega$. As a result, the Dirichlet problem is reduced to an integral equation with an unknown density defined in the thin boundary layer. An approximate solution to the latter integral equation generates a rather simple new numerical algorithm solving the $3D$ Dirichlet problems. This algorithm is different from the finite difference, finite element, or boundary element methods. It can be called a boundary layer element method. Examples of its accuracy are presented. All the results are obtained not just for the operator $I - \Delta$ but also for an arbitrary elliptic differential operator in $\mathbb{R}^n$ of an even order with constant coefficients, as well as for boundary value problems in interior and exterior domains of $\mathbb{R}^n$.

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1. Introduction

Some prerequisites, such as definitions of the Fourier transform, Sobolev spaces and so on, are given in the Appendix for the reader's convenience.

Let I be the identity operator, Δ the Laplacian, $\mathcal{S}(\mathbb{R}^n)$ the usual space of rapidly decreasing functions, and $H^s(\mathbb{R}^n)$ the standard Sobolev spaces (cf. Appendix). The latter are natural spaces for the solvability of elliptic boundary value problems.

It is known that the operator $I-\Delta$ acting in the whole space $\mathbb{R}^n$ generates two isomorphisms: $I-\Delta : \mathcal{S}(\mathbb{R}^n) \simeq \mathcal{S}(\mathbb{R}^n)$ and $I-\Delta : H^s(\mathbb{R}^n) \simeq H^{s-2}(\mathbb{R}^n)$, $s \in \mathbb{R}$. Moreover, the solution to the equation $(I-\Delta)u(x) = f(x)$ $(x \in \mathbb{R}^n)$ can be represented by an explicit formula

$$u(x) = \mathcal{F}^{-1}_{\xi\to x}\left(1+|\xi|^2\right)^{-1}\mathcal{F}_{x\to\xi} f(x), \tag{1}$$

or by

$$u(x) = \int_{\mathbb{R}^n} \Phi(x-\xi) f(\xi)\, d\xi,$$

where $x = (x_1, ..., x_n) \in \mathbb{R}^n$, $\xi = (\xi_1, ..., \xi_n) \in \mathbb{R}^n$, $|\xi| = \sqrt{\xi_1^2 + ... + \xi_n^2}$, $\mathcal{F}_{x\to\xi}$ is the Fourier transform, and $\mathcal{F}^{-1}_{\xi\to x}$ its inverse.

Let $\Omega := \Omega_+$ be a bounded connected open set in $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$ (C^∞-class surface) and $\overline{\Omega}$ its closure, i.e., $\overline{\Omega} = \Omega \cup \partial\Omega$. We denote by $\Omega_- := \mathbb{R}^n \setminus \overline{\Omega}$ the exterior of the domain Ω.

Near $\partial\Omega$ a normal vector field $\mathbf{n}(x) = (n_1(x), ..., n_n(x))$ is defined, as follows: for $x_0 \in \partial\Omega$, $\mathbf{n}(x_0)$ is the unit normal to $\partial\Omega$, pointing towards the exterior of Ω. We set

$$\mathbf{n}(x) := \mathbf{n}(x_0) \text{ for } x \text{ of the form } x = x_0 - s\mathbf{n}(x_0) =: \zeta(x_0, s),$$

where $x_0 \in \partial\Omega$, $s \in (-\varepsilon, \varepsilon)$. Here $\varepsilon > 0$ is taken to be so small that the representation of x in terms of $x_0 \in \partial\Omega$ and $s \in (-\varepsilon, \varepsilon)$ is unique and smooth, i.e., ζ is bijective and C^∞ with C^∞ inverse, from $\partial\Omega \times (-\varepsilon, \varepsilon)$ to the set $\zeta(\partial\Omega \times (-\varepsilon, \varepsilon)) \subset \mathbb{R}^n$. We denote by $\Omega_+^{bl} := \zeta(\partial\Omega \times (0, \varepsilon))$ the interior boundary layer to $\partial\Omega$, and by $\Omega_-^{bl} := \zeta(\partial\Omega \times (-\varepsilon, 0))$ the interior boundary layer to $\partial\Omega$. Let us note that $\Omega_+^{bl} \subset \Omega$ and $\Omega_-^{bl} \subset \complement\Omega$.

Let us consider the interior/exterior Dirichlet boundary value problem for the diffusion equation with dissipation

$$\begin{cases} (I-\Delta)u(x) = 0, & x \in \Omega_\pm, \\ u|_{\partial\Omega} = g(x), & x \in \partial\Omega. \end{cases} \tag{2}$$

Let the function

$$\Phi(x) = \begin{cases} (n-2)^{-1}\,\sigma_n^{-1}|x|^{2-n}e^{-|x|} & n>2, \\ (2\pi)^{-1}\,K_0(|x|) & n=2 \end{cases}$$

be the fundamental solution to the equation $(I-\Delta)\,u=f$, where

$$\sigma_n = 2\pi^{n/2}/\Gamma(n/2)\,, \quad \Gamma(z)=\int_0^\infty e^{-t}t^{z-1}dt, \quad K_0(x)=\int_0^\infty e^{-x\cosh t}dt.$$

In this article, we show that any solution to (2) is represented in the form of the volume potential with a density supported in the arbitrarily thin boundary layer (i.e., ε can be arbitrarily small). The boundary layer is exterior for Ω_+ and interior for Ω_-, i.e.,

$$u(x)=\int_{\Omega_\mp^{bl}}\Phi(x-\xi)\,f(\xi)\,d\xi \qquad x\in\Omega_\pm.$$

As a consequence, the interior/exterior Dirichlet problem for the homogeneous equation $(I-\Delta)\,u(x)=0$ $(x\in\Omega_\pm)$ is reduced to an integral equation

$$g(x)=\int_{\Omega_\mp^{bl}}\Phi(x-\xi)\,f(\xi)\,d\xi \qquad x\in\partial\Omega$$

with unknown density $f(\xi)$ defined in a thin exterior/interior boundary layer of $\partial\Omega$. An approximate solution of the latter integral equation generates a rather simple new numerical algorithm solving the $2D$ and $3D$ exterior/interior Dirichlet problem. The algorithm is different from the finite difference, finite element or boundary element methods, and can be called a boundary layer element method.

In the next section, instead of $I-\Delta$, we consider a more general case, i.e., an arbitrary elliptic differential operator P of even order $2m$ with constant coefficients, which is invertible in the whole space $\mathbb{R}^n$. The case of constant coefficients has been chosen here just because of the fact that in this case the fundamental solution $\Phi(x)$ can be found explicitly.

2. Preliminaries

Let P be a linear differential operator in $\mathbb{R}^n$ of order $2m$ $(m \in \mathbb{N}_+)$, with constant coefficients $a_\alpha \in \mathbb{C}$, i.e.

$$P := P(D) := \sum_{|\alpha| \leq 2m} a_\alpha D^\alpha. \tag{3}$$

We consider two equations

$$Pu(x) = f(x), \quad \text{either } x \in \Omega_+ \text{ or } x \in \Omega_-, \tag{4}$$

in short

$$Pu(x) = f(x), \quad x \in \Omega_\pm.$$

We investigate the solvability of (4) in the standard Sobolev spaces $H^m(\Omega_\pm)$ (cf. Appendix for the definition of the Sobolev spaces).

The polynomials

$$p(\xi) := \sum_{|\alpha| \leq 2m} a_\alpha \xi^\alpha \quad \text{and} \quad p_0(x, \xi) := \sum_{|\alpha| = 2m} a_\alpha \xi^\alpha$$

are called the symbol and the principal symbol of P, respectively.

It is known (cf. [13, Proposition 7.2]) that the map $P(D) : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is continuous, where $\mathcal{S}(\mathbb{R}^n)$ is the usual space of rapidly decreasing functions (cf. Appendix).

We will use the following assumptions.

Assumption 1. The operator P is md-elliptic (cf. [1, p. 23]). For the operator P with constant coefficients, this means that $p(\xi) \neq 0$ for all $\xi \in \mathbb{R}^n$.

Assumption 2$_-$. The operator P is md-properly elliptic (cf. [1, Assumption 1, p. 40]), i.e., the polynomial in the complex variable z, $p(\xi', z)$ has exactly m zeroes $\tau_1(\xi'), ..., \tau_m(\xi')$ with a positive imaginary part for all $\xi' \in \mathbb{R}^{n-1}$.

We conclude from Assumptions 1 and 2$_-$ that the polynomial $p(\xi', z)$ has no real zeroes as well as exactly m zeroes with a negative imaginary part for all $\xi' \in \mathbb{R}^{n-1}$. Since in $\mathbb{R}^n$ for $n \geq 2$ one can map two arbitrary, linearly independent vectors to two given linearly independent vectors by means of non-singular linear transformation, we obtain, similarly to [12, Section 10], the following equivalent statement of Assumption 2$_-$.

If $\xi, \eta \in \mathbb{R}^n$ are linearly independent vectors, the polynomial in the complex variable z, $p(\xi + z\eta)$ has exactly m zeroes with positive imaginary parts.

Assumption 2_+. The operator P is properly elliptic, i.e., the polynomial in the complex variable z, $p_0(\xi', z)$ has exactly m zeroes $\tau_1(\xi'), ..., \tau_m(\xi')$ with positive imaginary parts for all $\xi' \in \mathbb{R}^{n-1}$.

If $P(D)$ satisfies Assumption 1, then it is an invertible operator, i.e., the inverse bounded operator $P^{-1}(D) : H^{s-2m}(\mathbb{R}^n) \to H^s(\mathbb{R}^n)$ exists for all $s \in \mathbb{R}$. Moreover, the inverse operator can be represented by the Fourier transform $\mathcal{F}_{x\to\xi}$, and its inverse $\mathcal{F}^{-1}_{\xi\to x}$ as follows:

$$P^{-1}(D) = \mathcal{F}^{-1}_{\xi\to x}\frac{1}{p(\xi)}\mathcal{F}_{x\to\xi}\,. \tag{5}$$

The operator $P(D)$, acting on the space $C_0^\infty(\Omega)$ of finite smooth functions over Ω, can be extended to an operator generated by the boundary value problem with homogeneous boundary conditions. Using the approach of [10], [7], [6], the latter operator can be extended to an isomorphism acting between corresponding Sobolev spaces of positive and negative order over Ω. In particular, the operator $P(D)$, which is properly elliptic in the usual sense (cf. [12, Section 10]), is one-to-one mapping $C_0^\infty(\Omega) \to C_0^\infty(\Omega)$ and can be extended to the operator generated by the Dirichlet boundary value problem with homogeneous boundary conditions to an isomorphism (see [7])

$$H_0^m(\Omega) \times H^{-1/2}(\partial\Omega) \times H^{-3/2}(\partial\Omega) \times ... \times H^{-m+1/2}(\partial\Omega) \to H_0^{-m}(\Omega)\ ,$$

where $H_0^m(\Omega)\,,\ \left(H_0^{-m}(\Omega)\right)$ is a subspace of $H^m(\mathbb{R}^n)\,,\ (H^{-m}(\mathbb{R}^n))$ of functions (distributions) supported in $\overline{\Omega}$. However, the inverse operator in this case does not have as simple a form as (5) has.

The aim of the present article is to show that there exists another extension $\mathcal{P}_\pm$ to the operator $P(D) : C_0^\infty(\Omega_\pm) \to C_0^\infty(\Omega_\pm)$, which is an isomorphism of the space $H^m(\Omega_\pm)$ onto $H_0^{-m}(\Omega_\pm)$ and such that the inverse operator is represented by a formula similar to (5).

It is the simplicity of formula (5) that necessitates constant coefficients for the operator P (is the simplicity of the formula (5)). For the operator P with variable coefficients, we would have to use a more complicated analog of the formula (5).

In comparison with usual boundary value problems, here the extension $\mathcal{P}_\pm$ is more complicated than the operator generated by the elliptic boundary value problem. Conversely, the inverse operator $\mathcal{P}_\pm^{-1}$ is much more simple than the operators solving the usual boundary value problems, which are generated by the Green's function. The exact construction of the operator $\mathcal{P}_\pm$ is given in the next section.

We denote by $S_{1,0}^{r}(\mathbb{R}^n)$ the space consisting of functions (symbols) $p(\xi) \in C^{\infty}(\mathbb{R}^n)$ satisfying (for some $r \in \mathbb{R}$) the following estimates for some $r \in \mathbb{R}$: $\left|D_{\xi}^{\alpha} p(\xi)\right| \leq C_{\alpha} \langle\xi\rangle^{r-|\alpha|}$, for all $\alpha \in \mathbb{N}^n$, where C_α is a constant depending on α.

Let $\mathcal{D}$ denote the vector-operator $\mathcal{D} = \left(1, D_{\mathbf{n}}, ..., D_{\mathbf{n}}^{m-1}\right)$.

Assumption $3_{\pm}$. Let the boundary value problem

$$\begin{cases} P(D)\, v(x) = 0, & x \in \Omega_{\pm}\,, \\ \mathcal{D}v|_{\partial\Omega} = \underline{\chi}\,, \end{cases} \tag{6}$$

be uniquely solvable respectively in Ω_+ or in Ω_- for all smooth vector-functions $\underline{\chi}$ defined below in an Auxiliary Theorem.

Auxiliary Theorem. I. *If P is a pseudodifferential operator with symbol $p(\xi)$, i.e. $P :=\mathrm{OP}_x(p) := \mathcal{F}_{\xi\to x}^{-1} p(\xi)\, \mathcal{F}_{x\to\xi}$, where $p(\xi) \in S_{1,0}^{r}(\mathbb{R}^n)$, then P is continuous for all $s \in \mathbb{R}$, i.e., $P : H^s(\mathbb{R}^n) \to H^{s-r}(\mathbb{R}^n)$.*

II. *If the operator P satisfies Assumptions $1, 2_+, 3_+$ or $1, 2_-, 3_-$, then, for any $s \in \mathbb{R}$, there exists a bounded (Poisson) operator*

$$\mathbf{K}_{\pm} : \prod_{j=0}^{m-1} H^{s+2m-j-1/2}(\partial\Omega) \to H^{s+2m}(\Omega_{\pm})$$

which gives a unique solution $v = \mathbf{K}_{\pm}\underline{\chi}$ to the boundary value (6) problem with

$$\underline{\chi} = (\chi_1, ..., \chi_m) \in \prod_{j=0}^{m-1} H^{s+2m-j-1/2}(\partial\Omega)\,.$$

More precisely, the operator $v = \mathbf{K}_{\pm}\underline{\chi}$ solves the problem with $s < 0$ in the sense that $v = \mathbf{K}_{\pm}\underline{\chi}$ is the limit in the space $H^{s+2m}(\Omega_{\pm})$ of a sequence v_n in $H^{2m}(\Omega_{\pm})$ with

$$P(D)\, v_n = 0, \qquad \mathcal{D}v|_{\partial\Omega} = \underline{\chi}_n, \quad \underline{\chi}_n \to \underline{\chi} \text{ in } \prod_{j=0}^{m-1} H^{s+2m-j-1/2}(\partial\Omega)$$

$$\text{for } n \to \infty.$$

The proof of statement I is known (cf. [13, Section 7.6]).

For Ω_+, statement II of the Auxiliary Theorem is a very particular case of the Grubb's result [3, Theorem 5.4]. For Ω_-, statement II of the Auxiliary Theorem is a particular case of the Schrohe's result [11, Theorem 2.19]. This particular form of statement II of the Auxiliary Theorem is used just for simplicity of exposition.

Example. It is not difficult to check that all the conditions of statement II of the Auxiliary Theorem hold for the following Dirichlet problem in Ω_+ as well as in Ω_- :

$$\begin{cases} (I-\Delta)\, v(x) = 0, & x \in \Omega_\pm\,, \\ v|_{\partial\Omega} = g \in H^{s+2m-1/2}(\partial\Omega)\,. \end{cases} \tag{7}$$

Therefore, for any $g \in H^{s+2m-1/2}(\partial\Omega)$, there exists a unique solution $v \in H^{s+2m}(\Omega_\pm)$ to (7).

3. Main Theorem

The principal result of the present paper is the following theorem.

Theorem. *If the differential operator $P(D)$ of even order $2m$ is given by (3) and satisfies the Assumptions 1, 2 and 3,*
then the mapping $\mathcal{S}_\pm : f_\pm \to v$ given by the formula

$$v(x) := \mathcal{S}_\pm f_\pm(x) := \int\limits_{\Omega^{bl}_\pm} \Phi(x-y)\, f_\pm(y)\, dy \qquad x \in \complement\Omega^{bl}_\pm, \quad f_\pm \in H_0^{-m}\left(\Omega^{bl}_\pm\right)$$

is a surjective mapping

$$\mathcal{S}_\pm : H_0^{-m}\left(\Omega^{bl}_\pm\right) \to \left\{ v \in H^m\left(\complement\Omega^{bl}_\pm\right) : Pv = 0 \text{ in } \complement\overline{\Omega^{bl}_\pm} \right\}.$$

Moreover, the function $v(x) := \mathcal{S}_\pm f_\pm(x)$ is a solution to the Dirichlet boundary value problem

$$\begin{cases} P(D)\, v(x) = 0, \quad x \in \complement\overline{\Omega^{bl}_\pm}, \quad v \in H^m\left(\complement\overline{\Omega^{bl}_\pm}\right), \\ (\mathcal{D}v_\mp)|_{\partial\Omega^{bl}_\pm} = g \in H^{m-1/2}\left(\partial\Omega^{bl}_\pm\right) \times H^{m-3/2}\left(\partial\Omega^{bl}_\pm\right) \times \ldots \times H^{1/2}\left(\partial\Omega^{bl}_\pm\right), \end{cases}$$

if and only if $f_\pm(x)$ is a solution to the integral equation

$$g(x) = \mathcal{D} \int\limits_{\Omega^{bl}_\pm} \Phi(x-y)\, f_\pm(y)\, dy \qquad x \in \partial\Omega^{bl}_\pm.$$

The theorem will be proved below after some preparations. The following corollary is an immediate consequence of the theorem.

Corollary 1. (Regularity) *If* $v \in H^m\left(\complement\overline{\Omega^{bl}_{\pm}}\right)$ *is a solution of the equation* $v(x) := \mathcal{S}_{\pm} f_{\pm}(x)$, *where* $f \in H_0^{-m}\left(\Omega^{bl}_{\pm}\right) \cap H^{\ell}(\mathbb{R}^n) \quad (\ell \geq -m)$, *then* $v \in H^{\ell+2m}\left(\complement\overline{\Omega^{bl}_{\pm}}\right)$.

Example. It is not difficult to check that the operator $I - \Delta$ satisfies Assumptions $1_-, 2_-$ and 3_- and therefore, by the Theorem, is an isomorphism of the space $H^1(\Omega_+)$ onto $H_0^{-1}(\Omega_+)$.

In the following, let $P(D)$ be an md-elliptic and md-properly elliptic differential operator of even order $2m$, i.e., the operator $P(D)$ is given by (3) and satisfies Assumptions 1_- and 2_- or Assumptions 1_+ and 2_+.

Notation. Let $F \in C^{\infty}(\Omega) \cap \mathcal{S}\left(\complement\Omega\right)$ (cf. Appendix) and have finite jumps F_k of its normal derivative of order k $(k = 0, 1, ...)$ on $\partial\Omega$. For $x_0 \in \partial\Omega$, we use the following notation:

$$F_0(x_0) := [F]_{\partial\Omega}(x_0) := \lim_{\varepsilon \to +0} \left(F(x_0 + \varepsilon\mathbf{n}) - F(x_0 - \varepsilon\mathbf{n})\right),$$

$F_k(x_0) := \left[D^k_{\mathbf{n}} F\right]_{\partial\Omega}(x_0)$, where $\mathbf{n} = \mathbf{n}(x_0)$ denotes the exterior unit normal vector to $\partial\Omega$ at point $x_0 \in \partial\Omega$ (cf. Appendix).

We denote by $\{D^{\alpha} F(x)\}$ the classical derivative in the points where it exists.

Let $\delta_{\partial\Omega}$ denote the Dirac measure concentrated on $\partial\Omega$, i.e., a distribution acting as $(\delta_{\partial\Omega}, \varphi) := \int_{\partial\Omega} \overline{\varphi}(x)\, dS, \quad \varphi(x) \in C_0^{\infty}(\mathbb{R}^n)$ where dS is the surface measure on $\partial\Omega$ and $\overline{\varphi}(x)$ means the complex conjugate number to $\varphi(x)$.

It is known that for any differential operator P of order $2m$ there exists a representation $P = \sum_{j=0}^{2m} P_j D^j_{\mathbf{n}}$, where P_j is a tangential differential operator (cf. Appendix) of the order $2m - j$.

The following Lemma 1 shows a well-known distributional way of writing Green's formula.

Lemma 1. *Using the previous notation the following equality holds for the distribution* PF*:*

$$PF = \{PF\} - i \sum_{j=0}^{2m} P_j \sum_{k=0}^{j-1} D^k_{\mathbf{n}} \left(F_{j-1-k}\delta_{\partial\Omega}\right). \tag{8}$$

Definition. (Operators $\mathcal{P}_{\pm}$ and $E_{\pm}$) Denoting in the sequel as $e_+ u,\ e_- v$, the extensions by zero to $\mathbb{R}^n$ of functions $u(x) \in C^{\infty}\left(\overline{\Omega^{bl}_{\pm}}\right)$, $v(x) \in \mathcal{S}\left(\complement\Omega^{bl}_{\pm}\right)$, we obtain by Lemma 1,

$$e_+(Pu) = P(e_+ u) - i \sum_{j=1}^{2m} P_j \sum_{k=0}^{j-1} D^k_{\mathbf{n}} \left(\left(D^{j-1-k}_{\mathbf{n}} u\right)\Big|_{\partial\Omega^{bl}_{\pm}} \cdot \delta_{\partial\Omega^{bl}_{\pm}} \right),$$

$$\left(u \in C^{\infty}\left(\overline{\Omega_{\pm}^{bl}}\right)\right), \quad (9)$$

$$e_{-}(Pv) = P(e_{-}v) + i\sum_{j=1}^{2m} P_j \sum_{k=0}^{j-1} D_{\mathbf{n}}^{k}\left(\left(D_{\mathbf{n}}^{j-1-k}v\right)\Big|_{\partial\Omega_{\pm}^{bl}} \cdot \delta_{\partial\Omega_{\pm}^{bl}}\right)$$
$$\left(v \in \mathcal{S}\left(\complement\Omega_{\pm}^{bl}\right)\right). \quad (10)$$

In view of the latter formulae, we define the action of the operators $\mathcal{P}_{+}$ and $\mathcal{P}_{-}$ as follows:

$$\mathcal{P}_{\pm}u := P(e_{\pm}u) \mp i\sum_{j=1}^{2m} P_j \sum_{k=0}^{j-1} D_{\mathbf{n}}^{k}\left(\phi_{j-k} \cdot \delta_{\partial\Omega_{\pm}^{bl}}\right), \quad u \in C^{\infty}\left(\overline{\Omega_{\pm}^{bl}}\right) \quad (11)$$

where $\phi_j := \left(D_{\mathbf{n}}^{j-1}u\right)\Big|_{\partial\Omega_{\pm}^{bl}} = r_{\partial\Omega_{\pm}^{bl}}D_{\mathbf{n}}^{j-1}u$ for $j = 1, ..., m$, and $\phi_j := \left(D_{\mathbf{n}}^{j-1}v\right)\Big|_{\partial\Omega_{\pm}^{bl}} = r_{\partial\Omega_{\pm}^{bl}}D_{\mathbf{n}}^{j-1}v$ for $j = m+1, ..., 2m$. Here v denotes the unique solution to the Dirichlet problem

$$\begin{cases} P(D)\,v(x) = 0, & x \in \complement\overline{\Omega_{\pm}^{bl}}, \\ r_{\partial\Omega_{\pm}^{bl}}D_{\mathbf{n}}^{j-1}v = \phi_j, & j = 1, ..., m\,. \end{cases} \quad (12)$$

This unique solution exists by Assumption $3_{\mp}$ and by the Auxiliary Theorem.

Considering v satisfying (12) with $\phi_{j+1} = r_{\partial\Omega}D_{\mathbf{n}}^{j}u$ $(j = 0, ..., m-1)$ as an extension-function for $u \in H^m\left(\Omega_{\pm}^{bl}\right)$, let us define two extension operators $E_{+} : H^m\left(\Omega_{+}^{bl}\right) \to H^m(\mathbb{R}^n)$ and $E_{-} : H^m\left(\Omega_{-}^{bl}\right) \to H^m(\mathbb{R}^n)$ as follows:

$$E_{\pm}u \in H^m(\mathbb{R}^n), \quad E_{\pm}u(x) := \begin{cases} u(x) \in H^m\left(\Omega_{\pm}^{bl}\right), & x \in \Omega_{\pm}^{bl}, \\ v(x) \in H^m\left(\complement\overline{\Omega_{\pm}^{bl}}\right), & x \in \complement\overline{\Omega_{\pm}^{bl}}. \end{cases} \quad (13)$$

To describe the range of the extension operator $E_{\pm}$, we consider the following subspace of $H^m(\mathbb{R}^n)$:

$$H_{E_{\pm}}^{m}(\mathbb{R}^n) := \left\{\{u, v\} \in H^m\left(\Omega_{\pm}^{bl}\right) \times H^m\left(\complement\overline{\Omega_{\pm}^{bl}}\right),\ v \text{ satisfies } (12)\right\}. \quad (14)$$

By (13), it is clear that $H_{E_{\pm}}^{m}(\mathbb{R}^n)$ is the range of the operator $E_{\pm}$.

Lemma 2. *Under conditions of Theorem, the operator* $E_{+} : H^m\left(\Omega_{+}^{bl}\right) \to H_{E_{+}}^{m}(\mathbb{R}^n)$ *as well as* $E_{-} : H^m\left(\Omega_{-}^{bl}\right) \to H_{E_{-}}^{m}(\mathbb{R}^n)$ *is an isomorphism.*

Proof of Lemma 2. Let us prove that $H^m_{E_\pm}(\mathbb{R}^n)$ is a closed subspace of the space $H^m\left(\Omega^{bl}_\pm\right) \times H^m\left(\complement\overline{\Omega^{bl}_\pm}\right)$. Indeed, if $\{u_k, v_k\} \in H^m_{E_\pm}(\mathbb{R}^n)$ is a Cauchy sequence, then $\{u_k\}$ is a Cauchy sequence in the complete space $H^m\left(\Omega^{bl}_\pm\right)$, and therefore there exists $u = \lim_{k\to\infty} u_k \in H^m\left(\Omega^{bl}_\pm\right)$. Moreover, we have for the traces of their normal derivatives $r_{\partial\Omega^{bl}_\pm} D^j_{\mathbf{n}} u = \lim_{k\to\infty} r_{\partial\Omega^{bl}_\pm} D^j_{\mathbf{n}} u_k \in H^{m-j-1/2}\left(\partial\Omega^{bl}_\pm\right)$, where $j = 0, 1, ..., m-1$. Substituting $\phi_{j+1} = r_{\partial\Omega^{bl}_\pm} D^j_{\mathbf{n}} u_k$ $(j = 0, ..., m-1)$ into the problem (12) we get a solution v_k. By Auxiliary Theorem, there exists $v \in H^m\left(\complement\overline{\Omega^{bl}_\pm}\right)$ such that

$$\begin{aligned} v &= \lim_{k\to\infty} v_k = \lim_{k\to\infty} \mathbf{K}_\pm r_{\partial\Omega^{bl}_\pm}\left(v_k, D^1_{\mathbf{n}} v_k, ..., D^{m-1}_{\mathbf{n}} v_k\right) \\ &= \lim_{k\to\infty} \mathbf{K}_\pm r_{\partial\Omega^{bl}_\pm}\left(u_k, D^1_{\mathbf{n}} u_k, ..., D^{m-1}_{\mathbf{n}} u_k\right) = \mathbf{K}_\pm r_{\partial\Omega}\left(u, D^1_{\mathbf{n}} u, ..., D^{m-1}_{\mathbf{n}} u\right). \end{aligned}$$

This means that $H^m_{E_\pm}(\mathbb{R}^n)$ is a closed subspace of the space $H^m\left(\Omega^{bl}_\pm\right) \times H^m\left(\complement\overline{\Omega^{bl}_\pm}\right)$.

Let us show that the kernel of $E_\pm$ is trivial. Indeed, if $u \in \operatorname{Ker} E_\pm$, then $U := \{u, v\} = E_\pm u = 0$, i.e. $u = 0$. This means that $\operatorname{Ker} E_\pm = \{0\}$.

Since $\operatorname{Ker} E_\pm = \{0\}$ and $H^m_{E_\pm}(\mathbb{R}^n)$ is the range of $E_\pm$, then the operator $E_\pm : H^m\left(\Omega^{bl}_\pm\right) \to H^m_{E_\pm}(\mathbb{R}^n)$ is an isomorphism. □

Lemma 3. *The mapping $\mathcal{P}_\pm$ defined in (11) can be extended to an isomorphism*

$$\mathcal{P}_\pm : H^m(\Omega^{bl}_\pm) \simeq H^{-m}_0(\Omega^{bl}_\pm)$$

such that $\mathcal{P}_\pm u = P E_\pm u, \quad u \in H^m(\Omega^{bl}_\pm)$ holds and the inverse operator $\mathcal{P}^{-1}_\pm : H^{-m}_0(\Omega^{bl}_\pm) \to H^m(\Omega^{bl}_\pm)$ is represented by the Fourier transform $\mathcal{F}_{x\to\xi}$ in the form $\left(\mathcal{P}^{-1}_\pm f\right)(x) = r_{\Omega^{bl}_\pm} \mathcal{F}^{-1}_{\xi\to x}\left(1/p(\xi)\right) \mathcal{F}_{x\to\xi} f(x)$, or in the form

$$\left(\mathcal{P}^{-1}_\pm f\right)(x) = \int_{\Omega^{bl}_\pm} \Phi(x-y) f_\pm(y)\, dy, \quad x \in \Omega^{bl}_\pm,$$

where $r_{\Omega^{bl}_\pm}$ is the restriction to $\Omega^{bl}_\pm$ and $\Phi(x) = \mathcal{F}^{-1}_{\xi\to x}(p(\xi))^{-1}$.

Proof of Lemma 3. We consider only the case of the operator $\mathcal{P}_+$, because the case of $\mathcal{P}_-$ is quite similar. For any $u \in C^\infty\left(\overline{\Omega^{bl}_+}\right)$ we consider the operator $\mathcal{P}_+$ defined by (11) and (12). By (9), (10), we have

$$0 = P(e_- v) + i \sum_{j=1}^{2m} P_j \sum_{k=0}^{j-1} D^k_{\mathbf{n}}\left(\phi_{j-k} \delta_{\partial\Omega^{bl}_+}\right), \tag{15}$$

$$e_+ (Pu) = P (e_+u) - i \sum_{j=1}^{2m} P_j \sum_{k=0}^{j-1} D_{\mathbf{n}}^k \left(\phi_{j-k} \delta_{\partial\Omega_+^{bl}} \right). \tag{16}$$

Representing (15) in the form

$$i \sum_{j=1}^{2m} P_j \sum_{k=0}^{j-1} D_{\mathbf{n}}^k \left(\phi_{j-k} \delta_{\partial\Omega_+^{bl}} \right) = -P (e_-v),$$

we can rewrite (16) as follows:

$$\mathcal{P}_+u := e_+ (Pu) = P (e_+u) + P (e_-v) = P (e_+u + e_-v) \ . \tag{17}$$

By (17), in view of (13) and (12), we have $\mathcal{P}_+u = PE_+u, \quad u \in C^\infty \left(\overline{\Omega_+^{bl}} \right)$. Taking into account the latter formula, we define

$$\mathcal{P}_+u := PE_+u, \quad u \in H^m \left(\Omega_+^{bl} \right).$$

By Assumption 1, the mapping $P : H^m (\mathbb{R}^n) \to H^{-m} (\mathbb{R}^n)$ is invertible. Let us show that mapping P is an isomorphism of $H_{E_+}^m (\mathbb{R}^n)$ onto $H_0^{-m} \left(\Omega_+^{bl} \right)$.

Let $E_+u \in H_{E_+}^m (\mathbb{R}^n)$ and φ be an arbitrary function from $C_0^\infty (\complement\overline{\Omega_+^{bl}})$. Let $P^* (D) := \sum_{|\alpha| \le 2m} \overline{a}_\alpha D^\alpha$ denote the operator, which is formally adjoint to P. Since $\operatorname{supp} \varphi$ and $\operatorname{supp} P^*\varphi \subset \complement\overline{\Omega_+^{bl}}$, then $(PE_+u, \varphi) = (E_+u, P^*\varphi) = \int_{\Omega_+^{bl}} u\overline{P^*\varphi}dx + \int_{\complement\Omega_+^{bl}} v\overline{P^*\varphi}dx = \int_{\complement\Omega_+^{bl}} v\overline{P^*\varphi}dx = (Pv, \varphi)$.

By (12), $(Pv, \varphi) = 0$. Therefore, $(PE_+u, \varphi) = 0 \quad$ for any $\varphi \in C_0^\infty (\complement\overline{\Omega_+^{bl}})$. This means that $\operatorname{supp} PE_+u \subseteq \overline{\Omega_+^{bl}}$, i.e., P is a bounded operator from $H_{E_+}^m (\mathbb{R}^n)$ to $H_0^{-m} \left(\Omega_+^{bl} \right)$.

Let us show that P maps the space $H_{E_+}^m (\mathbb{R}^n)$ onto $H_0^{-m} \left(\Omega_+^{bl} \right)$. Indeed, if $f \in H_0^{-m} \left(\Omega_+^{bl} \right) \subset H^{-m} (\mathbb{R}^n)$, then we have $U := P^{-1}f \in H^m (\mathbb{R}^n)$. Representing U in the form $U = \{u, v\}$, where $u \in H^m(\Omega_+^{bl})$, $v \in H^m(\complement\overline{\Omega_+^{bl}})$, we get $D_{\mathbf{n}}^j v \Big|_{\partial\Omega_+^{bl}} = D_{\mathbf{n}}^j u \Big|_{\partial\Omega_+^{bl}}$ for $j = 0, 1, ..., m-1$. Since $\operatorname{supp} f = \operatorname{supp} PU \subseteq \overline{\Omega_+^{bl}}$, then $Pv = 0$ in the sense of distributions, i.e., $(Pv, \varphi) = 0$ for any $\varphi \in C_0^\infty (\complement\overline{\Omega_+^{bl}})$.

To show that $U = \{u, v\} = E_+u$, we suppose that $E_+u = \{u, \widetilde{v}\}$. Let us show that $v = \widetilde{v}$. By (9), (10), (12), we have $P\widetilde{v} = 0$. Moreover, $D_{\mathbf{n}}^j \widetilde{v} \Big|_{\partial\Omega_+^{bl}} = D_{\mathbf{n}}^j u \Big|_{\partial\Omega_+^{bl}} \quad (j = 0, 1, ..., m-1)$. Therefore, we obtain that the function $w :=$

$v-\widetilde{v}\in H^{m}(\complement\overline{\Omega_{+}^{bl}})$ satisfies the following homogeneous Dirichlet boundary value problem:

$$Pw=0\ \text{ in }\Omega_{-},\quad D_{\mathbf{n}}^{j}w\big|_{\partial\Omega_{+}^{bl}}=0\quad(j=0,1,...,m-1)\,. \tag{18}$$

It is clear that symbol $p\,(\xi)$ of the operator P satisfies the condition of hypoellipticity

$$\text{if }\alpha\neq0,\ D_{\xi}^{\alpha}p\,(\xi)\,/p\,(\xi)\rightarrow0\quad\text{for }|\xi|\rightarrow\infty. \tag{19}$$

Under condition (19), by the well known result of L. Hörmander [4, Corollary 4.1.2.], we have that if $w\in H^{m}(\complement\overline{\Omega_{+}^{bl}})$ and $Pw\in C^{\infty}\left(\complement\overline{\Omega_{+}^{bl}}\right)$, then $w\in C^{\infty}\left(\complement\overline{\Omega_{+}^{bl}}\right)$. Since, by (18), $Pw=0$, then $w\in C^{\infty}\left(\complement\overline{\Omega_{+}^{bl}}\right)$, and moreover, we get $w=0$, i.e., $U=P^{-1}f=E_{+}u$. This means that P maps $H_{E_{+}}^{m}\left(\mathbb{R}^{n}\right)$ onto $H_{0}^{-m}\left(\Omega_{+}^{bl}\right)$.

It follows that P is an isomorphism from $H_{E_{+}}^{m}\left(\mathbb{R}^{n}\right)$ onto $H_{0}^{-m}\left(\Omega_{+}^{bl}\right)$. Therefore, by Lemma 2, the mapping $PE_{+}:H^{m}(\Omega_{+}^{bl})\rightarrow H_{0}^{-m}\left(\Omega_{+}^{bl}\right)$ is an isomorphism. □

Proof of Theorem. By Lemma 2 and Lemma 3, the function $v\,(x)$ can be represented as follows

$$v\,(x)=r_{\complement\overline{\Omega_{+}^{bl}}}E_{+}\mathcal{P}_{+}^{-1}f.$$

In view of (14), (12) and Lemma 2, $r_{\complement\overline{\Omega_{+}^{bl}}}E_{+}$ is a surjective mapping

$$H^{m}\left(\Omega_{+}^{bl}\right)\rightarrow\left\{v\in H^{m}\left(\complement\overline{\Omega_{+}^{bl}}\right):Pv=0\text{ in }\complement\overline{\Omega_{+}^{bl}}\right\}.$$

Therefore, $\mathcal{S}_{+}=r_{\complement\overline{\Omega_{+}^{bl}}}E_{+}\mathcal{P}_{+}^{-1}$ is a surjective mapping

$$\mathcal{S}_{+}:H_{0}^{-m}\left(\Omega_{+}^{bl}\right)\rightarrow\left\{v\in H^{m}\left(\complement\overline{\Omega_{+}^{bl}}\right):Pv=0\text{ in }\complement\overline{\Omega_{+}^{bl}}\right\}.$$

Since the mapping $\mathcal{S}_{+}$ is a surjective operator onto

$$\left\{v\in H^{m}\left(\complement\overline{\Omega_{+}^{bl}}\right):Pv=0\text{ in }\complement\overline{\Omega_{+}^{bl}}\right\},$$

we get immediately the statement concerning the Dirichlet problem.

For $\mathcal{S}_{-}$, the proof is just the same. □

4. Numerical Algorithm to the Dirichlet Problem

Let us construct an approximate solution to the Dirichlet problem

$$\begin{cases} (I-\Delta)\, v(x) = 0, \quad x \in \Omega, \quad v \in H^1(\Omega), \\ v|_{\partial\Omega} = g \in H^{1/2}(\partial\Omega). \end{cases} \tag{20}$$

We consider a rectangular mesh in the space $\mathbb{R}^n$ $(n = 2, 3)$ and choose only those cells which belong to the thin exterior boundary layer Ω_-^{bl}. Denoting these cells ω_k $(k = 1, ..., N)$ we consider their indicators $f_k(x) := 1$ for $x \in \omega_k$, $f_k(x) := 0$ otherwise. For $\omega_k \in \Omega_-^{bl}$, we introduce basic functions

$$F_k(x) := \int_{\mathbb{R}^n} \Phi(x-y)\, f_k(y)\, dy = \int_{\omega_k} \Phi(x-y)\, dy \quad x \in \mathbb{R}^n. \tag{21}$$

The functions $F_k(x)$ can be represented as $F_k = E_-^{bl}\mathcal{P}_-^{-1} f_k$. Let us note that since the functions f_k are linearly independent and the operators E_-^{bl} and $\mathcal{P}_-^{-1}$ are invertible, then the functions F_k are linearly independent.

It is not difficult to evaluate F_k numerically. Analogously to the boundary element method, the basic functions $F_k(x)$ can be called *boundary layer elements*. Looking for an approximate solution to (20) in the form $\sum_{k=1}^N c_k F_k(x)$ and using N different points $x_j \in \partial\Omega$ $(j = 1, ..., N)$, we construct a linear system with respect to c_k :

$$g(x_j) = \sum_{k=1}^{N} c_k F_k(x_j) \qquad (j = 1, ..., N). \tag{22}$$

Solving the latter system we obtain an approximate solution to the exterior problem (20) in the form

$$v(x) = \sum_{k=1}^{N} c_k F_k(x). \tag{23}$$

Similarly, replacing the exterior boundary layer Ω_-^{bl} by the interior one Ω_+^{bl}, we get a solution to the exterior Dirichlet problem in the form (23).

Numerical Examples. We will consider two numerical examples. Let us consider two functions

$$u_1(x) := \frac{\sinh r}{r \sinh 1}, \quad u_2(x) := \frac{x_3\,(r\cosh r - \sinh r)}{r^3\,(\cosh 1 - \sinh 1)}, \tag{24}$$

where $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, $r = \sqrt{x_1^2 + x_2^2 + x_3^2}$. Each function is a solution to the equation $\Delta u - u = 0, \quad 0 \leq r < 1$ with the corresponding boundary condition $u|_{r=1} = g_i(x) \quad i = 1, 2, \quad g_1(x) = 1, \; g_2(x) = x_3$ for $r = 1$. This means that the functions $u_i(x)$ are solutions to the Dirichlet boundary value problem in the unit ball. Let us construct numerical approximations of these solutions using the previous algorithm. Thus the numerical calculations are done for the case where Ω is the unit ball in $\mathbb{R}^3$. The algorithm was implemented in C. We consider a Cartesian cubic mesh, i.e., the cube $[-2, 2]^3 \subset \mathbb{R}^3$ covered by cubic cells

$$I_{ijk} = \left[-2 + \frac{i-1}{2}, -2 + \frac{i}{2}\right] \times \left[-2 + \frac{j-1}{2}, -2 + \frac{j}{2}\right] \times \left[-2 + \frac{k-1}{2}, -2 + \frac{k}{2}\right],$$

$i, j, k = 1, ..., 8$. Altogether there are 512 cells. Using this mesh we produce an exterior boundary layer for the unit ball which consists of 224 cells. We choose the same number of 224 boundary points belonging to the rays which connect the centers of the boundary layer cells with the origin. To calculate the coefficients $F_k(x_j)$ of the linear system (22) we evaluate the integrals (21) using the following known procedures [9, Section 4.5, 4.6]: the integral over a three-dimensional region as well as the ten-point Gaussian integration for an integral over the interval $[a, b]$. Solving the linear system (22) using the standard Gaussian elimination with pivoting [8, Chapter 6], we get the coefficients c_k. Then by (23), we obtain values of the approximate solution in different points of the boundary. Comparing these values of the approximate solution with the boundary values of the exact solutions (24) in many boundary points we obtain

$$\text{maximum error} = \begin{cases} 0.000023 & \text{for } u_1, \\ 0.000063 & \text{for } u_2. \end{cases}$$

The same values of the maximum relative error are also obtained. According to the known maximum principle, the maximum error over the inner points of the unit ball is not greater.

Thus, the presented numerical method gives a rather accurate approximation. The results obtained so far are quite encouraging but we have considered only relatively simple test cases. We plan to consider more sophisticated cases in the near future.

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Appendix

We denote by $\mathbb{R}$ the set of real numbers, and by $\mathbb{C}$ the set of complex numbers. Let $\mathbb{N} := \{0, 1, ...\}$, $\mathbb{N}_+ := \{1, 2, ...\}$, $\mathbb{R}^n := \{x = (x_1, ..., x_n) : x_i \in \mathbb{R},\ i = 1, ..., n\}$.

Let α be a multi-index, i.e., $\alpha := (\alpha_1, \ldots, \alpha_n)$, $\alpha_j \in \mathbb{N}$, $|\alpha| := \alpha_1 + \ldots + \alpha_n$, $i := \sqrt{-1}$; $D_j := i^{-1}\partial_{x_j} := i^{-1}\partial/\partial x_j$; $D^\alpha := D_1^{\alpha_1} D_2^{\alpha_2} \ldots D_n^{\alpha_n}$.

Let $\Omega = \Omega_+$ denote a bounded connected open set in $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$ (C^∞-class surface) and $\overline{\Omega}$ its closure, i.e., $\overline{\Omega} = \Omega \cup \partial\Omega$.

Let $C^\infty\left(\overline{\Omega}\right)$ be the space of infinitely differentiable up to the boundary functions over $\overline{\Omega}$.

We call differential operators *tangential* when for $x \in \zeta\left(\partial\Omega \times (-\delta, \delta)\right)$ they are of the form $Af = \sum_{j=1}^n a_j(x)\,\partial_{x_j} f(x) + a_0(x)\, f$ with $\sum_{j=1}^n a_j(x)\, n_j(x) = 0$, or are products of such operators.

The derivative along the normal $\mathbf{n} = (n_1, ..., n_n)$ is denoted $\partial_{\mathbf{n}} f := \sum_{j=1}^n n_j(x)\,\partial_{x_j} f(x)$ for $x \in \zeta\left(\partial\Omega \times (-\delta, \delta)\right)$. Let $D_{\mathbf{n}} := i^{-1}\partial_{\mathbf{n}}$.

Let $\Omega_- := \mathbb{R}^n \setminus \overline{\Omega}$ denote the exterior of the domain Ω, and $r_{\partial\Omega}$, r_Ω be respectively the restriction operators to $\partial\Omega$, $\Omega : r_{\partial\Omega} f := f|_{\partial\Omega}$, $r_\Omega f := f|_\Omega$.

Let $H^s(\mathbb{R}^n)$ $(s \in \mathbb{R})$ be the usual Sobolev space over $\mathbb{R}^n$:

$$H^s(\mathbb{R}^n) := \left\{ f \in \mathcal{S}' \ : \ \|f\|_{H^s(\mathbb{R}^n)} := \left\| \mathcal{F}^{-1}\left(1 + |\xi|^2\right)^{s/2} \mathcal{F} f \right\|_{L_2(\mathbb{R}^n)} < \infty \right\},$$

where $\mathcal{F}$ denotes the Fourier transform $f \mapsto \mathcal{F}_{x\to\xi} f(x) = \int_{\mathbb{R}^n} e^{-ix\xi} f(x) dx$, $\mathcal{F}^{-1}$ its inverse, and $\mathcal{S}' = \mathcal{S}'(\mathbb{R}^n)$ denote the space of tempered distributions which is dual to the usual space of rapidly decreasing functions $\mathcal{S}(\mathbb{R}^n)$.

Let $\mathcal{S}\left(\overline{\Omega}_-\right)$ be the space of restrictions of the elements $u \in \mathcal{S}(\mathbb{R}^n)$ to $\overline{\Omega}_-$. Let either $u \in C^\infty\left(\overline{\Omega}\right)$ or $u \in \mathcal{S}\left(\overline{\Omega}_-\right)$, then we set $\gamma_k u := r_{\partial\Omega} D_{\mathbf{n}}^k u = \left(D_{\mathbf{n}}^k u\right)\big|_{\partial\Omega}$.

We denote by $H^s(\Omega_\pm)$ $(0 \le s \in \mathbb{R})$ the space of restrictions of elements of $H^s(\mathbb{R}^n)$ to $\Omega_\pm$. The norms in the space are defined by the relation $\|f\| := \inf \|g\|_{H^s(\mathbb{R}^n)}$, where infimum is taken over all elements $g \in H^s(\mathbb{R}^n)$, which are equal to f in $\Omega_\pm$.

By $H_0^s(\Omega_\pm)$ $(s \in \mathbb{R})$, we denote the closed subspaces of the space $H^s(\mathbb{R}^n)$, which consists of the elements with supports in $\overline{\Omega}_\pm$.