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# STABILIZATION OF JUMP LINEAR SYSTEMS WITH PARTIAL OBSERVATION OF MARKOV MODE

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**Abstract:** This paper deals with the state feedback stabilization of Markov jump linear systems. Feedback synthesis procedure proposed in this paper is Markov mode independent (or partly dependent), which makes the method superior to many other works in that it could handle the situation of partial Markov mode observation. A numerical example is given the illustrate the result.

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## 1. Introduction

Markov jump linear systems, first appearing in the [6] and [9], are gaining more and more consideration in the passed decades. The systems involve both continuous ($x(t)$) and discrete ($r(t)$) state variables, making them capable of simulating stochastic abrupt changing phenomena in systems in application [7]. The systems contain both continuous and discrete components are known as

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hybrid systems. Moreover, the Markov jump linear systems are apparently a specific kind of hybrid systems.

The stabilization of the markov jump linear systems is initially presented [9] and intensively studied in resent years. The numerus methods are employed to deal with this problem include [7], [5], [3], [10], etc.

Most of the works on stabilization, however, are based on the complete Markov mode observation. Fewer works are dealing with the problems of partial or no observation. From the point of view of application, however, the partial observation of Markov mode is apparently more reasonable. As a matter of fact, many abrupt changes or sudden failure in real systems are hard to detect. In [1] the stability of closed loop system without observation of Markov mode is investigated, but a stabilization synthesis procedure is not given. In [8] only sufficient condition of stabilization is presented. To the best of the author's knowledge, there is no equivalent condition for stabilization of the system with partial observation is so far proposed.

In this paper, we deal with the stabilization problem of jump-linear systems with partial information on the Markov mode. A methodology of state-feedback control design is proposed here by solving a series of LMIs (Linear Matrix Inequalities). This approach is different from all the previous works in that the state feedback design could merely rely on partial Markov mode observation, and the condition of stabilization is equivalent to that of the situation where Markov mode could be exactly accessed.

The the rest of the paper is organized as following: In Section 2, the fundamental setups of the issue is introduced and a lemma is introduced; in Section 3 the main result is presented as a theorem and the corresponding corollary; finally in Section 4 a example is given.

The notations in the paper are quite standard. We denote $\begin{bmatrix} A & B \\ B^T & C \end{bmatrix}$ by $\begin{bmatrix} A & B \\ \star & C \end{bmatrix}$ for convenience.

## 2. Problem Statement

We consider the following Linear Markov Jump System (LMJS):

$$\dot{x}(t) = A_{r(t)}x(t) + B_{r(t)}u(t), \tag{1}$$

where $r(t)$ is a left continuous Markov process takes value at a finite set $S = \{1, 2, 3, ..., N\}$ with the generation matrix $\Pi = (\pi_{ij})_{i,j\in S}$, $x(t) \in R^n$ is the system state and is piece-wisely smooth enough. $u(t) \in R^m$ is the control input. $A_i$, $B_i$, $i \in S$ are matrices with appropriate dimensions.

**Definition 1.** (Mean Square Stability) We say that the system (1) is mean square stable(MSS) if

$$\lim_{t\to\infty} E|x(t)|^2 = 0\,.$$

We then quote the standard stability result for MJLS from to [7] and [2].

**Lemma 2.** *The following statements are equivalent:*

*1)The system (sys) is mean square stable when* $u \equiv 0$.

*2) There exist real, symmetric matrices* $P_i$, $i \in S$, *such that the following coupled Lyapunov equations hold ,*

$$A_i^T P_i + P_i A_i + \sum_{j=1}^{N} \pi_{ij} P_j < 0, \qquad P_i > 0, \quad \forall i \in S.$$

LMI technique is widely employed to find the feasible and(or) optimal matrix $P_i$. However, most previous discussions are, to a large extend, refined on the assumption that the Markov mode $r(t)$ is fully accessible. In fact this is a strict hypothesis without considering that many systems in application are too complex to extract the complete underlying Markov mode. As to partial observation situation, [2] proposed a method by simply letting $P_1 = P_2 = ... = P_N$, which makes the result rather conservative and reduces the problem to a trivial one.

## 3. Main Results

The main object in the letter is to find a state-feedback control law, making the close-loop system MSS. The feedback synthesis procedure is obtained under the situation that only part of the states of the Markov chain is accessible.

More precisely, we represent the Markov mode space by decomposing it into a series of sub-sets

$$S = S_1 \bigcup S_2 \bigcup ... \bigcup S_{N_{\mathcal{O}}}\,,$$

where $S_k \bigcap S_l = \emptyset \quad \forall k, l \in \{1, 2, ...N_{\mathcal{O}}\}$. We now introduce a finite set $\mathcal{P} = \{p_1, p_2, ..., p_{N_{\mathcal{O}}}\}$, where each element $p_j$ could be regarded as an "observed state". Once $p_j$ is observed, the real Markov mode, though could not be directly accessed, should be some element in $S_j$. In one extreme, if the $N_{\mathcal{O}} = N$ then the case with complete Markov mode observation is retrieved. In another extreme, if $N_{\mathcal{O}} = 1$ then the Markov mode is unobservable.

The following theorem gives a equivalent condition for the MSS, in light of the *descriptor* approach proposed in [4].

**Theorem 3.** *The following two statements are equivalent:*
*(I) There exist symmetric matrices $P_i$ that solve the following equation*

$$A_i^T P_i + P_i A_i + \sum_{i=1}^{N} \pi_{ij} P_i < 0, \qquad P_i > 0, \quad \forall i \in S\,. \tag{2}$$

*(II) There exit positive definite matrices $P_i$ and matrix $Q$ such that*

$$\begin{bmatrix} QA_i + A_i^T Q^T + \sum_{j=1}^{N} \pi_{ij} P_j & P_i - Q + A_i^T Q^T \\ \star & -Q - Q^T \end{bmatrix} < 0\,. \tag{3}$$

*Proof.* Following the similar argument as in [4], the system (1) with $u \equiv 0$ could be equally represented as

$$\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \dot{\xi} = \begin{bmatrix} 0 & I \\ A_i & -I \end{bmatrix} \xi\,, \tag{4}$$

where $\xi = [x^T, \dot{x}^T]^T$. Therefore, the Lyapunov function $V(x,i)x$ could be re-written as

$$V(x,i) = x^T P_i x = V(\xi, i) = \xi^T \begin{bmatrix} P_i & 0 \\ 0 & 0 \end{bmatrix} \xi\,. \tag{5}$$

Notice that $\begin{bmatrix} P_i & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} P_i & Q \\ 0 & Q \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} P_i & 0 \\ Q^T & Q^T \end{bmatrix}$, we have

$$\begin{aligned} \mathcal{L}V &= \dot{\xi}^T \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} P_i & 0 \\ Q^T & Q^T \end{bmatrix} \xi \\ &\quad + \xi^T \begin{bmatrix} P_i & Q \\ 0 & Q \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \dot{\xi} + \xi^T \begin{bmatrix} \sum_{j=1}^{N} \pi_{ij} P_j & 0 \\ 0 & 0 \end{bmatrix} \xi \\ &= \xi^T \begin{bmatrix} 0 & A_i^T \\ I & -I \end{bmatrix} \begin{bmatrix} P_i & 0 \\ Q^T & Q^T \end{bmatrix} \xi \\ &\quad + \xi^T \begin{bmatrix} P_i & Q \\ 0 & Q \end{bmatrix} \begin{bmatrix} 0 & I \\ A_i & -I \end{bmatrix} \xi + \xi^T \begin{bmatrix} \sum_{j=1}^{N} \pi_{ij} P_j & 0 \\ 0 & 0 \end{bmatrix} \xi\,, \end{aligned} \tag{6}$$

where $\mathcal{L}$ is the (weak) infinitesimal operator defined in [7].

Hence, considering the Lemma 2, the the system is MSS iff

$$\begin{bmatrix} 0 & A_i^T \\ I & -I \end{bmatrix} \begin{bmatrix} P_i & 0 \\ Q^T & Q^T \end{bmatrix} + \begin{bmatrix} P_i & Q \\ 0 & Q \end{bmatrix} \begin{bmatrix} 0 & I \\ A_i & -I \end{bmatrix} + \begin{bmatrix} \sum_{j=1}^N \pi_{ij} P_j & 0 \\ 0 & 0 \end{bmatrix} < 0. \quad (7)$$

The equation (3)is therefore obtained, and we complete the proof. □

The next corollary further modifies the theorem, making it more adaptable for the partial observation feed-back synthesis.

**Corollary 4.** *The equation (2) is equivalent to*

$$\begin{bmatrix} Q_l A_i + A_i^T Q_l^T + \sum_{j=1}^N \pi_{ij} P_j & P_i - Q_l + A_i^T Q_l^T \\ \star & -Q_l - Q_l^T \end{bmatrix} < 0 \quad (8)$$

*for all* $P_i > 0$, $i \in S_l$, $l = p_1, p_2, ..., p_{N_{\mathcal{O}}}$.

We then move to the stabilization synthesis for the system with partial observation of the Markov mode. The feed-back control law is given as:

$$u(t) = \left( \sum_{l \in P} K_l \mathbf{1}_{\{r(t) \in S_l\}} \right) x(t), \qquad t > 0\,,$$

where $\mathbf{1}_{\{\cdot\}}$ is the indicator function. Then the close-loop system is

$$\dot{x} = A_{r(t)} x(t) + B_{r(t)} K_j x(t), \qquad t > 0\,, \quad (9)$$

where $r(t) \in S_j$, $\quad j \in P$ and $K_j \in R^{m \times n}$. The main result of the paper is the following theorem.

**Theorem 5.** *The system (1) is MSS if the matrices* $P_i$ *and* $Q_l$ $Z_l$ *satisfy*

$$\begin{bmatrix} A_i Q_l + Q_l^T A_i^T + B_i Z_l + Z_l^T B_i^T + \sum_{j=1}^N \pi_{ij} P_j & P_i - Q_l + Q_l^T A_i^T + Z_l^T B_i^T \\ \star & -Q_l - Q_l^T \end{bmatrix} < 0, \quad (10)$$

$$P_i > 0.$$

*for all* $i \in S_l$, $l = p_1, p_2, ..., p_{N_{\mathcal{O}}}$. *Then the feed back control matrix is given as*

$$K_l = Z_l Q_l^{-1}$$

*Proof.* Let $\hat{A}_i = A_i + B_i K_l$ and substitute it into equation (8). We obtain

$$\begin{bmatrix} Q_l A_i + Q_l B_i K_l + A_i^T Q_l^T + \\ K_l^T B_i^T Q_l^T + \sum_{j=1}^{N} \pi_{ij} P_j & P_i - Q_l + A_i^T Q_l^T + K_l^T B_i^T Q_l^T \\ \star & -Q_l - Q_l^T \end{bmatrix} < 0. \quad (11)$$

Defining $M \doteq \begin{bmatrix} Q_l^{-1} & 0 \\ 0 & I \end{bmatrix}$ we pre- and post-multiply (11) by $M^T$ and $M$, respectively. Then we replace $Q_l^{-1}$ by $Q_l$ and $Q_l^{-1} P_i Q_l^{-T}$ by $P_i$, we have (10). □

## 4. Numerical Example

Consider a three-mode Markov jump system with

$$A_1 = \begin{bmatrix} -1 & 2/5 \\ 1/2 & -1/5 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -3/10 & -2/5 \\ 1 & 7/5 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} -1/5 & 0 \\ 2 & -1 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1/10 \\ 2 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1/5 \\ 3 \end{bmatrix}$$

and

$$B_3 = \begin{bmatrix} -\frac{7}{10} \\ 1/5 \end{bmatrix}.$$

The transition matrix of the Markov chain $r(t)$ is given as

$$\Pi = \begin{bmatrix} -5.0 & 1.0 & 4.0 \\ 3.500 & -7.500 & 4.0 \\ 2.0 & 4.500 & -6.500 \end{bmatrix}.$$

Applying the above mentioned methodology, we could obtain the state feedbacks (see Table 1) in different situations.

| Complete observation | Partial observation |
|---|---|
| $K_1 = \begin{bmatrix} -0.2576 & -0.4687 \end{bmatrix}$<br>$K_2 = \begin{bmatrix} -0.4374 & -0.8574 \end{bmatrix}$<br>$K_3 = \begin{bmatrix} 6.313 & -0.8761 \end{bmatrix}$ | $K_1 = \begin{bmatrix} -0.2796 & -0.3587 \end{bmatrix}$<br>$K_{\{2,3\}} = \begin{bmatrix} -0.02455 & -0.8254 \end{bmatrix}$ |

| Noobservation |
|---|
| $K_{\{1,2,3\}} = \begin{bmatrix} -0.2056 & -0.6460 \end{bmatrix}$ |

Table 1:

## 5. Conclusions

In this brief, a new method of feedback stabilization synthesis is discussed. The *descriptor* technique is used to decouple the the Markov mode and the feedback gain matrices, and the feedback law is therefore designed in way that there is no need to fully access the Markov mode. LMI technique is employed as tool to conveniently compute the feedback gains. Finally a numerical example is given to illustrate of the feasibility of the procedure.

Moreover, we notice the approach presented is not only confined to state feedback stabilization problem but could be also extended to some other broader areas such as JLQ (Jump Linear Quadratic) optimization problems and $H_\infty$ control and filtering for Markov linear jump systems, which are, in fact, under our further investigation.

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