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NONCREASY BANACH SPACES

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Abstract: In this paper, the properties of noncreasy and local uniformly noncreasy are investigated.

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1. Introduction

Let X be a Banach space. Denote by B_X and S_X the unit ball and the unit sphere of X respectively, and $D(x) = \{f \in X^* : f(x) = \|f\| = 1\}$. The concepts and background material on this paper can be found in [3]. We take a functional $x^* \in S_{X^*}$ and a scalar $\delta \in [0,1]$, set

$$S(x^*, \delta) = \{x \in B_X : x^*(x) \geq 1 - \delta\}.$$

There has appeared a lot of papers discussing the geometry of Banach spaces from the viewpoint of slices $S(x^*, \delta)$. Special attention has been paid to discuss the convexity and smoothness, for instance [9]. It is easy to see the uniform convexity and uniform smoothness can be easily characterized in terms of slices $S(x^*, \delta)$.

Theorem. *Let X be a Banach space, then X is uniformly convex if and only if for any $\varepsilon > 0$ there exists $\delta > 0$ such that* $\operatorname{diam} S(x^*, \delta) \leq \varepsilon$ *for each functional $x^* \in S_{X^*}$.*

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Let $x^*, y^* \in S_{X^*}$ and $\delta \in [0,1]$, we put $S(x^*, y^*, \delta) = S(x^*, \delta) \cap S(y^*, \delta)$, then we can easy to see the following theorem.

Theorem. *A Banach space X is uniformly smooth if and only if for any $\varepsilon > 0$ there exists $\delta > 0$ such that $S(x^*, y^*, \delta) = \Phi$ for any functionals $x^*, y^* \in S_{X^*}$ with $\|x^* - y^*\| \geq \varepsilon$.*

The aim of this paper is to study the property of the noncreasy introduced by S. Prus [9] and the local uniformly noncreasy which will be introduced below.

2. Noncreasy Banach Spaces

In [9], S. Prus defined the creasy and noncreasy as follows.

Definition 2.1. The unit sphere of Banach space X is said to has a crease if there are two distinct functionals $x^*, y^* \in S_{X^*}$ such that $\text{diam}S(x^*, y^*, 0) > 0$. Banach space X is called noncreasy if its unit sphere does not have a crease.

Let $x = (1, 1, 1, 0, ..., 0)$, $y = (1, 0, 1, 0, ..., 0) \in c_0$, $x^* = (1, 0, 0, 0, ..., 0)$, $y^* = (0, 0, 1, 0, ..., 0) \in l_1$, then $x, y \in S_{c_0}$ and $x^*, y^* \in S_{l_1}$ with $x^*(x) = x^*(y) = 1, y^*(x) = y^*(y) = 1$, so Banach space c_0 is creasy.

S. Prus [9] showed that if Banach space X is uniformly rotund (smooth) then X is uniformly noncreasy and a uniformly noncresy space is superreflexive. Furthermore, he proved that uniformly noncreasy spaces have the fixed point. In [5], Lin Bor-Luh and Shi Zhongrui determined conditions for an Orlicz function spaces with Orlicz/Luxemburg norm to be uniformly noncreasy or noncreasy. First of all, we list some elementary properties of the creasy.

Theorem 2.2. *If Banach space X is creasy, then X is neither strictly convex nor smooth.*

Proof. Since Banach space X is creasy, we have two distinct functionals $x^*, y^* \in S_{X^*}$ such that $\text{diam}S(x^*, y^*, 0) > 0$, so there exists $x, y \in B_X, x \neq y$ with $x^*(x) = 1, x^*(y) = 1, y^*(x) = 1$ and $y^*(y) = 1$, thus $x, y \in S_X, \frac{x+y}{2} \in S_X$ and $x^*, y^* \in D(x)$, hence X is neither strictly convex nor smooth. □

Theorem 2.3. *If Banach space X is noncreasy, then any $x \in S_X$, x must be a smooth point or extreme point of B_X.*

Proof. Let $z \in S_X$, and suppose that z is not a smooth point, then there exists $x^*, y^* \in S_{X^*}$, $x^* \neq y^*$, with $x^*(z) = y^*(z) = 1$. If there are $x, y \in S_X, x \neq y$, such that $z = \frac{x+y}{2}$, then we have $x^*(x) = x^*(y) = 1$ and $y^*(x) = y^*(y) = 1$, which contradicts $S(x^*, y^*, 0) = 0$, hence x is an extreme point of B_X. □

In l_∞, neither $x = (1, 1, 1, 0, ..., 0) \in S_{l_\infty}$ is an extreme point nor a smooth point, by above theorem, we know that l_∞ is creasy.

Let us introduce good extreme point and nice extreme point to discuss noncreasy.

Definition 2.4. Let $x \in S_X$, x is called a good extreme point of B_X, if $x = y$ whenever $f(y) = 1$ for all $f \in D(x)$.

Definition 2.5. Let $x \in S_X$, x is called a nice extreme point of B_X, if $x = y$ whenever $f(y) = 1$ for some $f \in D(x)$.

For the reader's convenience, we recall the definition of rotund point. Let X be Banach space, $x \in S_X$ is called rotund point in B_X if for any $y \in B_X$ with $\|x+y\| = 2$, then $x = y$. It is easy to see nice extreme point must be good extreme point, but good extreme point may be not nice extreme point, and if $x \in S_X$ is a good extreme point of B_X, then x is an extreme point of B_X. For example, in l_1, let $e_1 = (1, 0, ..., 0) \in S_{l_1}$, there exist $e_2 = (0, 1, 0, ..., 0) \in S_{l_1}$ and $x^* = (1, 1, 0, ..., 0) \in S_{l_\infty}$ such that $x^*(e_1) = x^*(e_2)$, so e_1 is a good extreme point of B_{l_1}, but it is not a nice extreme point.

It is easy to see the following theorem holds.

Theorem 2.6. *Let X be a Banach space X, then:*

(1) If X is noncreasy then every $x \in S_X$ is a smooth point or good extreme point of B_X.

(2) If every $x \in S_X$ is a smooth point or rotund point of B_X then X is noncreasy.

Proof. Suppose that Banach space X is noncreasy, let $x_0 \in S_X$, and suppose that x_0 is not a smooth point, then there exists $x^*, y^* \in S_{X^*}$, $x^* \neq y^*$, with $x^*(x_0) = y^*(x_0) = 1$, i.e. $x^*, y^* \in D(x_0)$. If there is some $y_0 \in S_X, x_0 \neq y_0$, such that $f(y_0) = 1$ for all $f \in D(x_0)$, then we have $x^*(x_0) = x^*(y_0) = 1$ and $y^*(x_0) = y^*(y_0) = 1$, so $x_0, y_0 \in S(x^*, 0) \cap S(y^*, 0)$, which contradicts $S(x^*, y^*, 0) = 0$, hence x_0 is a good extreme point of B_X.

Suppose that every $x \in S_X$ is a smooth point or rotund point of B_X, and X is creasy, then there are two distinct functionals $x^*, y^* \in S_{X^*}$ such that $\operatorname{diam} S(x^*, y^*, 0) > 0$. So there exist $x, y \in S_X, x \neq y$ such that $x, y \in S(x^*, 0) \cap S(y^*, 0)$, i.e. $x^*(x) = y^*(x) = 1$ and $x^*(y) = y^*(y) = 1$, by $x^*(x) = y^*(x) = 1$ and $x^* \neq y^*$, we know x is not smooth point. Since $x^*(y) = 1$ for $x^* \in D(x)$, so $\|x + y\| = 2$, by x is a rotund point, we have $x = y$, which contracts $x \neq y$, hence X is noncreasy. □

Theorem 2.7. *Let X be a Banach space, if there is a $x \in S_X$, x is neither smooth point nor extreme point, then there exits a two dimension subspace M of X which is neither smooth nor rotund.*

Proof. If we have a $x_0 \in S_X$, x_0 is neither smooth point nor extreme point, then there are $f \neq g, f, g \in D(x_0)$, and there exist $y \neq z, y, z \in S_X$, such that $x_0 = \frac{y+z}{2}$, so $f, g \in D(y)$ and $f, g \in D(z)$. Let $M = \operatorname{span}\{y, z\}$, then $y, z \in S_M$, it is easy to see that M is not rotund, and $y, z \in S_M$, y, z is not smooth point of M. Hence M s neither smooth nor rotund. □

We all know that Banach space X is rotund if and only if its two dimensional subspace is rotund, but it is not true for noncreasy. c_0 is creasy, and every two dimensional subspace of c_0 is noncreasy. It is easy to see in R^2, let $\|x\| = |x_1| + |x_2|$, then $(R^2, \|.\|)$ is noncreasy, and $(R^2, \|.\|)$ is not rotund and is not smooth.

In [9], S. Prus proved that if Banach space X_0, X_1 are uniformly convex and uniformly smooth, then the space $(X_0 \oplus X_1)_\infty$ and $(X_0 \oplus X_1)_1$ are uniformly noncreasy. Now a very natural question arises: If Banach space X_0, X_1 are smooth and rotund, then the space $(X_0 \oplus X_1)_\infty$ and $(X_0 \oplus X_1)_1$ are noncreasy or not?

Example 2.8. Let $x = (x_1, x_2, x_3) \in R^3$, $\|x\| = \max\{(|x_1|^2 + |x_2|^2)^{\frac{1}{2}}, |x_3|\}$, then it is easy to see $X_0 = R^2, \|(x_1, x_2)\| = (|x_1|^2 + |x_2|^2)^{\frac{1}{2}}$, $X_1 = R, \|x_3\| = |x_3|$, and $R^3 = (X_0 \oplus X_1)_\infty$. Let $x = (\sqrt{2}/2, \sqrt{2}/2, \sqrt{2}/2)$, then $\|x\| = 1$, $x = [(\sqrt{2}/2, \sqrt{2}/2, 3\sqrt{2}/5) + (\sqrt{2}/2, \sqrt{2}/2, 2\sqrt{2}/5)]/2$, so x is not an extreme point. Set $f = (\sqrt{2}/2, \sqrt{2}/2, 0), g = (0, \sqrt{2}/2, \sqrt{2}/2)$, then $f, g \in D(x)$, thus x is not smooth point, hence $(R^3, \|.\|)$ is creasy.

3. Locally Uniformly Noncreasy Banach Spaces

We recall the definition of uniformly noncreasy [9].

Definition 3.1. A Banach space X is uniformly noncreasy provided that for every $\varepsilon > 0$ there is $\delta > 0$ such that if $x^*, y^* \in S_{X^*}$ and $\|x^* - y^*\| \geq \varepsilon$, then

$$\operatorname{diam} S(x^*, y^*, \delta) \leq \varepsilon .$$

Next, we introduce the notion of locally uniformly noncreasy.

Definition 3.2. A Banach space X is locally uniformly noncreasy provided that for each $x \in S_X$, for any $\varepsilon > 0$ there is $\delta = \delta(x, \varepsilon) > 0$ such that if $x^* \in D(x), y^* \in S_{X^*}$ and $\|x^* - y^*\| \geq \varepsilon$, then

$$\operatorname{diam} S(x^*, y^*, \delta) \leq \varepsilon .$$

Theorem 2.3. *Let X be a Banach space, if X is locally uniformly rotund, then X is locally uniformly noncreasy.*

Proof. Suppose that X is not locally uniformly noncreasy, then there exist $x_0 \in S_X$, $\varepsilon_0 > 0$ such that for any $\delta_n = \frac{1}{n}$, we have $x_n^* \in D(x_0), y_n^* \in S_{X^*}$ with $\|x_n^* - y_n^*\| \geq \varepsilon_0$ and diam $S(x_n^*, y_n^*, \delta_n) \geq \varepsilon_0$. So we have $x_n, y_n \in B_X$, such that $x_n^*(x_n) \geq 1 - \frac{1}{n}, x_n^*(y_n) \geq 1 - \frac{1}{n}, y_n^*(x_n) \geq 1 - \frac{1}{n}, y_n^*(x_n) \geq 1 - \frac{1}{n}$ and $\|x_n - y_n\| \geq \varepsilon_0$. Thus $\|x_n + x_0\| \geq x_n^*(x_n) + x_n^*(x_0) \geq 1 - \frac{1}{n} + 1 \to 2$, since X is locally uniformly rotund, we have $\|x_n - x_0\| \to 0$. Similarly, we also have $\|y_n - x_0\| \to 0$, but it contradicts $\|x_n - y_n\| \geq \varepsilon_0$, hence Banach space X is locally uniformly noncreasy. □

With the similar technique, it is easy to prove the following theorem.

Theorem 3.4. *Let X be a Banach space, if X^* is locally uniformly rotund, then X is locally uniformly noncreasy.*

Theorem 3.5. *Let X be a Banach space, if X is locally uniformly noncreasy, then for any $x \in S_X, x$ is an extreme point of B_X or a strongly smooth point.*

Proof. Suppose that $x_0 \in S_X$ and x_0 is neither an extreme point of the unit ball nor a strongly smooth point, then we have $\varepsilon_1 > 0, y, z \in S_X, \|y - z\| \geq \varepsilon_1$, such that $x_0 = \frac{y+z}{2}$, and there exist $x_n^* \in S_{X^*}$, such that $x_n^*(x_0) \to 1$, with $\|x_n^* - x^*\| \geq \varepsilon_2$, where $x^* \in D(x_0)$. Thus for x_0 and $\varepsilon_0 = \min\{\varepsilon_1, \varepsilon_2\} > 0$, for any $\delta > 0$, there exist $n_0 \in N$, such that whenever $n > n_0$, we have $x_n^*(x_0) > 1 - \delta$, and $\|x_n^* - x^*\| \geq \varepsilon_0$, so $x_n^*(y) > 1 - \delta$, $x_n^*(z) > 1 - \delta$, thus $y, z \in S(x_n^*, \delta)$. Obviously, we have $y, z \in S(x^*, \delta)$, so we know $y, z \in S(x_n^*, x^*, \delta)$, $\|x_n^* - x^*\| \geq \varepsilon_0$ and $\|y - z\| \geq \varepsilon_0$, but it contracts X is locally uniformly noncreasy. □

Corollary 3.6. *Let X be a Banach space, if X is locally uniformly noncreasy and unit ball B_X has no extreme point, then X is strongly smooth space.*

We recall the definition of strong convex point [10]. Let X be a Banach space. $x \in S_X$ is called a strong convex point of B_X if for any $x_n \in B_X, x^*(x_n) \to 1$ for all $x^* \in D(x)$, then $\lim\ x_n = x$.

Theorem 3.7. *Let X be a Banach space, if X is locally uniformly noncreasy, then for any $x \in S_X$, x is a smooth point or a strongly convex point.*

Proof. Suppose that $x_0 \in S_X$ is neither a smooth point nor a strong convex point, then there exists $x^*, y^* \in D(x_0)$, such that $\|x^* - y^*\| \geq \varepsilon_1$ for some $\varepsilon_1 > 0$, and we have $x_n \in S_X$, such that for all $x^* \in D(x_0)$, $x^*(x_n) \to 1$, and $\|x_n - x_0\| \geq \varepsilon_2$. Set $\varepsilon_0 = \min\{\varepsilon_1, \varepsilon_2\}$, then for any $\delta > 0$, there is $n_0 \in N$, such that $x^*(x_n) > 1 - \delta$, and $y^*(x_n) > 1 - \delta$ for all $n > n_0$, so $x_n \in S(x^*, y^*, \delta)$. Obviously, $x_0 \in S(x^*, y^*, 0) \subseteq S(x^*, y^*, \delta)$, thus diam $S(x^*, y^*, \delta) \geq \varepsilon_0$, which gives us a contradiction. □

We recall Banach space X is 2UR if any sequence in its unit sphere such that $lim\|x_m + x_n\| = 2$ is a Cauchy sequence [1].

Theorem 3.8. *Let X be a 2UR Banach space, if X is smooth and rotund, then X is locally uniformly noncreasy.*

Proof. Suppose that X is not locally uniformly noncreasy, then there exist $x_0 \in S_X$, $\varepsilon_0 > 0$ such that for any $\delta_n = \frac{1}{n}$, we have $x_n^* \in D(x_0), y_n^* \in S_{X^*}$

with $\|x_n^* - y_n^*\| \geq \varepsilon_0$ and diam $S(x_n^*, y_n^*, \delta_n) \geq \varepsilon_0$. So we have $x_n, y_n \in B_X$, such that $x_n^*(x_n) \geq 1 - \frac{1}{n}, x_n^*(y_n) \geq 1 - \frac{1}{n}, y_n^*(x_n) \geq 1 - \frac{1}{n}, y_n^*(x_n) \geq 1 - \frac{1}{n}$ and $\|x_n - y_n\| \geq \varepsilon_0$. Since x is a smooth point, thus $x_n^* = x^* \in D(x_0)$ for all $n \in N$.

By $x^*(x_n) \geq 1 - \frac{1}{n}, x^*(y_n) \geq 1 - \frac{1}{n}$, we have $x_n, y_n \in S(x^*, \frac{1}{n})$. Since X is 2UR, we have x_n converges to x_0, and y_n converges to y_0, thus $x_0, y_0 \in S_X$, and $x^*(x_0) = x^*(y_0) = 1$, by X is rotund, we know that $x_0 = y_0$, but it contracts $\|x_n - y_n\| \geq \varepsilon_0$, hence Banach space X is locally uniformly noncreasy. □

Rolewicz [8] introduced the notion of drop property for Banach spaces. For any x not belong to B_X, the drop determined by x is the set $D(x, B_X) = \mathrm{conv}(\{x\} \bigcup B_X)$. A Banach space X has the drop property if for every closed set C disjoint with B_X there exists an element $x \in C$ such that

$$D(x, B_X) \bigcap C = \{x\}\,.$$

Theorem 3.9. *Let X be a Banach space has drop property, if X is smooth and rotund, then X is locally uniformly noncreasy.*

Proof. Suppose that X is not locally uniformly noncreasy, then there exist $x_0 \in S_X$, $\varepsilon_0 > 0$ such that for any $\delta_n = \frac{1}{n}$, we have $x_n^* \in D(x_0), y_n^* \in S_{X^*}$ with $\|x_n^* - y_n^*\| \geq \varepsilon_0$ and diam $S(x_n^*, y_n^*, \delta_n) \geq \varepsilon_0$. So we have $x_n, y_n \in B_X$, such that $x_n^*(x_n) \geq 1 - \frac{1}{n}, x_n^*(y_n) \geq 1 - \frac{1}{n}, y_n^*(x_n) \geq 1 - \frac{1}{n}, y_n^*(x_n) \geq 1 - \frac{1}{n}$ and $\|x_n - y_n\| \geq \varepsilon_0$. Since x is a smooth point, thus $x_n^* = x^* \in D(x)$ for all $n \in N$.

By $x^*(x_n) \geq 1 - \frac{1}{n}, x^*(y_n) \geq 1 - \frac{1}{n}$, we have $x_n, y_n \in S(x^*, \frac{1}{n})$. Since Banach space X has drop property, X is reflexive, so we have x_{n_k} weak converges to x_0, and y_{n_k} weak converges to y_0, thus $x_0, y_0 \in S_X$, and $x^*(x_0) = x^*(y_0) = 1$, by X is rotund, we know that $x_0 = y_0$, by Banach space has drop property is (H), we know x_{n_k} converges to x_0 in norm, and y_{n_k} converges to x_0 in norm, it contracts $\|x_n - y_n\| \geq \varepsilon_0$, hence Banach space X is locally uniformly noncreasy. □

We recall that Banach space X is said to be nearly uniformly convex [4], if for every $\varepsilon > 0$, there exists $\delta \in (0, 1)$, such that for every sequence $\{x_n\} \in B_X$ with sep ($\{x_n\}) > \varepsilon$, we have

$$\mathrm{conv}\,(\{x_n\}) \bigcap (1 - \delta) B_X \neq \Phi$$

By S. Prus [9], we know that the space $(R \bigoplus l_2)_\infty$ is uniformly noncreasy, but it is not nearly uniformly convex. Since Banach space X is nearly uniformly convex imply X has the drop property, so it is easy to see that the following corollary holds.

Corollary 3.10. *If Banach space X is nearly uniformly convex and smooth and rotund, then X is locally uniformly noncreasy.*

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