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2-ASSOCIATIVE AND 2-COASSOCIATIVE HOPF ALGEBRAS OF PLANAR BINARY BITREES

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Abstract: In this paper the 2-associative and 2-coassociative Hopf algebras on planar binary bitrees are constructed. Also, we prove that is possible to endow 2-coassociative coalgebra to 2-coassociative bialgebra.

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1. Introduction

In [4] it has been demonstrated, that it is possible to renormalize the propagators in quantum electrodynamics using planar binary bitrees, where a few Hopf algebra structures, like photon-electron propagator Hopf algebra and charge Hopf algebra are constructed. Motivated by structures of 2-associative algebras, introduced in [6], we try to upgrade some of mentioned structures of planar binary bitrees.

Our intention is to consider two products and two coproducts (the concatenation, the shuffle, the deconcatenation and the shuffle coproduct) and to observe structures we can obtain by theirs combinations. One part of this task (2-associative bialgebra) is considered in [6], and here it will be defined and considered a dual structure, the structure of a 2-coassociative bialgebra. So, in this paper we introduce the coalgebras with two coassociative cooperations, which will be referred to as 2-coassociative coalgebras. Then we study the

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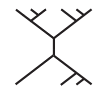

Figure 1: Planar binary bitree

2-coassociative coalgebras equipped with product operation.

Concerning the use of planar binary bitrees, we recall that they have appeared in [2] where it was constructed addition and multiplication on bitrees closely related to addition and multiplication on the integers. Furthermore, as we mentioned, they were used in [4] for renormalization of propagators in quantum electrodynamics, and in [3] for construction of prefix and Huffman prefix code and coding English alphabet in a different way from already established ones.

This paper is organized as follows. In Section 2 we recall basic facts about planar binary bitrees. In Section 3 and Section 4, we consider the structures of 2-associative and 2-coassociative Hopf algebra of planar binary bitrees. Finally, in Section 5 we construct the structure of the shuffle Hopf algebra on planar binary bitrees which is generalization of the Hopf algebra structure on planar binary trees introduced in [7].

Notation. In the following sections it will be supposed that all vector spaces and algebras are defined over the same field K. The vector space spanned by arbitrary set X will be denoted by KX, and the free associative algebra on X (non-commutative polynomials) by $K\langle X\rangle$.

2. About Planar Binary Bitrees

2.1. Definition and Grading

A *planar binary bitree* (p.b. bitree) is an oriented planar graph which contains the upper and lower binary tree whose roots are connected by the edge. This edge is called the root of the planar binary bitree. In every p.b. tree, each internal vertex has two leaves and one root.

The number of internal vertices from the planar binary bitree is called the *degree* of the planar binary bitree, and an ordered pair of numbers of internal vertices from the upper- and lower-trees is called the *bidegree* of the p.b. bitree.

By $\mathbb{Y}_n$ the set of p.b. bitrees of degree n is denoted. Generally speaking, the set $\mathbb{Y}_n$ has c_{n+1} elements, where $c_n = \frac{(2n)!}{n!(n+1)!}$ is the so-called Catalan number (the set of p.b. trees of degree n has c_n elements).

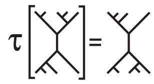

Figure 2: Involution τ

Trees of degree n, denoted by Y_n, which appear in [1], [7] and [8], will be called upper-trees in the future. Observe that $Y_n \subset \mathbb{Y}_n$.

By definition a *planar binary bigrove* of degree n is a non-empty subset of $\mathbb{Y}_n$. We will refer to a bigrove as a disjoint union of bitrees.

2.2. Cutting, Grafting and Involution

On p.b. bitrees the operation of *cutting* (and the inverse operation of *welding*) can be defined, so that by cutting a p.b. bitree of bidegree (k, l) we get the upper-tree of degree k and the lower-tree of degree l. For any bitree x we have $x = x_u \updownarrow x_d$.

For p.b. bitrees a symmetry τ around the axis perpendicular to the root is defined. The symmetry τ maps the p.b. bitree of the bidegree (n, m) to the p.b. bitree of the bidegree (m, n).

2.3. Definition of Addition

The sum of two p.b. bitrees x and y is:

$$x + y = (x_u + y_u) \updownarrow (x_d + y_d),$$

where $x_d + y_d = \tau\ (\tau(x_d) + \tau(y_d))$.

Addition is extended to bigroves by distributivity on both sides:

$$(\cup_j x_j) + (\cup_k y_k) := \cup_{j,k}(x_j + y_k).$$

The defined addition is associative, with the neutral element (bitree | of degree 0), but not commutative.

3. 2-Associative Hopf Algebra of Planar Binary Bitrees

3.1. Introduction

We recall the definitions of Hopf algebra, 2-associative algebra, 2-associative bialgebra and unital infinitesimal bialgebra.

Definition 1. A Hopf algebra $(\mathcal{H}, *, \Delta, \eta, \varepsilon, S)$ is a vector space $\mathcal{H}$ equipped with an associative product $* : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H}$, a unit $\eta : K \to \mathcal{H}$, and a coassociative coproduct $\Delta : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$, a counit $\varepsilon : \mathcal{H} \to K$ such that $*$ and η are morphisms of coalgebras or, equivalently, Δ and ε are morphisms of algebras. An antipode for $\mathcal{H}$ is a linear map $S : \mathcal{H} \to \mathcal{H}$ such that $S \star \mathrm{Id} = \eta \circ \varepsilon = \mathrm{Id} \star S$, where $\star$ is convolution, for two given linear maps $f, g : \mathcal{H} \to \mathcal{H}$ defined by $f \star g : * \circ (f \otimes g) \circ \Delta$.

Definition 2. A *2-associative algebra* over K is a vector space V equipped with two associative operations $(x, y) \mapsto x * y$ and $(x, y) \mapsto x \cdot y$. A 2-associative algebra is said to be *unital* if there is an element 1 which is a unit for both operations. Unless otherwise stated we suppose that the 2-associative algebras are unital.

Definition 3. A *unital infinitesimal bialgebra* $(\mathcal{H}, \cdot, \Delta)$ is a vector space $\mathcal{H}$ equipped with a unital associative product $\cdot$ and a counital coassociative coproduct Δ which are related by the *unital infinitesimal relation*:

$$\Delta(x \cdot y) = (x \otimes 1) \cdot \Delta(y) + \Delta(x) \cdot (1 \otimes y) - x \otimes y,$$

where the product $\cdot$ on $\mathcal{H} \otimes \mathcal{H}$ is given by

$$(x \otimes y) \cdot (x' \otimes y') := x \cdot x' \otimes y \cdot y'.$$

Definition 4. A *2-associative bialgebra* (resp. 2-associative Hopf algebra) $(\mathcal{H}, *, \cdot, \Delta)$ is a vector space $\mathcal{H}$ equipped with two operations $*$ and $\cdot$ and one cooperation Δ, such that:

- $(\mathcal{H}, *, \Delta)$ is a bialgebra (resp. Hopf algebra),
- $(\mathcal{H}, \cdot, \Delta)$ is a unital infinitesimal bialgebra.

3.2. 2-Associative Hopf Algebra of Planar Binary Bitrees

In this section we furnish the set of planar binary bitrees with a structure of 2-associative Hopf algebra.

Theorem 5. *The quadruple* $(K\langle\mathbb{Y}\rangle/(1-|), *, \cdot, \Delta)$ *is a 2-associative Hopf algebra.*

Proof. Let $K\langle\mathbb{Y}\rangle/(1-|)$ be a free associative algebra on the set of bitrees where we identify the formal unit 1 with the bitree $|$. This algebra is equipped with the associative product called concatenation:

$$t_1 \dots t_i \cdot t_{i+1} \dots t_n := t_1 \dots t_n.$$

Dually, the vector space $K\langle\mathbb{Y}\rangle/(1-|)$ equipped with the coassociative co-operation Δ called *deconcatenation*,

$$\Delta(t_1\cdots t_n)=\sum_{i=0}^{n} t_1\cdots t_i\otimes t_{i+1}\cdots t_n$$

form a structure of a coalgebra.

It is known that the tensor coalgebra is a Hopf algebra for the product induced by the shuffles (c.f. [5], p.68):

$$t_1\dots t_p * t_{p+1}\dots t_{p+q}:=\sum_{\sigma} t_{\sigma_1}\dots t_{\sigma_{p+q}},$$

where the sum is extended to all permutations σ which are (p,q)-shuffles.

So, the triple $(K\langle\mathbb{Y}\rangle/(1-|), *, \Delta)$ is a Hopf algebra, and we only need to prove that $(K\langle\mathbb{Y}\rangle/(1-|), \cdot, \Delta)$ is a unital infinitesimal bialgebra.

Let us compute $\Delta(x\cdot y)$ for $x=t_1\dots t_k$ and $y=t_{k+1}\dots t_n$:

$$\begin{aligned}\Delta(x\cdot y)&=\Delta(t_1\dots t_n)=\sum_{i=0}^{n} t_1\dots t_i\otimes t_{i+1}\dots t_n\\ &=\sum_{i=0}^{k} t_1\dots t_i\otimes t_{i+1}\dots t_n - t_1\dots t_k\otimes t_{k+1}\dots t_n+\sum_{i=k}^{n} t_1\dots t_i\otimes t_{i+1}\dots t_n\\ &=\Delta(t_1\dots t_k)\cdot(1\otimes t_{k+1}\dots t_n)-t_1\dots t_k\otimes t_{k+1}\dots t_n+(t_1\dots t_k\otimes 1)\\ &\qquad\cdot\Delta(t_{k+1}\dots t_n)=(x\otimes 1)\cdot\Delta(y)-x\otimes y+\Delta(x)\cdot(1\otimes y).\quad\square\end{aligned}$$

Remark 6. The constructed 2-associative Hopf algebra of planar binary bitrees is isomorphic to the so-called 2-associative shuffle Hopf algebra over tensor shuffle algebra $T^{sh}(V)$, introduced in [6].

4. 2-Coassociative Hopf Algebra of Planar Binary Bitrees

4.1. Introduction

In this section we introduce the algebras with two coassociative cooperations, which will be called 2-coassociative coalgebras. Then we study the 2-coassociative coalgebras equipped with a product operation.

The concept of a 2-coassociative coalgebra is dual to the concept of a 2-associative algebra in the following sense. Paraphrasing the definition of a 2-associative algebra we define a 2-coassociative coalgebra in the following way.

Definition 7. A *2-coassociative coalgebra* ($2coas(V)$) over K is a vector space V equipped with two coassociative cooperations Δ and δ. A 2-coassociative coalgebra is said to be *counital* if there is a counit common for both cooperations. Unless otherwise stated we suppose that the 2-coassociative coalgebras are counital.

Definition 8. A $2coas(V)$ is cofree if V is primitive for both cooperations and if one coalgebra from $2coas(V)$ is isomorphic to the tensor coalgebra over V.

It is clear that a cofree $2coas(V)$ is graded and of the form $2coas(V) = \oplus_{n\geq 0} 2coas(V)_n$, where $2coas(V)_0 = K$, $2coas(V)_1 = V$, etc.

Definition 9. A *2-coassociative bialgebra* (resp. 2-coassociative Hopf algebra) $(\mathcal{H}, \cdot, \Delta, \delta)$ is a vector space $\mathcal{H}$ equipped with two cooperations Δ and δ and one operation $\cdot$, such that

- $(\mathcal{H}, \cdot, \delta)$ is a bialgebra (resp. Hopf algebra),
- $(\mathcal{H}, \cdot, \Delta)$ is a unital infinitesimal bialgebra.

4.2. 2-Coassociative Hopf Algebra of Planar Binary Bitrees

In this section we construct the structure of 2-coassociative Hopf algebra of planar binary bitrees.

Theorem 10. *The quadruple ($K\langle \mathbb{Y} \rangle/(1 - |)$, $\cdot$, Δ, δ) is a 2-coassociative Hopf algebra.*

Proof. Let $K\langle \mathbb{Y} \rangle/(1 - |)$ be a vector space equipped with the coassociative cooperation Δ,

$$\Delta(t_1 \cdots t_n) = \sum_{i=0}^{n} t_1 \cdots t_i \otimes t_{i+1} \cdots t_n,$$

and the coassociative cooperation δ,

$$\delta(t_1 \cdots t_n) = 1 \otimes t_1 \cdots t_n + \sum_{i=1}^{n-1} \sum_{\sigma} t_{\sigma(1)} \cdots t_{\sigma(i)} \otimes t_{\sigma(i+1)} \cdots t_{\sigma(n)} + t_1 \cdots t_n \otimes 1,$$

where σ is a $(p, n-p)$-shuffle. Then $K\langle \mathbb{Y} \rangle/(1 - |)$ with these two coassociative cooperations form a structure of 2-coassociative coalgebra. In both coalgebras hold $\varepsilon(t) = 0$ and $\Delta(t) = \delta(t) = 1 \otimes t + t \otimes 1$ for all $t \in \mathbb{Y}$. Obviously, $\varepsilon(t_1 \ldots t_n) = 0$ for all $t_1, \ldots, t_n \in \mathbb{Y}$, because of multiplicativity of ε. If we take the coalgebra structure determined by the cooperation Δ and furnish it with the concatenation product:

$$t_1 \dots t_i \cdot t_{i+1} \dots t_n := t_1 \dots t_n,$$

then we get a Hopf algebra structure. The usual procedure of endowing a tensor algebra to bialgebra structure is described in [5].

An antipod in this bialgebra is defined by:

$$S(|) = | \quad \text{and} \quad S(t_1 \dots t_n) = (-1)^n t_n \dots t_1.$$

So, the triple $(K\langle\mathbb{Y}\rangle/(1-|), \cdot, \delta)$ is a Hopf algebra, and we already know that $(K\langle\mathbb{Y}\rangle/(1-|), \cdot, \Delta)$ is a unital infinitesimal bialgebra. □

Remark 11. The constructed bialgebra is cofree, i.e. as a coalgebra, it is isomorphic to a tensor coalgebra over the space of primitive elements of corresponding vector space.

Proposition 12. *There exists a unique operation $\mu(x,y) := x \cdot y$ on the cofree 2-coassociative coalgebra $2coas(V)$ which makes it into a 2-coassociative bialgebra.*

Proof. We define $\mu : 2coas(V) \otimes 2coas(V) \to 2coas(V)$ by the following requiremarkents:

- $\mu(1 \otimes 1) = 1,$
- $\mu(v \otimes 1) = \mu(1 \otimes v) = v, \quad \forall\ v \in V,$
- $\mu(v \otimes w) = v \otimes w, \quad \forall\ v, w \in V,$
- $(\mu \otimes \mu)(\mathrm{Id} \otimes \tau \otimes \mathrm{Id})(\Delta(x) \otimes \Delta(y)) = \Delta(\mu(x \otimes y)),$
- $(\mu \otimes \mu)(x \otimes 1 \otimes \delta(y)) + (\mu \otimes \mu)(\delta(x) \otimes 1 \otimes y)) - x \otimes y = \delta(\mu(x \otimes y)).$

Let us prove that the product is well-defined by induction on the degree of the elements in $2coas(V) = \oplus_{n \geq 0} 2coas(V)_n$. It is already defined on $2coas(V)_2 = V \otimes V$. Suppose that μ is defined up to $2coas(V)_{n-1}$. Any element of $2coas(V)_n$ is of the form $x \otimes y$ for elements x and y of degree strictly smaller then n. Then the product is given by the required formulas.

Note that the fourth requiremarkent claims that the product is a homomorphism of coalgebras what is a precondition for existence of a Hopf algebra structure. Similarly, the fifth requiremarkent implies an existence of the infinitesimal bialgebra.

Since the only relations in $2coas(V)$ are the coassociativity of Δ and δ, and counitality for both coproducts, we need to verify that

$$\begin{aligned}
(\mu\otimes\mu)((\Delta\otimes\mathrm{Id})\Delta(x)) &= (\mu\otimes\mu)((\mathrm{Id}\otimes\Delta)\Delta(x)),\\
(\mu\otimes\mu)((\delta\otimes\mathrm{Id})\delta(x)) &= (\mu\otimes\mu)((\mathrm{Id}\otimes\delta)\delta(x)),\\
\mu((\mathrm{Id}\otimes\varepsilon)\Delta(x)) &= \mu((\varepsilon\otimes\mathrm{Id})\Delta(x)),\\
\mu((\mathrm{Id}\otimes\varepsilon)\delta(x)) &= \mu((\varepsilon\otimes\mathrm{Id})\delta(x)).
\end{aligned}$$

First of all, we prove the last two lines which have similar prove:

$$\mu((\mathrm{Id}\otimes\varepsilon)\Delta(x)) = \mu((\mathrm{Id}\otimes\varepsilon)(\sum_{(x)} x'\otimes x'')) = \mu(\sum_{(x)} x'\otimes\varepsilon(x'')) = \mu(x\otimes 1),$$

$$\mu((\varepsilon\otimes\mathrm{Id})\Delta(x)) = \mu((\varepsilon\otimes\mathrm{Id})(\sum_{(x)} x'\otimes x'')) = \mu(\sum_{(x)} \varepsilon(x')\otimes x'') = \mu(1\otimes x).$$

Now we verify the first one:

$$\begin{aligned}
&(\mu\otimes\mu)((\Delta\otimes\mathrm{Id})\Delta(x))\\
&\quad= (\mu\otimes\mu)((\Delta\otimes\mathrm{Id})(\sum_{(x)} x'\otimes x'')) = (\mu\otimes\mu)(\sum_{(x)}(\Delta(x')\otimes(1\otimes x'')))\\
&\quad= \sum_{(x)}\Delta(\mu(x'\otimes x'')) = \sum_{(x)}(\mu\otimes\mu)((x'\otimes 1)\otimes\Delta(x''))\\
&\quad= (\mu\otimes\mu)(\sum_{(x)}(x'\otimes\Delta(x''))) = (\mu\otimes\mu)((\mathrm{Id}\otimes\Delta)(\sum_{(x)} x'\otimes x''))\\
&\quad= (\mu\otimes\mu)((\mathrm{Id}\otimes\Delta)\Delta(x)).
\end{aligned}$$

Finally, let us check the second one. On the left side we have

$$\begin{aligned}
&((\delta\otimes\mathrm{Id})\delta)(\mu(x\otimes y))\\
&\quad= (\delta\otimes\mathrm{Id})[(\mu\otimes\mu)((x\otimes 1)\otimes\delta y) + (\mu\otimes\mu)(\delta y\otimes(1\otimes y)) - x\otimes y]\\
&\quad= (\delta\otimes\mathrm{Id})[\mu(x\otimes 1)\otimes\mu\delta y + \mu\delta x\otimes\mu(1\otimes y) - x\otimes y]\\
&\quad= \delta(\mu(x\otimes 1))\otimes\mu\delta y + \delta(\mu\delta x)\otimes\mu(1\otimes y) - \delta x\otimes y\\
&\quad= x\otimes 1\otimes\mu\delta y + \mu\delta x\otimes 1\otimes\mu\delta y - x\otimes 1\otimes\mu\delta y\\
&\qquad + \mu\delta x\otimes 1\otimes y + \delta x\otimes y - \mu\delta x\otimes 1\otimes y - \delta x\otimes y.
\end{aligned}$$

and on the right site

$$
\begin{aligned}
&((\mathrm{Id}\otimes\delta)\delta)(\mu(x\otimes y))\\
&\quad=(\mathrm{Id}\otimes\delta)[(\mu\otimes\mu)((x\otimes 1)\otimes\delta y)+(\mu\otimes\mu)(\delta y\otimes(1\otimes y))-x\otimes y]\\
&\quad=(\mathrm{Id}\otimes\delta)[\mu(x\otimes 1)\otimes\mu\delta y+\mu\delta x\otimes\mu(1\otimes y)-x\otimes y]\\
&\quad=\mu(x\otimes 1)\otimes\delta(\mu\delta y)+\mu\delta x\otimes\delta(\mu(1\otimes y))-x\otimes\delta y\\
&\quad=x\otimes\delta y+\mu\delta x\otimes 1\otimes\mu\delta y-\mu\delta x\otimes 1\otimes y\\
&\qquad+x\otimes 1\otimes\mu\delta y+\mu\delta x\otimes 1\otimes y-x\otimes 1\otimes\mu\delta y-x\otimes\delta y.
\end{aligned}
$$

We see that they are equal. Hence μ is well defined. It follows that $(2coas(V),\mu,\Delta,\delta)$ is a 2-coassociative bialgebra. □

5. Shuffle Hopf Algebra of Planar Binary Bitrees

5.1. Introduction

If we take structures of the algebra with the shuffle product and the coalgebra with the shuffle coproduct, it can be shown that it is not possible to acquire a structure of a bialgebra, because the shuffle coproduct is not homomorphism of algebras. Therefore we consider modified shuffle product and coproduct on the set of planar binary bitrees and construct Hopf algebra structure on bitrees. This structure represent generalization of the corresponding Hopf algebra structure on planar binary trees.

Let us recall some basic facts about planar binary trees. The Hopf algebra structure of planar binary trees was investigated in [7]. In this article J.-L. Loday and M.O. Ronco showed that the set of planar binary trees Y_∞, or more precisely, the graded vector space

$$k[Y_\infty]:=\oplus_{n\geq 0}k[Y_n],$$

has the structure of a graded Hopf algebra. The product was defined by

$$X^y * X^z := X^{y+z},\qquad y,z\in Y_\infty$$

and restriction of coproduct on Y_n by the formula

$$\Delta(x)=\sum_{j,k}\big((x^l_{(j)}+x^r_{(k)})\otimes(x^l_{(n-j)}\vee x^r_{(m-k)})\big)+x\otimes 0,$$

where

$$\Delta(x^l)=\sum_j x^l_{(j)}\otimes x^l_{(n-j)},\quad \Delta(x^r)=\sum_k x^r_{(k)}\otimes x^r_{(m-k)}$$

and $\Delta(0)=0\otimes 0$.

Remark 13. We called this algebra the Shuffle Hopf algebra of planar binary trees, since the product in this algebra is derived from the shuffle product on a symmetric group.

5.2. Shuffle Hopf Algebra of Planar Binary Bitrees

Let K be a commutative ring. We introduce $K[\mathbb{Y}]$, the vector space with basis $X^y, y \in \mathbb{Y}$. We define the product analogically to [7]:

$$X^y * X^z = X^{y+z},$$

since $y+z$ need not be a bitree but a bigrove, $X^{y_1 \cup y_2} := X^{y_1} + X^{y_2}$.

Before defining the coproduct, we define the tensor product

$$(x_u \updownarrow x_d) \otimes (y_u \updownarrow y_d) := (x_u \otimes y_u) \updownarrow (x_d \otimes y_d),$$

and extend this formula by distributivity on both sides with respect to the addition:

$$\Big(\sum_i (x_u^i \otimes y_u^i)\Big) \updownarrow \Big(\sum_j (x_d^j \otimes y_d^j)\Big) = \sum_{i,j} \Big((x_u^i \updownarrow x_d^j) \otimes (y_u^i \updownarrow y_d^j)\Big).$$

Let the restriction of the coproduct on the set of planar binary bitrees $\mathbb{Y}$ be given by the following relation:

$$\Delta(x) = \Delta(x_u) \updownarrow \Delta(x_d),$$

where $\Delta(x_d) = (\tau \otimes \tau)\big(\Delta(\tau(x_d))\big)$.

The counit in a coalgebra is the map $\varepsilon : k[\mathbb{Y}] \to k$, defined by relations

$$\varepsilon(X^{|}) = 1, \quad \text{i} \quad \varepsilon(X^x) = 0 \ \ \forall\, | \neq x \in \mathbb{Y}.$$

Theorem 14. *The sixtuple ($k[\mathbb{Y}]$, $*$, η, Δ, ε, S) is a graded Hopf algebra, non-commutative and non-cocommutative.*

Proof. The associativity of the product follows from the associativity of addition on planar binary bitrees, the neutral element is bitree $X^{|}$ and the coassociativity of the coproduct follows from the definition of the coproduct on p.b. trees and the coassociativity of the coproduct on p.b. trees:

$$\begin{aligned}(\Delta \otimes \mathrm{id})\Delta(x) = (\Delta \otimes \mathrm{id})\Delta(x_u \updownarrow x_d) &= (\Delta \otimes \mathrm{id})\big(\Delta(x_u)\big) \updownarrow \big(\Delta(x_d)\big) \\ &= \big((\Delta \otimes \mathrm{id})\Delta(x_u)\big) \updownarrow \big((\Delta \otimes \mathrm{id})\Delta(x_d)\big)\end{aligned}$$

$$= \big((\mathrm{id}\otimes\Delta)\Delta(x_u)\big) \updownarrow \big((\mathrm{id}\otimes\Delta)\Delta(x_d)\big)$$
$$= (\mathrm{id}\otimes\Delta)\big(\Delta(x_u)\big) \updownarrow \big(\Delta(x_d)\big) = (\mathrm{id}\otimes\Delta)\Delta(x_u \updownarrow x_d) = (\mathrm{id}\otimes\Delta)\Delta(x)\,.$$

The coproduct Δ is a homomorphism of algebras:

$$\Delta(x+y) = \Delta\big((x+y)_u \updownarrow (x+y)_d\big) = \big(\Delta(x+y)_u\big) \updownarrow \big(\Delta(x+y)_d\big)$$
$$= \big(\Delta(x_u)+\Delta(y_u)\big) \updownarrow \big(\Delta(x_d)+\Delta(y_d)\big) = \Delta(x_u \updownarrow x_d) + \Delta(y_u \updownarrow y_d)$$
$$= \Delta(x) + \Delta(y).$$

The antipode S is the algebra anti-homomorphism automatically defined on generators by the recursive formula $S(|) = |$ and

$$S(x) = -x - \sum_{P(x)} S(x_1)x_2 = -x - \sum_{P(x)} x_1 S(x_2),$$

where $P(x) = \Delta(x) - x \otimes | - | \otimes x$. For upper trees, antipod $S(x_u)$ is defined in [7], and for lower-trees we define: $S(x_d) = -\tau(S(\tau(x_d)))$. □

Remark 15. The defined Hopf algebra of planar binary bitrees is isomorphic to tensor product of Hopf algebra on planar binary trees with itself.

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